In all of the examples that we have seen so far, we have calculated the distribution of a random variable, Y, which is defined as a function of another random variable, X.

What about the case where we define a random variable, Z, as a function of multiple random variables?

For example, here is the function of two random variables.

How can we find a distribution of Z?

The general methodology is exactly the same.

We somehow calculate the CDF of the random variable Z and then differentiate to find its PDF.

Let us illustrate this methodology with a simple example.

So suppose that X and Y are independent random variables and each one of them is uniform on the unit interval.

So their joint distribution is going to be a uniform PDF on the unit square.

We're interested in the random variable, which is defined as the ratio of Y divided by X.

So we will now calculate the CDF of Z and then differentiate.

It is useful to work in terms of a diagram.

This is essentially our sample space, the unit square.

The PDF of X is 1 on the unit interval.

The PDF of Y is 1 on the unit interval.

Because of independence, the joint PDF is the product of their individual PDFs.

So the joint PDF is equal to 1 throughout this unit square.

So now let us write an expression for the CDF of Z, which, by definition, is the probability that the random variable Z, which in our case is Y divided by X, is less than or equal than a certain number, little z.

What is the probability of this event?

Let us consider a few different cases.

Suppose that z is negative.

What is the probability that this ratio is negative?

Well, since X and Y are non-negative numbers, there's no way that the ratio is going to be negative.

So if little z is a negative number, the probability of this event is going to be equal to 0.

This is the easier case.

Now suppose that z is a positive number.

Let us draw a line that has a slope of little z.

y/z being less than or equal to little z is the same as saying that y is less than or equal to little z times x.

This is the line on which y is equal to z times x.

So below that line, y is going to be less than or equal to z times x.

So the event of interest is actually this triangle here.

And the probability of this event, since we're dealing with a uniform distribution on the unit square, is just the area of this triangle.

Now, since this line rises at slope z, this point here, this intercept is at z.

And so the sides of the triangle are 1 and z.

And so this formula here gives us the value of the CDF for the case where little z is positive.

And the same formula would also be true if z also were equal to 0, in which case, we get 0 probability.

But is this correct for all positive z's?

Well, not really.

This calculation was based on this picture.

And in this picture, this line intercepted this side of the unit square.

And for that to happen, this slope must be less than or equal to 1.

So this formula is only correct in the case where we have a slope of less than or equal to 1.

And now we need to deal with the remaining case in which little z is strictly larger than 1.

In this case, we get a somewhat different picture.

If we draw a line with slope, again, little z, because little z is bigger than 1, it's going to intercept this side of the rectangle.

Now, the event that Y/X is less than or equal to little z is, again, the event that the pair, X, Y, lies below this line that has a slope of z.

So all we need is to find the area of this region.

One way of finding the area of this region is to take the area of the entire unit square, which is equal to 1, and subtract the area of this triangle.

What is the area of this triangle?

Well, since this line has a slope of z, in order for it to rise to a value of 1, x must be equal to 1 over little z.

Therefore, this side of the triangle is 1/z.

And therefore, the area of the triangle is 1/2 times 1/z, which is this expression here.

And so we have found the value of the CDF for all possible choices of little z.

We can draw the CDF.

And the picture is as follows.

For z negative, the CDF is equal to 0.

For z between 0 and 1, the CDF rises linearly at a slope of 1/2.

And so when z is equal to 1, the CDF has risen to a value of 1/2.

And then as z goes to infinity, this term disappears and the CDF will converge to 1.

So it converges to 1 monotonically but in a non-linear fashion.

So we get a picture of this type.

The next step, the final step, is to differentiate the CDF and obtain the PDF.

In this region, the CDF is constant, so its derivative is going to be equal to 0.

In this region, the CDF is linear, so its derivative is equal to this factor of 1/2.

So the CDF is equal to 1/2 for z's between 0 and 1.

And finally, in this region, this is the formula for the CDF.

When we take the derivative, we get the expression 1 over 2z squared, which is a function that decreases as z goes to infinity.

So it has a shape like this one.

So we have completed the solution to this problem.

We found the CDF, and we found the corresponding PDF.

This methodology works more generally for more complicated functions of X and Y and for more complicated distributions for X and Y. Of course, when the functions or the distributions are more complicated, the calculus involved and the geometry may require a lot more work.

But conceptually, the methodology is exactly the same.