GILBERT STRANG: OK. This is our last look at the first order linear differential equation that you see up here. The dy dt is ay, that's the interest rate growing in the bank example. y is our total balance. And q of t is our deposits or withdrawals.

Only one change. We allow the interest rate a to change with time. This we didn't see before. Now we will get a formula. It will be a formula we had before when a was constant. And now we'll see it. It looks a little messier, but the point is, it can be done. We can solve that equation by a new way.

So that's really the other point. Everybody in the end likes these integrating factors. And I will call it m. And let me show you what it is and how it works. What it is is the solution to the null equation, with a minus sign. With a minus sign. dM dt equals minus a of tM. No source term. We can solve that equation.

If a is constant-- and I'll keep that case going because that's the one with simple, recognizable formulas. If a is a constant, we're looking for the function M whose derivative is minus aM. And that function is e to the minus at.

The derivative brings down the minus a that we want. In case a is varying, we can still to solve this equation. It will still be an exponential of minus something. But what we have to put here when I take the derivative of M, the derivative of that will come down. So I want the integral of a here. And then the derivative of the integral is a minus a, coming down as it should.

So I want minus the integral of a. And can I introduce dummy variables, say a of T dT, just to make the notation look right. OK. You see that, again, the derivative of M is always with an exponential. It's always the exponential times the derivative of the exponent. And the derivative of that exponent is minus a. Because by the fundamental theorem of calculus, if I integrate a and take its derivative, I get a again. And it's that a that I want.

Now, why do I want this M? How does it work? Here's the reason M succeeds. Look at the derivative of M times y. That's a product. So I'll use the product rule. I get the derivative of y times M, and then I get the derivative of M times y. But the derivative of M is minus a of tM, so I better put the derivative of M is minus a of tM times y.

But what have I got here? Factor out an M and that's just dy dt minus ay, dy dt minus ay is q.

So when I factor out the M, I just have q. All together, this is M times q. Look, my differential equation couldn't look nicer. Multiplying by M made it just tell us that a derivative is a right-hand side. To solve that equation, we just integrate both sides.

So if you'll allow me to take that step, integrate both sides and see what I've got, that will give us the formula we know when we're in the constant case, and the formula we've never seen when t is varying. And then I'll do an example. Let me do an example right away.

Suppose a of t, instead of being constant, is growing. The economy is really in hyperinflation. Take that example if a of t is, let's say, 2t. Interest rate started low and moves up, then growth is going to be faster and faster as time goes on. And what will be the integral of 2t?

The integral of 2t is t squared, so M, in that case, will be e to the minus a t squared. Sorry, there's no a anymore. a is just the 2t. e to the minus t squared. With a minus sign, it's dropping fast. In a minute, we'll have a plus sign there and we'll see the growth. Do you see that this is the integrating factor when a of t happens to be 2t?

OK. Now I come back to this equation and integrate both sides to get the answer. OK. All right. The integral of My, of the derivative, the integral of the derivative is just M of t y of t minus M of 0 y of 0. That's the integral on the left side. And on the right side, I have the integral of M times q from 0 to t.

And again, I'm going to put in an integration variable different from t just to keep things straight. OK. So now I've got a formula for y. It involves the M. Actually, the y is multiplied by M, I better divide by-- first of all, do we remember what M of 0 is?

That's the growth factor at 0. It's just 1. Nothing's happened. It's the exponential of 0 in our formulas for M. M of 0 is 1. That's where M starts. So M of 0 is 1. I can remove that.

OK. And now-- oh, let me put that on the other side so this will be equals y of 0 plus that. OK. And now if I divide by M, I have my answer. So those are the steps. Find the integrating factor. Do the integration, which is now made easy because I have a perfect derivative whose integral I just have to integrate. And then put in what M is, and divide by it so that I get y.

OK. So I'm dividing by M. So what is 1 over M? Well, M has this minus sign in the exponent. 1 over M will have a plus sign. M here has e to the minus t squared. 1 over M will be e to the plus t squared.

So when I divide by M, I get y of t. This will be 1 over M. That will be e to the plus the integral of a of t dt y of 0. That's the null solution. That's the solution that's growing out of y of 0. And now I have plus the integral from 0 to t of-- remember, I'm dividing by M. And that's e to the plus the integral from 0 to s of a times q of s ds.

OK. Oh, just a moment. I'm dividing by M and I had an M there. Oh, wait a minute. I haven't got it right here. So I want to know what is M at time s divided by M at time t? So this was the integral from 0 to s. This is an integral from 0 to t. And both are in the exponent.

This is-- can I say it here? This is a e to the-- divided by M is the integral from 0 to t. And then I'm multiplying by e to the minus the integral from 0 to s. The rule for exponents is if I have a product of two exponentials, I add the exponents. When I add this to this, this knocks off the lower half of the integral. I'm left with the integral from s to t of a.

So this was an integral of a minus an integral of a. Let me do our example. Our example up here. Example-- M of t will be-- when a is equal to 2t, this was the example a equal to 2t. The first time we've been able to deal with a varying interest rate. So the integral of 2t is t squared. From the-- is e to the t squared. And I subtract the lower limit, s squared. That's the growth factor.

That's the growth factor from time s to time t. When a was constant, that exponent was just a times t minus s. That told us the time. But now, a is varying and the growth factor between s and t is e to the t squared minus s squared. So that's what goes in here. Let me-- that's the growth factor.

May I just put it in here? In this example, it's e to the t squared minus s squared. Instead of e to the a t minus s, I now have t squared minus s squared, because I had an integral of a of t, and a is not constant anymore. This is my example. And I don't know if you like this formula. Can I just describe it again?

This was an integral from 0 to t, so that would be-- this part would be e to the t squared. That's the growth factor that multiplies the initial deposit. The growth factor that multiplies the later deposit is e to the t squared minus s squared. And we allow deposits all the way from s equals 0 to t. So when we add those up, we get that sum.

We've solved an equation that we hadn't been able to solve before. That's a small triumph in differential equations. Small, admittedly. I'd rather move next to non-linear equations, which we have not touched. And that's a big deal. Thank you.