CHAPTER 15 VECTOR CALCULUS

15.1 Vector Fields (page 554)

An ordinary function assigns a value $f(x)$ to each point x. A vector field assigns a ***vector*** $\mathbf{F}(x,y)$ to each point (x,y). Think of the vector as going out from the point (not out from the origin). The vector field is like a head of hair! We are placing a straight hair at every point. Depending on how the hair is cut and how it is combed, the vectors have different lengths and different directions.

The vector at each point (x,y) has two components. Its horizontal component is $M(x,y)$, its vertical component is $N(x,y)$. *The vector field is* $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Remember: A vector from every point.

1. Suppose all of the vectors $\mathbf{F}(x,y)$ have length 1, and their directions are *outward* (or *radial*). Find their components $M(x,y)$ and $N(x,y)$.

   - At a point like (3,0) on the x axis, the outward direction is the x direction. The vector of length 1 from that point is $\mathbf{F}(3,0) = \mathbf{i}$. This vector goes outward *from the point.* At (0,2) the outward vector is $\mathbf{F}(0,2) = \mathbf{j}$. At the point $(-2,0)$ it is $\mathbf{F}(-2,0) = -\mathbf{i}$. (The *minus* x direction is outward.) At every point (x,y), the outward direction is parallel to $x\mathbf{i} + y\mathbf{j}$. This is the "position vector" $\mathbf{R}(x,y)$.

   We want an outward spreading field of **unit** vectors. So divide the position vector $\mathbf{R}$ by its length:

   $$\mathbf{F}(x,y) = \frac{\mathbf{R}(x,y)}{|\mathbf{R}(x,y)|} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}. \quad \text{This special vector field is called } \mathbf{u}_r.$$

   The letter $\mathbf{u}$ is for "unit," the subscript r is for "radial." No vector is assigned to the origin, because the outward direction there can't be decided. Thus $\mathbf{F}(0,0)$ is not defined (for this particular field). Then we don't have to divide by $r = \sqrt{x^2+y^2} = 0$ at the origin.

   To repeat: The field of outward unit vectors is $\mathbf{u}_r = \frac{\mathbf{R}}{r}$. Another way to write it is $\mathbf{u}_r = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$. The components are $M(x,y) = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$ and $N(x,y) = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$.

2. Suppose again that all the vectors $\mathbf{F}(x,y)$ are unit vectors. But change their directions to be *perpendicular* to $\mathbf{u}_r$. The vector at (3,0) is $\mathbf{j}$ instead of $\mathbf{i}$. Find a formula for this "***unit spin field.***"

   - We want to take the vector $\mathbf{u}_r$, at each point (x,y) except the origin, and ***turn that vector by*** 90°. The turn is counterclockwise and the new vector is called $\mathbf{u}_\theta$. It is still a unit vector, and its dot product with $\mathbf{u}_r$ is zero. Here it is, written in two or three different ways:

   $$\begin{aligned} \mathbf{u}_\theta &= -\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j} \text{ is perpendicular to } \mathbf{u}_r = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j} \\ \mathbf{u}_\theta &= \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j} = \frac{1}{r}(-y\ \mathbf{i} + x\ \mathbf{j}) = \mathbf{S}/r. \end{aligned}$$

   Where $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ was the position field (outward), $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$ is the ***spin field*** (around the origin). The lengths of $\mathbf{R}$ and $\mathbf{S}$ are both $r = \sqrt{x^2+y^2}$, increasing as we move outward. For unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ we divide by this length.

   Two other important fields are $\mathbf{R}/r^2$ and $\mathbf{S}/r^2$. At each point one is still outward (parallel to $\mathbf{R}$) and the other is "turning" (parallel to $\mathbf{S}$). But now the lengths decrease as we go outward. The length of $\frac{\mathbf{R}}{r^2}$ is $\frac{r}{r^2}$ or $\frac{1}{r}$. This is closer to a typical men's haircut.

So far we have six vector fields: three radial fields $\mathbf{R}$ and $\mathbf{u}_r = \mathbf{R}/r$ and $\mathbf{R}/r^2$ and three spin fields $\mathbf{S}$ and $\mathbf{u}_\theta = \mathbf{S}/r$ and $\mathbf{S}/r^2$. The radial fields point along ***rays***, out from the origin. The spin fields are tangent to ***circles***, going around the origin. These rays and circles are the ***field lines*** or ***streamlines*** for these particular vector fields.

The field lines give the direction of the vector $\mathbf{F}(x, y)$ at each point. The length is not involved (that is why $\mathbf{S}$ and $\mathbf{S}/r$ and $\mathbf{S}/r^2$ all have the same field lines). The direction is tangent to the field line so the slope of that line is $dy/dx = N(x, y)/M(x, y)$.

3. Find the field lines (streamlines) for the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$.

   - Solve $dy/dx = N/M = -y/x$ by separating variables. We have $\frac{dy}{y} = -\frac{dx}{x}$. Integration gives

$$\ln y = -\ln x + C. \text{ Therefore } \ln x + \ln y = C \text{ or } \ln xy = C \text{ or } xy = c.$$

The field lines $xy = c$ are hyperbolas. The vectors $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ are tangent to those hyperbolas.

4. ***A function*** $f(x, y)$ ***produces*** **a** ***gradient field*** $\mathbf{F}$. Its components are $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. This field has the special symbol $\mathbf{F} = \nabla f$. Describe this gradient field for the particular function $f(x, y) = xy$.

   - The partial derivatives of $f(x, y) = xy$ are $\partial f/\partial x = y$ and $\partial f/\partial y = x$. Therefore the gradient field is $\nabla \mathbf{f} = y\mathbf{i} + x\mathbf{j}$. Not a radial field and not a spin field.

Remember that the gradient vector gives the direction in which $f(x, y)$ changes fastest. This is the "steepest direction." Tangent to the curve $f(x, y) = c$ there is no change in f. ***Perpendicular to the curve there is maximum change.*** This is the gradient direction. So the gradient field $y\mathbf{i} + x\mathbf{j}$ of Problem 4 is perpendicular to the field $x\mathbf{i} - y\mathbf{j}$ of Problem 3.

In Problem 3, the field is tangent to the hyperbolas $xy = c$. In Problem 4, the field is perpendicular to those hyperbolas. The hyperbolas are called ***equipotential lines*** because the "potential" xy is "equal" (or constant) along those curves $xy = c$.

Read-throughs and selected even-numbered solutions :

A vector field assigns a **vector** to each point (x, y) or (x, y, z). In two dimensions $\mathbf{F}(x, y) = \mathbf{M(x,y)i+N(x,y)j}$. An example is the position field $\mathbf{R} = \mathbf{x}\,\mathbf{i} + \mathbf{y}\,\mathbf{j}(+\mathbf{z}\,\mathbf{k})$. Its magnitude is $|\mathbf{R}| = \mathbf{r}$ and its direction is **out from the origin.** It is the gradient field for $f = \frac{1}{2}(\mathbf{x^2} + \mathbf{y^2})$. The level curves are **circles,** and they are **perpendicular** to the vectors $\mathbf{R}$.

Reversing this picture, the spin field is $\mathbf{S} = -\mathbf{y}\,\mathbf{i} + \mathbf{x}\,\mathbf{j}$. Its magnitude is $|\mathbf{S}| = \mathbf{r}$ and its direction is **around the origin.** It is not a gradient field, because no function has $\partial f/\partial x = -\mathbf{y}$ and $\partial f/\partial y = \mathbf{x}$. $\mathbf{S}$ is the velocity field for flow going **around the origin.** The streamlines or **field** lines or integral **curves** are **circles.** The flow field $\rho\mathbf{V}$ gives the rate at which **mass** is moved by the flow.

A gravity field from the origin is proportional to $\mathbf{F} = \mathbf{R}/\mathbf{r^3}$ which has $|\mathbf{F}| = \mathbf{1}/\mathbf{r^2}$. This is Newton's **inverse** square law. It is a gradient field, with potential $f = \mathbf{1}/\mathbf{r}$. The equipotential curves $f(x, y) = c$ are **circles.** They are **perpendicular** to the field lines which are **rays.** This illustrates that the **gradient** of a function $f(x, y)$ is **perpendicular** to its level curves.

The velocity field $y\,\mathbf{i} + x\,\mathbf{j}$ is the gradient of $f = \mathbf{xy}$. Its streamlines are **hyperbolas.** The slope dy/dx of a streamline equals the ratio $\mathbf{N/M}$ of velocity components. The field is **tangent** to the streamlines. Drop a leaf onto the flow, and it goes along **a streamline.**

2 $x\mathbf{i} + \mathbf{j}$ is the gradient of $f(x,y) = \frac{1}{2}\mathbf{x}^2 + \mathbf{y}$, which has **parabolas** $\frac{1}{2}x^2 + y = c$ as equipotentials (they open down). The streamlines solve $dy/dx = 1/x$ (this is N/M). So $y = \ln x + C$ gives the streamlines.

6 $x^2\mathbf{i} + y^2\mathbf{j}$ is the gradient of $f(x,y) = \frac{1}{3}(\mathbf{x}^3 + \mathbf{y}^3)$, which has closed curves $x^3 + y^3 =$ constant as equipotentials. The streamlines solve $dy/dx = y^2/x^2$ or $dy/y^2 = dx/x^2$ or $\mathbf{y}^{-1} = \mathbf{x}^{-1}$ + constant.

14 $\frac{\partial f}{\partial x} = \mathbf{2x - 2}$ and $\frac{\partial f}{\partial y} = \mathbf{2y}; \mathbf{F} = (2x-2)\mathbf{i} + 2y\mathbf{j}$ leads to circles $(x-1)^2 + y^2 = c$ around the center (1,0).

26 $f(x,y) = \frac{1}{2}\ln(\mathbf{x}^2 + \mathbf{y}^2) = \ln\sqrt{\mathbf{x}^2 + \mathbf{y}^2}$. This comes from $\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$ or $f = \int \frac{x\,dx}{x^2+y^2}$.

32 From the gradient of $y - x^2$, $\mathbf{F}$ must be $-\mathbf{2xi} + \mathbf{j}$ (or this is $-\mathbf{F}$).

36 $\mathbf{F}$ is the gradient of $f = \frac{1}{2}\mathbf{ax}^2 + \mathbf{bxy} + \frac{1}{2}\mathbf{cy}^2$. The equipotentials are **ellipses** if $ac > b^2$ and **hyperbolas** if $ac < b^2$. (If $ac = b^2$ we get straight lines.)

15.2 Line Integrals (page 562)

The most common line integral is along the x axis. We have a function $y(x)$ and we integrate to find $\int y(x)dx$. Normally this is just called an integral, without the word "line." But now we have functions defined at every point in the xy plane, so we can integrate along curves. A better word for what is coming would be "curve" integral.

Think of a curved wire. The density of the wire is $\rho(x,y)$, possibly varying along the wire. Then the total mass of the wire is $\int \rho(x,y)ds$. This is a line integral or curve integral (or wire integral). Notice $ds = \sqrt{(dx)^2 + (dy)^2}$. We use dx for integrals along the x axis and dy up the y axis and ds for integrals along other lines and curves.

1. A circular wire of radius R has density $\rho(x,y) = x^2y^2$. How can you compute its mass $M = \int \rho\ ds$?

   - Describe the circle by $x = R\cos t$ and $y = R\sin t$. You are free to use θ instead of t. The point is that we need a parameter to describe the path and to compute ds:

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt = \sqrt{R^2\sin^2 t + R^2\cos^2 t}\ dt = R\ dt.$$

   Then the mass integral is $M = \int x^2y^2\ ds = \int_{t=0}^{2\pi}(R^2\cos^2 t)(R^2\sin^2 t)R\ dt$. I won't integrate.

This chapter is about vector fields. But we integrate scalar functions (like the density ρ). So if we are given a vector $\mathbf{F}(x,y)$ at each point, we take its dot product with another vector – to get an ordinary scalar function to be integrated. Two dot products are by far the most important:

$$\mathbf{F}\cdot\mathbf{T}\,ds = \text{component of } \mathbf{F} \textit{ tangent} \text{ to the curve} = M\ dx + N\ dy$$
$$\mathbf{F}\cdot\mathbf{n}\,ds = \text{component of } \mathbf{F} \textit{ normal} \text{ to the curve} = M\ dy - N\ dx$$

The unit tangent vector is $\mathbf{T} = \frac{d\mathbf{R}}{ds}$ in Chapter 12. Then $\mathbf{F}\cdot\mathbf{T}\ ds$ is $\mathbf{F}\cdot d\mathbf{R} = (M\ \mathbf{i} + N\ \mathbf{j})\cdot(dx\ \mathbf{i} + dy\ \mathbf{j})$. This dot product is $M\ dx + N\ dy$, which we integrate. Its integral is the ***work*** done by $\mathbf{F}$ along the curve. Work is force times distance, but the distance is measured parallel to the force. This is why the tangent component $\mathbf{F}\cdot\mathbf{T}$ goes into the work integral.

2. Compute the work by the force $\mathbf{F} = x\mathbf{i}$ around the unit circle $x = \cos t,\ y = \sin t$.

   - Work $= \int M\ dx + N\ dy = \int x\ dx + 0\ dy = \int_{t=0}^{2\pi} (\cos t)(-\sin t\ dt)$. This integral is zero!

3. Compute the work by $\mathbf{F} = x\mathbf{i}$ around a square: along $y = 0$, up $x = 1$, back along $y = 1$, back down $x = 0$.

   - Along the x axis, the direction is $\mathbf{T} = \mathbf{i}$ and $\mathbf{F} \cdot \mathbf{T} = x\mathbf{i} \cdot \mathbf{i} = x$. Work $= \int_0^1 x\ dx = \frac{1}{2}$.

   Up the line $x = 1$, the direction is $\mathbf{T} = \mathbf{j}$. Then $\mathbf{F} \cdot \mathbf{T} = x\mathbf{i} \cdot \mathbf{j} = 0$. No work.

   Back along $y = 1$ the direction is $\mathbf{T} = -\mathbf{i}$. Then $\mathbf{F} \cdot \mathbf{T} = -x$. The work is $\int \mathbf{F} \cdot \mathbf{T} ds = \int -x\ dx = -\frac{1}{2}$.

   Note! You might think the integral should be $\int_1^0 (-x) dx = +\frac{1}{2}$. Wrong. Going left, ds is $-dx$.

   The work down the y axis is again zero. $\mathbf{F} = x\mathbf{i}$ is perpendicular to the movement $d\mathbf{R} = \mathbf{j}\ dy$. So $\mathbf{F} \cdot \mathbf{T} = 0$.

   Total work around square $= \frac{1}{2} + 0 - \frac{1}{2} + 0 =$ *zero.*

4. Does the field $\mathbf{F} = x\mathbf{i}$ do zero work around *every* closed path? If so, why?

   - *Yes*, the line integral $\int \mathbf{F} \cdot \mathbf{T}\ ds = \int x\ dx + 0\ dy$ is always zero around closed paths. The antiderivative is $f = \frac{1}{2}x^2$. ***When the start and end are the same point*** P ***the definite integral is*** $f(P) - f(P) = 0$.

   We used the word "antiderivative." From now on we will say "***potential function***." This is a function $f(x, y)$ – if it exists – such that $df = M\ dx + N\ dy$:

   The potential function has $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. Then $\int_P^Q M\ dx + N\ dy = \int_P^Q df = f(Q) - f(P)$.

   The field $\mathbf{F}(x, y)$ is the ***gradient*** of the potential function $f(x, y)$. Our example has $f = \frac{1}{2}x^2$ and $\mathbf{F} = \nabla f = x\mathbf{i}$. Conclusion: Gradient fields are conservative. The work around a *closed* path is zero.

5. Does the field $\mathbf{F} = y\mathbf{i}$ do zero work around every closed path? If not, why not?

   - This is not a gradient field. There is no potential function that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. We are asking for $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = 0$ which is impossible. The work around the unit circle $x = \cos t,\ y = \sin t$ is

   $$\int M\ dx + N\ dy = \int y\ dx = \int_{t=0}^{2\pi} (\sin t)(-\sin t\ dt) = -\pi. \text{ Not zero!}$$

   Important A gradient field has $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. Every function has equal mixed derivatives $\frac{\partial^2 f}{\partial x\ \partial y} = \frac{\partial^2 f}{\partial y\ \partial x}$. Therefore the gradient field has $\partial M/\partial y = \partial N/\partial x$. This is the quick test "**D**" for a gradient field. $\mathbf{F} = y\mathbf{i}$ fails this test as we expected, because $\partial M/\partial y = 1$ and $\partial N/\partial x = 0$.

Read-throughs and selected even-numbered solutions :

Work is the **integral** of $\mathbf{F} \cdot d\mathbf{R}$. Here $\mathbf{F}$ is the **force** and $\mathbf{R}$ is the **position**. The **dot** product finds the component of $\mathbf{F}$ in the direction of movement $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j}$. The straight path $(x, y) = (t, 2t)$ goes from **(0,0)** at $t = 0$ to **(1,2)** at $t = 1$ with $d\mathbf{R} = dt\,\mathbf{i} + \mathbf{2dt}\,\mathbf{j}$.

Another form of $d\mathbf{R}$ is $\mathbf{T}ds$, where $\mathbf{T}$ is the **unit tangent** vector to the path and the arc length has $ds = \sqrt{(\mathbf{dx/dt})^2 + (\mathbf{dy/dt})^2}$. For the path $(t, 2t)$, the unit vector $\mathbf{T}$ is $(\mathbf{i} + \mathbf{2j})/\sqrt{5}$ and $ds = \sqrt{\mathbf{5}}dt$. For $\mathbf{F} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{F} \cdot \mathbf{T}\ ds$ is still $\mathbf{5}dt$. This $\mathbf{F}$ is the gradient of $f = \mathbf{3x + y}$. The change in $f = 3x + y$ from (0,0) to (1,2) is **5**.

When $\mathbf{F} = \text{grad } f$, the dot product $\mathbf{F} \cdot d\mathbf{R}$ is $(\partial f/\partial x)dx + (\partial \mathbf{f}/\partial \mathbf{y})\mathbf{dy} = df$. The work integral from P to Q is $\int df = \mathbf{f(Q) - f(P)}$. In this case the work depends on the **endpoints** but not on the **path.** Around a closed path the work is **zero.** The field is called **conservative.** $\mathbf{F} = (1+y)\,\mathbf{i} + x\,\mathbf{j}$ is the gradient of $f = \mathbf{x} + \mathbf{xy}$. The work from (0,0) to (1,2) is **3**, the change in potential.

For the spin field $\mathbf{S} = -\mathbf{y}\,\mathbf{i} + \mathbf{x}\,\mathbf{j}$, the work does depend on the path. The path $(x,y) = (3\cos t, 3\sin t)$ is a circle with $\mathbf{S}\cdot d\mathbf{R} = -\mathbf{y}\,\mathbf{dx} + \mathbf{x}\,\mathbf{dy} = \mathbf{9}\,\mathbf{dt}$. The work is $\mathbf{18\pi}$ around the complete circle. Formally $\int g(x,y)ds$ is the limit of the sum $\sum \mathbf{g(x_i, y_i)\Delta s_i}$.

The four equivalent properties of a conservative field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ are **A: zero work around closed paths, B: work depends only on endpoints, C: gradient field, D:** $\partial M/\partial y = \partial N/\partial x$. Test **D** is passed by $\mathbf{F} = (y+1)\mathbf{i} + x\mathbf{j}$. The work $\int \mathbf{F}\cdot d\mathbf{R}$ around the circle $(\cos t, \sin t)$ is **zero.** The work on the upper semicircle equals the work on **the lower semicircle (clockwise).** This field is the gradient of $f = \mathbf{x} + \mathbf{xy}$, so the work to $(-1,0)$ is $\mathbf{-1}$ **starting from (0,0).**

4 Around the square $0 \le x, y \le 3$, $\int_3^0 y\,dx = -9$ along the top (backwards) and $\int_0^3 -x\,dy = -9$ up the right side. All other integrals are zero: answer $\mathbf{-18}$. By Section 15.3 this integral is always $-2 \times$ area.

8 Yes The field $x\mathbf{i}$ is the gradient of $f = \frac{1}{2}x^2$. Here $M = x$ and $N = 0$ so $\int_P^Q M dx + N dy = f(Q) - f(P)$. More directly: up and down movement has no effect on $\int x\,dx$.

10 Not much. Certainly the limit of $\Sigma(\Delta s)^2$ is zero.

14 $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ and $\mathbf{F}$ is the gradient of $f = xe^y$. Then $\int \mathbf{F}\cdot d\mathbf{R} = f(Q) - f(P) = \mathbf{-1}$.

18 $\frac{\mathbf{R}}{r^n}$ has $M = \frac{x}{(x^2+y^2)^{n/2}}$ and $\frac{\partial M}{\partial y} = -xny(x^2+y^2)^{-(n/2)-1}$. This agrees with $\frac{\partial N}{\partial x}$ so $\frac{\mathbf{R}}{r^n}$ is a **gradient field for all n.** The potential is $\mathbf{f} = \frac{\mathbf{r^{2-n}}}{\mathbf{2-n}}$ or $\mathbf{f} = \mathbf{\ln r}$ when $n = 2$.

32 $\frac{\partial N}{\partial x} \ne \frac{\partial M}{\partial y}$ (not conservative): $\int x^2y\,dx + xy^2dy = \int_0^1 2t^3dt = \mathbf{\frac{1}{2}}$ but $\int_0^1 t^2(t^2)dt + t(t^2)^2(2t\,dt) = \mathbf{\frac{17}{35}}$.

34 The potential is $f = \mathbf{\frac{1}{2}\ln(x^2+y^2+1)}$. Then $f(1,1) - f(0,0) = \mathbf{\frac{1}{2}\ln 3}$.

15.3 Green's Theorem (page 571)

The last section studied line integrals of $\mathbf{F}\cdot\mathbf{T}ds$. This section connects them to *double integrals.* The work can be found by integrating around the curve (with ds) or *inside the curve* (with $dA = dx\,dy$). The connection is by Green's Theorem. The theorem is for integrals around ***closed curves***:

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

We see again that this is zero for gradient fields. Their test is $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ so the double integral is immediately zero.

1. Compute the work integral $\int M\,dx + N\,dy = \int y\,dx$ for the force $\mathbf{F} = y\mathbf{i}$ around the unit circle.

   - Use Green's Theorem with $M = y$ and $N = 0$. The line integral equals the double integral of $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1$. Integrate -1 over a circle of area π to find the answer $-\pi$. This agrees with Question 5 in the previous section of the Guide. It also means that the true-false Problem 15.2.44c has answer "False."

     Special case If $M = -\frac{1}{2}y$ and $N = \frac{1}{2}x$ then $\partial N/\partial x = \frac{1}{2}$ and $-\partial M/\partial y = \frac{1}{2}$. Therefore Green's Theorem is $\frac{1}{2}\int_C -y\,dx + x\,dy = \iint_R(\frac{1}{2} + \frac{1}{2})dx\,dy =$ area of R.

2. Use that special case to find the area of a triangle with corners (0,0), (x_1, y_1), and (x_2, y_2).

- We have to integrate $\frac{1}{2}\int -y\ dx + x\ dy$ around the triangle to get the area. The first side has $x = x_1 t$ and $y = y_1 t$. As t goes from 0 to 1, the point (x, y) goes from (0,0) to (x_1, y_1). The integral is $\int(-y_1 t)(x_1 dt) + (x_1 t)(y_1 dt) = 0$. Similarly the line integral between (0,0) and (x_2, y_2) is zero.
  The third side has $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$. It goes from (x_1, y_1) to (x_2, y_2).

$$\frac{1}{2}\int -y\ dx + x\ dy = \text{ (substitute } x \text{ and } y \text{ simplify) } = \frac{1}{2}\int_0^1 (x_1y_2 - x_2y_1)dt = \frac{1}{2}(x_1y_2 - x_2y_1).$$

This is the area of the triangle. It is half the parallelogram area = $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ in Chapter 11.

Green's Theorem also applies to the flux integral $\int \mathbf{F}\cdot\mathbf{n}\ ds$ ***around a closed curve*** C. Now we are integrating $M\ dy - N\ dx$. By changing letters in the first form (the work form) of Green's Theorem, we get the second form (the flux form):

$$\text{Flow through curve } = \int_C M\ dy - N\ dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy.$$

We are no longer especially interested in gradient fields (which give zero work). Now we are interested in source-free fields (which give zero flux). The new test is $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$. This quantity is the ***divergence*** of the field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. A source-free field has *zero divergence*.

3. Is the position field $\mathbf{F} = x\ \mathbf{i} + y\ \mathbf{j}$ source-free? If not, find the flux $\int \mathbf{F}\cdot\mathbf{n}\ ds$ going out of a unit square.

   - The divergence of this $\mathbf{F}$ is $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$. The field is not source-free. The flux is not zero. Green's Theorem gives flux $= \iint 2\ dx\ dy = 2 \times$ area of region $= 2$, for a unit square.

4. Is the field $\mathbf{F} = x\ \mathbf{i} - y\ \mathbf{j}$ source-free? Is it also a gradient field?

   - *Yes* and *yes*. The field has $M = x$ and $N = -y$. It passes both tests. Test **D** for a gradient field is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is $0 = 0$. Test **H** for a source-free field is $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1 - 1 = 0$.

   This gradient field has the potential function $f(x, y) = \frac{1}{2}(x^2 - y^2)$. This source-free field also has a ***stream function*** $g(x, y) = xy$. The stream function satisfies $\frac{\partial g}{\partial y} = M$ and $\frac{\partial g}{\partial x} = -N$. Then $g(x, y)$ is the antiderivative for the flux integral $\int M\ dy - N\ dx$. When it goes around a closed curve from P to P, the integral is $g(P) - g(P) = 0$. This is what we expect for source-free fields, with stream functions.

5. Show how the combination of "conservative" plus "source-free" leads to Laplace's equation for f (and g).

   - $f_{xx} + f_{yy} = M_x + N_y$ because $f_x = M$ and $f_y = N$. But source-free means $M_x + N_y = 0$.

Read-throughs and selected even-numbered solutions :

The work integral $\oint M\ dx + N\ dy$ equals the double integral $\iint(\mathbf{N_x - M_y})\mathbf{dx\,dy}$ by **Green's** Theorem. For $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ the work is **zero.** For $\mathbf{F} = \mathbf{xj}$ and $\mathbf{-yi}$ the work equals the area of R. When $M = \partial f/\partial x$ and $N = \partial f/\partial y$, the double integral is zero because $\mathbf{f_{xy} = f_{yx}}$. The line integral is zero because $\mathbf{f(Q) = f(P)}$ **when** $\mathbf{Q = P}$ **(closed curve)**. An example is $\mathbf{F = y\,i + x\,j}$. The direction on C is **counterclockwise** around the outside and **clockwise** around the boundary of a hole. If R is broken into very simple pieces with crosscuts between them, the integrals of $\mathbf{M\ dx + N\ dy}$ cancel along the crosscuts.

Test **D** for gradient fields is $\partial\mathbf{M}/\partial\mathbf{y} = \partial\mathbf{N}/\partial\mathbf{x}$. A field that passes this test has $\oint \mathbf{F}\cdot d\mathbf{R} = \mathbf{0}$. There is a solution to $f_x = \mathbf{M}$ and $f_y = \mathbf{N}$. Then $df = M\ dx + N\ dy$ is an **exact** differential. The spin field $\mathbf{S}/r^2$ passes test

D except at $\mathbf{r} = \mathbf{0}$. Its potential $f = \theta$ increases by $\mathbf{2\pi}$ going around the origin. The integral $\iint (N_x - M_y)dx\,dy$ is not zero but $\mathbf{2\pi}$.

The flow form of Green's heorem is $\oint_C \mathbf{M\,dy - N\,dx} = \iint_R (\mathbf{M_x + N_y})\mathbf{dx\,dy}$. The normal vector in $\mathbf{F}\cdot\mathbf{n}ds$ points **out across C** and $|\mathbf{n}| = \mathbf{1}$ and $\mathbf{n}\,ds$ equals $dy\,\mathbf{i} - dx\,\mathbf{j}$. The divergence of $M\mathbf{i} + N\mathbf{j}$ is $\mathbf{M_x + N_y}$. For $\mathbf{F} = x\mathbf{i}$ the double integral is $\iint \mathbf{1\,dt} =$ **area**. There is a source. For $\mathbf{F} = y\,\mathbf{i}$ the divergence is **zero**. The divergence of $\mathbf{R}/r^2$ is zero except at $\mathbf{r} = \mathbf{0}$. This field has a **point** source.

A field with no source has properties **E = zero flux through C, F = equal flux across all paths from P to Q, G = existence of stream function, H=** zero divergence. The stream function g satisfies the equations $\partial g/\partial \mathbf{y} = \mathbf{M}$ **and** $\partial \mathbf{g}/\partial \mathbf{x} = -\mathbf{N}$. Then $\partial M/\partial x + \partial N/\partial y = 0$ because $\partial^2 g/\partial x \partial y = \partial^2 \mathbf{g}/\partial \mathbf{y}\,\partial \mathbf{x}$. The example $\mathbf{F} = y\mathbf{i}$ has $g = \frac{1}{2}\mathbf{y^2}$. There is not a potential function. The example $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ has $g = \mathbf{xy}$ and also $f = \frac{1}{2}\mathbf{x^2} - \frac{1}{2}\mathbf{y^2}$. This f satisfies Laplace's equation $\mathbf{f_{xx} + f_{yy} = 0}$, because the field $\mathbf{F}$ is both **conservative and source-free.** The functions f and g are connected by the **Cauchy-Riemann** equations $\partial f/\partial x = \partial g/\partial y$ and $\partial \mathbf{f}/\partial \mathbf{y} = -\partial \mathbf{g}/\partial \mathbf{x}$.

4 $\oint y\,dx = \int_0^1 t(-dt) = -\frac{1}{2}; M = y, N = 0, \iint(-1)dx\,dy = -$ area $= -\frac{1}{2}$.

12 Let R be the square with base from a to b on the x axis. Set $\mathbf{F} = f(x)\mathbf{j}$ so $M = 0$ and $N = f(x)$. The line integral $\oint M dx + N dy$ is $(\mathbf{b}-\mathbf{a})\mathbf{f}(\mathbf{b})$ up the right side **minus** $(\mathbf{b}-\mathbf{a})\mathbf{f}(\mathbf{a})$ down the left side. The double integral is $\iint \frac{df}{dx}dx\,dy = (\mathbf{b}-\mathbf{a})\int_\mathbf{a}^\mathbf{b} \frac{\mathbf{df}}{\mathbf{dx}}\mathbf{dx}$. Green's Theorem gives equality; cancel $b - a$.

16 $\oint \mathbf{F}\cdot\mathbf{n}ds = \int xy\,dy = \frac{1}{2}$ up the right side of the square where $\mathbf{n} = \mathbf{i}$ (other sides give zero). Also $\int_0^1\int_0^1 (y+0)dx\,dy = \frac{1}{2}$.

22 $\oint \mathbf{F}\cdot\mathbf{n}\,ds$ is the same through a square and a circle because the difference is $\iint(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})dx\,dy = \iint \text{div}\,\mathbf{F}dx\,dy = 0$ over the region in between.

30 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 3y^2 - 3y^2 = 0$. Solve $\frac{\partial g}{\partial y} = 3xy^2$ for $g = \mathbf{xy^3}$ and check $\frac{\partial g}{\partial x} = y^3$.

38 $g(Q) = \int_P^Q \mathbf{F}\cdot\mathbf{n}ds$ starting from $g(P) = 0$. Any two paths give the same integral because forward on one and back on the other gives $\oint \mathbf{F}\cdot\mathbf{n}\,ds = 0$, provided the tests $E - H$ for a stream function are passed.

15.4 Surface Integrals (page 581)

The length of a curve is $\int ds$. The area of a surface is $\iint dS$. Curves are described by functions $y = f(x)$ in single-variable calculus. Surfaces are described by functions $z = f(x,y)$ in multivariable calculus. When you have worked with ds, you see dS as the natural next step:

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\,dx \qquad \text{and} \qquad dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,dx\,dy$$

The basic step dx is along the x axis. The extra $(\frac{dy}{dx})^2$ in ds accounts for the extra length when the curve slopes up or down. Similarly $dx\,dy$ is the area dA down in the base plane. The extra $(\frac{\partial z}{\partial x})^2$ and $(\frac{\partial z}{\partial y})^2$ account for the extra area when the surface slopes up or down.

1. Find the length of the line $x + y = 1$ cut off by the axes $y = 0$ and $x = 0$. The line segment goes from (1,0) to (0,1). Find the area of the plane $x + y + z = 1$ cut off by the planes $z = 0$ and $y = 0$ and $x = 0$. This is a triangle with corners at (1,0,0) and (0,1,0) and (0,0,1).

   - The line $y = 1 - x$ has $\frac{dy}{dx} = -1$. Therefore $ds = \sqrt{1+(-1)^2}\,dx = \sqrt{2}\,dx$. The integral goes from $x = 0$ to $x = 1$ along the base. The length is $\int_0^1 \sqrt{2}\,dx = \sqrt{2}$. Check: The line from (1,0) to (0,1) certainly has length $\sqrt{2}$.
   - The plane $z = 1 - x - y$ has $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -1$. Therefore $dS = \sqrt{1+(-1)^2+(-1)^2}\,dx\,dy = \sqrt{3}\,dx\,dy$. ***The integral is down in the xy plane!*** The equilateral triangle in the sloping plane is over a right triangle in the base plane. Look only at the xy coordinates of the three corners: (1,0) and (0,1) and (0,0). Those are the corners of the *projection* (the "shadow" down in the base). This shadow triangle has area $\frac{1}{2}$. The surface area above is:

   $$\text{area of sloping plane} = \iint_{\text{shadow}} dS = \iint_{\text{base area}} \sqrt{3}\,dx\,dy = \sqrt{3}\cdot\frac{1}{2}.$$

   Check: The sloping triangle has sides of length $\sqrt{2}$. That is the distance between its corners (1,0,0) and (0,1,0) and (0,0,1). An equilateral triangle with sides $\sqrt{2}$ has area $\sqrt{3}/2$.

   All these problems have three steps : *Find dS. Find the shadow. Integrate dS over the shadow.*

2. Find the area on the plane $x + 2y + z = 4$ which lies inside the vertical cylinder $x^2 + y^2 = 1$.

   - The plane $z = 4 - x - 2y$ has $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -2$. Therefore $dS = \sqrt{1+(-1)^2+(-2)^2}\,dx\,dy = \sqrt{6}\,dx\,dy$. The shadow in the base is the inside of the circle $x^2 + y^2 = 1$. This unit circle has area π. So the surface area on the sloping plane above it is $\iint \sqrt{6}\,dx\,dy = \sqrt{6} \times$ area of shadow $= \sqrt{6}\,\pi$.

   The region on that sloping plane is an *ellipse*. This is automatic when a plane cuts through a circular cylinder. The area of an ellipse is πab, where a and b are the half-lengths of its axes. The axes of this ellipse are hard to find, so the new method that gave area $= \sqrt{6}\pi$ is definitely superior.

3. Find the surface area on the sphere $x^2 + y^2 + z^2 = 25$ between the horizontal planes $z = 2$ and $z = 4$.

   - The lower plane cuts the sphere in the circle $x^2 + y^2 + 2^2 = 25$. This is $r^2 = 21$. The upper plane cuts the sphere in the circle $x^2 + y^2 + 4^2 = 25$. This is $r^2 = 9$. *The shadow in the xy plane is the ring between $r = 3$ and $r = \sqrt{21}$.*

   The spherical surface has $x^2 + y^2 + z^2 = 25$. Therefore $2x + 2z\frac{\partial z}{\partial x} = 0$ or $\frac{\partial z}{\partial x} = -\frac{x}{z}$. Similarly $\frac{\partial z}{\partial y} = -\frac{y}{z}$:

   $$dS = \sqrt{1 + (-\frac{x}{z})^2 + (-\frac{y}{z})^2}\,dx\,dy = \frac{\sqrt{x^2+y^2+z^2}}{z}\,dx\,dy = \frac{5}{z}\,dx\,dy.$$

   Remember $z = \sqrt{25 - x^2 - y^2} = \sqrt{25 - r^2}$. Integrate $\frac{5}{z}$ over the shadow (the ring) using r and θ :

   $$\begin{aligned}\text{Surface area} &= \iint_{\text{ring}} \tfrac{5}{z}\,dx\,dy = \int_0^{2\pi}\int_3^{\sqrt{21}} \tfrac{5\,r\,dr\,d\theta}{\sqrt{25-r^2}} \\ &= (-5)(2\pi)\sqrt{25-r^2}\,\Big]_3^{\sqrt{21}} = -10\pi(2-4) = 20\pi.\end{aligned}$$

Surface equations with parameters Up to now the surface equation has been $z = f(x, y)$. This is restrictive. Each point (x, y) in the base has only one point above it in the surface. A complete sphere is not allowed. We solved a similar problem for curves, by allowing a parameter: $x = \cos t$ and $y = \sin t$ gave a complete circle. For surfaces we need *two parameters* u and v. Instead of $x(t)$ we have $x(u, v)$. Similarly $y = y(u, v)$ and $z = z(u, v)$. As u and v go over some region R, the points (x, y, z) go over the surface S.

For a circle, the parameter t is really the angle θ. For a sphere, the parameters u and v are the angle ϕ down from the North Pole and the angle θ around the Equator. These are just spherical coordinates from Section 14.4: $x = \sin u \cos v$ and $y = \sin u \sin v$ and $z = \cos u$. In this case the region R is $0 \le u \le \pi$ and $0 \le v \le 2\pi$. Then the points (x, y, z) cover the surface S of the unit sphere $x^2 + y^2 + z^2 = 1$.

We still have to find dS! The general formula is equation (7) on page 575. For our x, y, z that equation gives $dS = \sin u\, du\, dv$. (In spherical coordinates you remember the volume element $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. We are on the surface $\rho = 1$. And the letters ϕ and θ are changed to u and v.) This good formula for dS is typical of good coordinate systems – equation (7) is not as bad as it looks.

Integrate dS over the base region R in uv space to find the surface area above.

4. Find the surface area (known to be 4π) of the unit sphere $x^2 + y^2 + z^2 = 1$.

- Integrate dS over R to find $\int_0^{2\pi} \int_0^{\pi} \sin\ u\, du\, dv = (2\pi)(-\cos u)]_0^{\pi} = 4\pi$.

5. Recompute Question 3, the surface area on $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

- This sphere has radius $\sqrt{25} = 5$. Multiply the points (x, y, z) on the unit sphere by 5:

$$x = 5 \sin u \cos v \text{ and } y = 5 \sin u \sin v \text{ and } z = 5 \cos u \text{ and } dS = 25 \sin u\, du\, dv.$$

Now find the region R. The angle v (or θ) goes around from 0 to 2π. Since $z = 5\cos u$ goes from 2 to 4, the angle u is between $\cos^{-1}\frac{2}{5}$ and $\cos^{-1}\frac{4}{5}$. Integrate dS over this region R and compare with 20π above:

$$\text{Surface area } = \iint_R 25 \sin u\, du\, dv = 25(2\pi)(-\cos u)] = 50\pi(\frac{4}{5} - \frac{2}{5}) = 20\pi.$$

6. Find the surface area of the cone $z = 1 - \sqrt{x^2 + y^2}$ above the base plane $z = 0$.

- *Method 1:* Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and dS. Integrate over the shadow, a circle in the base plane. (Set $z = 0$ to find the shadow boundary $x^2 + y^2 = 1$.) The integral takes 10 steps in Schaum's Outline. The answer is $\pi\sqrt{2}$.
- *Method 2:* Use parameters. Example 2 on page 575 gives $x = u \cos v$ and $y = u\ \sin v$ and $z = u$ and $dS = \sqrt{2}u\, du\, dv$. The cone has $0 \le u \le 1$ (since $0 \le z \le 1$). The angle v (alias θ) goes from 0 to 2π. This gives the parameter region R and we integrate in one step:

$$\text{cone area } = \int_0^{2\pi} \int_0^1 \sqrt{2}\ u\, du\, dv = (2\pi)(\sqrt{2})(\frac{1}{2}) = \pi\sqrt{2}.$$

The discussion of surface integrals ends with the calculation of ***flow through a surface.*** We are given the flow field $\mathbf{F}(x, y, z)$ – a vector field with three components $M(x, y, z)$ and $N(x, y, z)$ and $P(x, y, z)$. The flow *through* a surface is $\iint \mathbf{F} \cdot \mathbf{n}\, dS$. where $\mathbf{n}$ is the unit normal vector to the surface.

For the surface $z = f(x,y)$, you would expect a big square root for dS. It is there, but it is cancelled by a square root in $\mathbf{n}$. We divide the usual normal vector $\mathbf{N} = -\frac{\partial f}{\partial x}\,\mathbf{i} - \frac{\partial f}{\partial y}\,\mathbf{j} + \mathbf{k}$ by its length to get the unit vector $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. That length is the square root that cancels. This leaves

$$\text{Flow through surface } = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{N}\,dx\,dy = \iint_R \left(-M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right)dx\,dy.$$

The main job is to find the shadow region R in the xy plane and integrate. The "shadow" is the range of (x,y) down in the base, while (x,y,z) travels over the surface. With parameters u and v, the shadow region R is in the uv plane. It gives the range of parameters (u,v) as the point (x,y,z) travels over the surface S. This is not the easiest section in the book.

Read-throughs and selected even-numbered solutions :

A small piece of the surface $z = f(x,y)$ is nearly **flat**. When we go across by dx, we go up by $(\partial \mathbf{z}/\partial \mathbf{x})\mathbf{dx}$. That movement is $\mathbf{A}dx$, where the vector $\mathbf{A}$ is $\mathbf{i} + \mathbf{dz/dx\,k}$. The other side of the piece is $\mathbf{B}dy$, where $\mathbf{B} = \mathbf{j} + (\partial \mathbf{z}/\partial \mathbf{y})\mathbf{k}$. The cross product $\mathbf{A}\times\mathbf{B}$ is $\mathbf{N} = -\partial \mathbf{z}/\partial \mathbf{x}\,\mathbf{i} - \partial \mathbf{z}/\partial \mathbf{y}\,\mathbf{j} + \mathbf{k}$. The area of the piece is $dS = |\mathbf{N}|dx\,dy$. For the surface $x = xy$, the vectors are $A = \iint\sqrt{\mathbf{1+x^2+y^2}}\mathbf{dx\,dy}$ and $N = -\mathbf{y\,i} - \mathbf{x} + \mathbf{k}$. The area integral is $\iint dS = \mathbf{i} + \mathbf{y\,k}$.

With parameters u and v, a typical point on a 45° cone is $x = u\cos v, y = \mathbf{u\,sinv}, z = \mathbf{u}$. A change in u moves that point by $\mathbf{A\,du} = (\mathbf{cos\,v\,i} + \mathbf{sinv\,j} + \mathbf{k})du$. The change in v moves the point by $\mathbf{B}dv = (-\mathbf{u\,sinv\,i} + \mathbf{u\,cos\,v\,j})\mathbf{dv}$. The normal vector is $\mathbf{N} = \mathbf{A}\times\mathbf{B} = -\mathbf{u\,cos\,v\,i} - \mathbf{u\,sinv\,j} + \mathbf{u\,k}$. The area is $dS = \sqrt{\mathbf{2}}\,\mathbf{u}du\,dv$. In this example $\mathbf{A}\cdot\mathbf{B} - \mathbf{0}$ so the small piece is a **rectangle** and $dS = |\mathbf{A}||\mathbf{B}|du\,dv$.

For flux we need $\mathbf{n}dS$. The **unit normal** vector $\mathbf{n}$ is $\mathbf{N} = \mathbf{A}\times\mathbf{B}$ divided by $|\mathbf{N}|$. For a surface $z = f(x,y)$, the product $\mathbf{n}dS$ is the vector $\mathbf{N}\,dx\,dy$ (to memorize from table). The particular surface $z = xy$ has $\mathbf{n}dS = (-\mathbf{yi} - \mathbf{xj} + \mathbf{k})dx\,dy$. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the flux through $z = xy$ is $\mathbf{F}\cdot\mathbf{n}dS = -\mathbf{xy}\,dx\,dy$.

On a 30° cone the points are $x = 2u\cos v, y = 2u\sin v, z = u$. The tangent vectors are $\mathbf{A} = \mathbf{2\,cosv\,i} + \mathbf{2\,sinv\,j} + \mathbf{k}$ and $\mathbf{B} = -\mathbf{2\,u\,sin\,v\,i} + \mathbf{2u\,cos\,v\,j}$. This cone has $\mathbf{n}dS = \mathbf{A}\times\mathbf{B}\,du\,dv = (-\mathbf{2u\,cos\,v\,i} - \mathbf{2\,u\,sinv\,j} + \mathbf{4\,u\,k})\mathbf{du\,dv}$. For $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, the flux element through the cone is $\mathbf{F}\cdot\mathbf{n}dS = \mathbf{zero}$. The reason for this answer is **that F is along the cone**. The reason we don't compute flux through a Möbius strip **is that N cannot be defined (the strip is not orientable)**.

2 $\mathbf{N} = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ and $dS = \sqrt{\mathbf{1+4x^2+4y^2}}\mathbf{dx\,dy}$. Then $\iint dS = \int_0^{2\pi}\int_2^{\sqrt{8}}\sqrt{1+4r^2}\,r\,dr\,d\theta = \frac{\pi}{6}(\mathbf{33^{3/2} - 17^{3/2}})$.

8 $\mathbf{N} = -\frac{x\mathbf{i}}{r} - \frac{y\mathbf{i}}{r} + \mathbf{k}$ and $dS = \frac{x^2+y^2+r^2}{r^2}dx\,dy = \sqrt{\mathbf{2}}\,\mathbf{dx\,dy}$. Then area $= \int_0^{2\pi}\int_a^b\sqrt{2}r\,dr\,d\theta = \sqrt{\mathbf{2}}\pi(\mathbf{b^2 - a^2})$.

16 On the sphere $dS = \sin\phi\,d\phi\,d\theta$ and $g = x^2 + y^2 = \sin^2\phi$. Then $\int_0^{2\pi}\int_0^{\pi/2}\sin^3\phi\,d\phi\,d\theta = 2\pi(\frac{2}{3}) = \frac{\mathbf{4\pi}}{\mathbf{3}}$.

20 $\mathbf{A} = v\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{B} = u\mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{N} = \mathbf{A}\times\mathbf{B} = -2\mathbf{i} + (u+v)\mathbf{j} + (v-u)\mathbf{k}, dS = \sqrt{4 + 2u^2 + 2v^2}du\,dv$.

24 $\iint \mathbf{F}\cdot\mathbf{n}dS = \int_0^{2\pi}\int_2^{\sqrt{8}} -r^3 dr\,d\theta = -24\pi$.

30 $\mathbf{A} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - 2r\mathbf{k}, \mathbf{B} = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}, \mathbf{N} = \mathbf{A}\times\mathbf{B} = 2r^2\cos\theta\,\mathbf{i} + 2r^2\sin\theta\,\mathbf{j} + r\mathbf{k}$, $\iint \mathbf{k}\cdot\mathbf{n}\,dS = \iint \mathbf{k}\cdot\mathbf{N}du\,dv = \int_0^{2\pi}\int_0^a r\,dr\,d\theta = \pi\mathbf{a^2}$ as in Example 12.

15.5 The Divergence Theorem (page 588)

This theorem says that the total source inside a volume V equals the total flow through its closed surface S. We need to know how to measure the *source* and how to measure the *flow out*. Both come from the vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ which assigns a flow vector to every point inside V and on S :

$$\text{Source } = \text{divergence of } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$\text{Flow out } = \text{normal component of } \mathbf{F} = \mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{N}/|\mathbf{N}|$$

The balance between source and outward flow is the *Divergence Theorem.* It is Green's Theorem in 3 dimensions:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\; dS = \iiint_V (\text{divergence of } \mathbf{F})\; dV = \iiint_V (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z})\; dx\; dy\; dz.$$

An important special case is a "source-free" field. This means that the divergence is zero. Then the integrals are zero and the total outward flow is zero. There may be flow out through one part of S and flow in through another part – they must cancel when div $\mathbf{F} = 0$ inside V.

1. For the field $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ in the unit ball $x^2 + y^2 + z^2 \leq 1$, compute both sides of the Divergence Theorem.

   - The divergence is $\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial 0}{\partial z} = 0$. This source-free field has $\iiint$ div $\mathbf{F}\; dV = 0$.
   - The normal vector to the unit sphere is radially outward: $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Its dot product with $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ gives $\mathbf{F} \cdot \mathbf{n} = 2xy$ for flow out through S. This is not zero, but *its integral is zero.* One proof is by symmetry: $2xy$ is equally positive and negative on the sphere. The direct proof is by integration (use spherical coordinates):

$$\iint 2xy\; dS = \int_0^{2\pi} \int_0^{\pi} 2(\sin\phi\; \cos\theta)(\sin\phi\; \sin\theta) \sin\phi\; d\phi\; d\theta = 0 \text{ because } \int_0^{2\pi} 2\cos\theta\; \sin\theta\; d\theta = 0.$$

   To emphasize: The flow out of *any volume* V is zero because this field has divergence = 0. For strange shapes we can't do the surface integral. But the volume integral is still $\iiint 0\; dV$.

2. (This is Problem 15.5.8) Find the divergence of $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and the flow out of the sphere $\rho = a$.

   - Divergence $= \frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$. The triple integral for the total source is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} 3\rho^2(\rho^2 \sin\phi\; d\rho\; d\phi\; d\theta) = 3(\frac{a}{5})^5(2)(2\pi) = 12\pi a^5/5.$$

   - Flow out has $\mathbf{F} \cdot \mathbf{n} = (x^3, y^3, z^3) \cdot (\frac{x}{a}, \frac{y}{a}, \frac{z}{a}) = \frac{x^4+y^4+z^4}{a}$. The integral of $\frac{z^4}{a}$ over the sphere $\rho = a$ is

$$\int_0^{2\pi} \int_0^{\pi} \frac{1}{a}(a\cos\phi)^4(a^2 \sin\phi\; d\phi\; d\theta) = -2\pi a^5 \frac{\cos^5\phi}{5}]_0^{\pi} = \frac{4\pi a^5}{5}.$$

   By symmetry $\frac{x^4}{a}$ and $\frac{y^4}{a}$ have this same integral. So multiply by 3 to get the same $12\pi a^5/5$ as above.

The text assigns special importance to the vector field $\mathbf{F} = \mathbf{R}/\rho^3$. This is radially outward (remember $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$). The length of $\mathbf{F}$ is $|\mathbf{R}|/\rho^3 = \rho/\rho^3 = 1/\rho^2$. This is the ***inverse-square law*** – the force of gravity from a point mass at the origin decreases like $1/\rho^2$.

The special feature of this radial field is to have zero divergence – except at one point. This is not typical of radial fields: $\mathbf{R}$ itself has divergence $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$. But dividing $\mathbf{R}$ by ρ^3 gives a field with div $\mathbf{F} = 0$. (Physically: No divergence where there is no mass and no source of gravity.) The exceptional point is (0,0,0), where there *is* a mass. That point source is enough to produce 4π on both sides of the divergence theorem (provided S encloses the origin). The point source has strength 4π. The divergence of $\mathbf{F}$ is 4π times a "delta function."

The other topic in this section is the vector form of two familiar rules: the ***product rule*** for the derivative of $u(x)v(x)$ and the reverse of the product rule which is ***integration by parts***. Now we are in 2 or 3 dimensions and v is a vector field $\mathbf{V}(x, y, z)$. The derivative is replaced by the divergence or the curl. We just use the old product rule on uM and uN and uP. Then collect terms:

$$\text{div}(u\mathbf{V}) = u \text{ div}\mathbf{V} + (\text{grad } u) \cdot \mathbf{V} \qquad \text{curl}(u\mathbf{V}) = u \text{ curl } \mathbf{V} + (\text{grad } u) \times \mathbf{V}.$$

Read-throughs and selected even-numbered solutions :

In words, the basic balance law is **flow in = flow out.** The flux of $\mathbf{F}$ through a surface S is the double integral $\iint \mathbf{F} \cdot \mathbf{n}d\mathbf{S}$. The divergence of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is $\mathbf{M_x} + \mathbf{N_y} + \mathbf{P_z}$. It measures the **source at the point.** The total source is the triple integral $\iiint$ div $\mathbf{F}$ $\mathbf{dV}$. That equals the flux by the **Divergence** Theorem.

For $\mathbf{F} = 5z\mathbf{k}$ the divergence is **5**. If V is a cube of side a then the triple integral equals $\mathbf{5a^3}$. The top surface where $z = a$ has $\mathbf{n} = \mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n} = 5a$. The bottom and sides have $\mathbf{F} \cdot \mathbf{n} =$ **zero**. The integral $\iint \mathbf{F} \cdot \mathbf{n}dS\mathbf{5a^3}$.

The field $\mathbf{F} = \mathbf{R}/\rho^3$ has div $\mathbf{F} = 0$ except **at the origin.** $\iint \mathbf{F} \cdot \mathbf{n}dS$ equals 4π over any surface around the origin. This illustrates Gauss's Law: **flux =** 4π **times source strength.** The field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ has div $\mathbf{F} = \mathbf{0}$ and $\iint \mathbf{F} \cdot \mathbf{n}dS = \mathbf{0}$. For this $\mathbf{F}$, the flux out through a pyramid and in through its base are **equal.**

The symbol ∇ stands for $(\partial/\partial\mathbf{x})\mathbf{i} + (\partial/\partial\mathbf{y})\mathbf{j} + (\partial/\partial\mathbf{z})\mathbf{k}$. In this notation div $\mathbf{F}$ is $\nabla \cdot \mathbf{F}$. The gradient of f is $\nabla\mathbf{f}$. The divergence of grad f is $\nabla \cdot \nabla\mathbf{f}$ **or** $\nabla^2\mathbf{f}$. The equation div grad $f = 0$ is **Laplace's** equation.

The divergence of a product is $\text{div}(u\mathbf{V}) = \mathbf{u\,div\,V + (grad\,u) \cdot V}$. Integration by parts in 3D is $\iiint u \text{ div } \mathbf{V} dx\,dy\,dz = -\iiint \mathbf{V} \cdot \mathbf{grad\,u\,dx\,dy\,dz} + \iint \mathbf{u\,V \cdot n\,dS}$. In two dimensions this becomes $\iint \mathbf{u}(\partial\mathbf{M}/\partial\mathbf{x} + \partial\mathbf{N}/\partial\mathbf{y})\mathbf{dx\,dy} = -\int(\mathbf{M}\,\partial\mathbf{u}/\partial\mathbf{x} + \mathbf{N}\,\partial\mathbf{u}/\partial\mathbf{y})\mathbf{dx\,dy} + \int \mathbf{u\,V \cdot n\,ds}$. In one dimension it becomes **integration by parts**. For steady fluid flow the continuity equation is div $\rho\mathbf{V} = -\partial\rho/\partial\mathbf{t}$.

14 $\mathbf{R} \cdot \mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{i} = x = 1$ on one face of the box. On the five other faces $\mathbf{R} \cdot \mathbf{n} = 2, 3, 0, 0, 0$. The integral is $\int_0^3 \int_0^2 1dy\,dz + \int_0^3 \int_0^1 2dx\,dz + \int_0^2 \int_0^1 3dx\,dy = \mathbf{18}$. Also div $\mathbf{R} = 1 + 1 + 1 = 3$ and $\int_0^3 \int_0^2 \int_0^1 3dx\,dy\,dz = \mathbf{18}$.

18 grad $f \cdot \mathbf{n}$ is the **directional** derivative in the normal direction $\mathbf{n}$ (also written $\frac{\partial f}{\partial n}$). The Divergence Theorem gives $\iiint$ div (grad f) $dV = \iint$ grad $f \cdot \mathbf{n}dS = \iint \frac{\partial f}{\partial n} dS$. But we are given that div (grad f) $= f_{xx} + f_{yy} + f_{zz}$ is zero.

26 When the density ρ is constant (incompressible flow), the continuity equation becomes div $\mathbf{V} = 0$. If the flow is irrotational then $\mathbf{F} =$ grad f and the continuity equation is div (ρ grad f) $= -d\rho/dt$. If also $\rho =$ constant, then **div grad f = 0**: Laplace's equation for the "potential."

30 The boundary of a solid ball is a sphere. A sphere has no boundary. Similarly for a cube or a cylinder – the boundary is a closed surface and **that surface's boundary** is empty. This is a crucial fact in topology.

15.6 Stokes' Theorem and the Curl of F (page 595)

The curl of **F** measures the "spin". A spin field in the plane is $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$, with third component $P = 0$:

$$\text{curl } \mathbf{S} = (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + (1+1)\mathbf{k}.$$

The curl is 2 **k**. It points along the spin axis (the z axis, perpendicular to the plane of spin). Its magnitude is 2 times the rotation rate. This special spin field **S** gives a rotation counterclockwise in the xy plane. It is counterclockwise because **S** points that way, and also because of the right hand rule: thumb in the direction of curl **F** and fingers "curled in the direction of spin." Put your right hand on a table with thumb upward along **k**.

Spin fields can go around any axis vector **a**. The field $\mathbf{S} = \mathbf{a} \times \mathbf{R}$ does that. Its curl is 2 **a** (after calculation). Other fields have a curl that changes direction from point to point. Some fields have no spin at all, so their curl is *zero. These are gradient fields!* This is a key fact:

$$\text{curl } \mathbf{F} = \mathbf{0} \text{ whenever } \mathbf{F} = \text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

You should substitute those three partial derivatives for M, N, P and see that the curl formula gives zero. ***The curl of a gradient is zero.*** The quick test **D** for a gradient field is curl $\mathbf{F} = \mathbf{0}$.

The twin formula is that *the divergence of a curl is zero.* The quick test **H** for a source-free "curl field" is div $\mathbf{F} = 0$. A gradient field can be a gravity field or an electric field. A curl field can be a magnetic field.

1. Show that $\mathbf{F} = yz\,\mathbf{i} + xy\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$ passes the quick test **D** for a gradient field (a conservative field). The test is curl $\mathbf{F} = \mathbf{0}$. Find the potential function f that this test guarantees: **F** equals the gradient of f.

   - The curl of this **F** is $(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k} = (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = \mathbf{0}$. The field passes test **D**. There must be a function whose partial derivatives are M, N, P : $\frac{\partial f}{\partial x} = yz$ and $\frac{\partial f}{\partial y} = xz$ and $\frac{\partial f}{\partial z} = xy + 2z$ lead to the potential function $f = xyz + z^2$.

   We end with ***Stokes' Theorem.*** It is like the original Green's Theorem, where work around a plane curve C was equal to the double integral of $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$. Now the curve C can go out of the plane. The region inside is a curved surface – also not in a plane. The field **F** is three-dimensional – its component $P\mathbf{k}$ goes out of the plane. Stokes' Theorem has a line integral $\int \mathbf{F} \cdot d\mathbf{R}$ (work around C) equal to a surface integral:

$$\int_C M\,dx + N\,dy + P\,dz = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS = \iint_{\text{base}} (\text{curl } \mathbf{F}) \cdot \mathbf{N}\,dx\,dy.$$

   If the surface is $z = f(x,y)$ then its normal is $\mathbf{N} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$ and dz is $\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Substituting for **N** and dz reduces the 3-dimensional theorem of Stokes to the 2-dimensional theorem of Green.

2. (Compare 15.6.12) For $\mathbf{F} = \mathbf{i} \times \mathbf{R}$ compute both sides in Stokes' Theorem when C is the unit circle.

   - The cross product $\mathbf{F} = \mathbf{i} \times \mathbf{R}$ is $\mathbf{i} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = y\mathbf{k} - z\mathbf{j}$. This spin field has curl $\mathbf{F} = 2\mathbf{i}$. either substitute $M = 0, N = -z, P = y$ in the curl formula or remember that curl $(\mathbf{a} \times \mathbf{R}) = 2\mathbf{a}$. Here $\mathbf{a} = \mathbf{i}$.

The line integral of $\mathbf{F} = y\mathbf{k} - z\mathbf{j}$ around the unit circle is $\int 0\,dx - z\,dy + y\,dz$. All those are zero because $z = 0$ for the circle $x^2 + y^2 = 1$ in the xy plane. ***The left side of Stokes' Theorem is zero for this F.***

The double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 2\mathbf{i} \cdot \mathbf{n}$ is certainly zero if the surface S is the flat disk inside the unit circle. The normal vector to that flat surface is $\mathbf{n} = \mathbf{k}$. Then $2\mathbf{i} \cdot \mathbf{n}$ is $2\mathbf{i} \cdot \mathbf{k} = 0$.

The double integral of $2\mathbf{i} \cdot \mathbf{n}$ is zero even if the surface S is not flat. S can be a mountain (always with the unit circle as its base boundary). The normal $\mathbf{n}$ out from the mountain can have an $\mathbf{i}$ component, so $2\mathbf{i} \cdot \mathbf{n}$ can be non-zero. But Stokes' Theorem says: The *integral* of $2\mathbf{i} \cdot \mathbf{n}$ over the mountain is zero. That is the theorem. The mountain integral must be zero because the baseline integral is zero.

Read-throughs and selected even-numbered solutions :

The curl of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector $(\mathbf{P_y - N_z})\mathbf{i} + (\mathbf{M_z - P_x})\mathbf{j} + (\mathbf{N_x - M_y})\mathbf{k}$. It equals the 3 by 3 determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial \mathbf{x} & \partial/\partial \mathbf{y} & \partial/\partial \mathbf{z} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{vmatrix}$ The curl of $x^2\mathbf{i} + z^2\mathbf{k}$ is **zero**. For $\mathbf{S} = y\mathbf{i} - (x+z)\mathbf{j} + y\mathbf{k}$ the curl is $\mathbf{2i - 2k}$. This $\mathbf{S}$ is a **spin** field $\mathbf{a} \times \mathbf{R} = \frac{1}{2}(\text{curl } \mathbf{F}) \times \mathbf{R}$, with axis vector $\mathbf{a} = \mathbf{i} - \mathbf{k}$. For any gradient field $f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ the curl is **zero**. That is the important identity curl grad f = **zero**. It is based on $f_{xy} = f_{yx}$ and $\mathbf{f_{xz} = f_{zx}}$ and $\mathbf{f_{yz} = f_{zy}}$. The twin identity is **div curl F = 0**.

The curl measures the **spin (or turning)** of a vector field. A paddlewheel in the field with its axis along $\mathbf{n}$ has turning speed $\frac{1}{2}\mathbf{n} \cdot$ **curl F**. The spin is greatest when $\mathbf{n}$ is in the direction of **curl F**. Then the angular velocity is $\frac{1}{2}|\text{curl } \mathbf{F}|$. Stokes' Theorem is $\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$. The curve C is the **boundary** of the surface S. This is **Green's** Theorem extended to **three** dimensions. Both sides are zero when $\mathbf{F}$ is a gradient field because **the curl is zero.**

The four properties of a conservative field are $\mathbf{A}: \oint \mathbf{F} \cdot d\mathbf{R} = 0$ **and** $\mathbf{B}: \int_{\mathbf{P}}^{\mathbf{Q}} \mathbf{F} \cdot d\mathbf{R}$ **depends only on P and Q and C : F is the gradient of a potential function f(x, y, z) and D : curl F = 0.** The field $y^2z^2\mathbf{i} + 2xy^2z\mathbf{k}$ fails test **D**. This field is the gradient of **no f**. The work $\int \mathbf{F} \cdot d\mathbf{R}$ from (0,0,0) to (1,1,1) is $\frac{3}{5}$ **along the straight path x = y = z = t**. For every field $\mathbf{F}$, $\iint$ curl $\mathbf{F} \cdot \mathbf{n}dS$ is the same out through a pyramid and up through its base **because they have the same boundary, so $\oint \mathbf{F} \cdot d\mathbf{R}$ is the same.**

14 $\mathbf{F} = (x^2 + y^2)\mathbf{k}$ so curl $\mathbf{F} = 2(y\mathbf{i} - x\mathbf{j})$. (Surprise that this $\mathbf{F} = \mathbf{a} \times \mathbf{R}$ has curl $\mathbf{F} = 2\mathbf{a}$ even with nonconstant $\mathbf{a}$.) Then $\oint \mathbf{F} \cdot d\mathbf{R} = \iint$ curl $\mathbf{F} \cdot \mathbf{n}dS = 0$ since $\mathbf{n} = \mathbf{k}$ is perpendicular to curl $\mathbf{F}$.

18 If curl $\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is the gradient of a potential: $\mathbf{F}$ = grad f. Then div $\mathbf{F} = 0$ is div grad $f = 0$ which is Laplace's equation.

24 Start with one field that has the required curl. (Can take $\mathbf{F} = \frac{1}{2}\mathbf{i} \times \mathbf{R} = -\frac{z}{2}\mathbf{j} + \frac{y}{2}\mathbf{k}$). Then add any $\mathbf{F}$ with curl zero (particular solution plus homogeneous solution as always). The fields with curl $\mathbf{F} = \mathbf{0}$ are gradient fields $\mathbf{F}$ = grad f, since curl grad = $\mathbf{0}$. Answer: $\mathbf{F} = \frac{1}{2}\mathbf{i} \times \mathbf{R} +$ any grad f.

26 $\mathbf{F} = y\mathbf{i} - x\mathbf{k}$ has curl $\mathbf{F} = \mathbf{j} - \mathbf{k}$. (a) Angular velocity $= \frac{1}{2}$ curl $\mathbf{F} \cdot \mathbf{n} = \frac{1}{2}$ if $\mathbf{n} = \mathbf{j}$. (b) Angular velocity $= \frac{1}{2}|\text{curl}\mathbf{F}| = \frac{\sqrt{2}}{2}$ (c) Angular velocity = 0.

36 curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & xyz \end{vmatrix} = \mathbf{i}(xz) + \mathbf{j}(1 - yz) + \mathbf{k}(1)$ and $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. So curl $\mathbf{F} \cdot \mathbf{n} = x^2z + y - y^2z + z$. By symmetry $\iint x^2z\,dS = \iint y^2z dS$ on the half sphere and $\iint y dS = 0$. This leaves $\iint z\,dS = \int_0^{2\pi}\int_0^{\pi/2} \cos\phi(\sin\phi\,d\phi\,d\theta) = \frac{1}{2}(2\pi) = \pi$.

38 (The expected method is trial and error) $\mathbf{F} = 5yz\mathbf{i} + 2xy\mathbf{k} +$ any grad f.

15 Chapter Review Problems

Review Problems

R1 For $f(x,y) = x^2 + y^2$ what is the gradient field $\mathbf{F} = \nabla f$? What is the unit field $\mathbf{u} = \mathbf{F}/|\mathbf{F}|$?

R2 Is $\mathbf{F}(x,y) = \cos x\ \mathbf{i} + \sin x\ \mathbf{j}$ a gradient field? Draw the vectors $\mathbf{F}$ (0,0), $\mathbf{F}(0,\pi), \mathbf{F}(\pi,0), \mathbf{F}(\pi,\pi)$.

R3 Is $\mathbf{F}(x,y) = \cos y\ \mathbf{i} + \sin x\ \mathbf{j}$ a gradient field or not? Draw $\mathbf{F}$(0,0), $\mathbf{F}(0,\pi), \mathbf{F}(\pi,0), \mathbf{F}(\pi,\pi)$.

R4 Is $\mathbf{F}(x,y) = \cos x\ \mathbf{i} + \sin y\ \mathbf{j}$ a gradient field? If so find a potential $f(x,y)$ whose gradient is $\mathbf{F}$.

R5 Integrate $-y\ dx + x\ dy$ around the unit circle $x = \cos t, y = \sin t$. Why twice the area?

R6 Integrate the gradient of $f(x,y) = x^3 + y^3$ around the unit circle.

R7 With Green's Theorem find an integral around C that gives the area inside C.

R8 Find the flux of $\mathbf{F} = x^2\mathbf{i}$ through the unit circle from both sides of Green's Theorem.

R9 What integral gives the area of the surface $z = f(x,y)$ above the square $|x| \le 1$, $|y| \le 1$?

R10 Describe the cylinder given by $x = 2\cos v, y = 2\sin v$, and $z = u$. Is (2, 2, 2) on the cylinder? What parameters u and v produce the point (0, 2, 4)?

Drill Problems

D1 Find the gradient $\mathbf{F}$ of $f(x,y,z) = x^3 + y^3$. Then find the divergence and curl of $\mathbf{F}$.

D2 What integral gives the surface area of $z = 1 - x^2 - y^2$ above the xy plane?

D3 What is the area on the sloping plane $z = x + y$ above a base area (or shadow area) A?

D4 Write down Green's Theorem for $\int M dx + N dy = work$ and $\int M dy - N dx = flux$. Write down the flux form with $M dy - N dx$.

D5 Write down the Divergence Theorem. Say in words what it balances.

D6 If $\mathbf{F} = y^2\ \mathbf{i} + (1 + 2x)y\ \mathbf{j}$ is the gradient of $f(x,y)$, find the potential function f.

D7 When is the area of a surface equal to the area of its shadow on $z = 0$? (Surface is $z = f(x,y)$).

D8 For $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j} + 5\mathbf{k}$, the flux $\iint \mathbf{F} \cdot \mathbf{n} dS$ equals ___ times the volume inside S.

D9 What vector field is the curl of $\mathbf{F} = xyz\ \mathbf{i}$? Find the gradient of that curl.

D10 What vector field is the gradient of $f = xyz$? Find the curl of that gradient.