# CHAPTER 9 POLAR COORDINATES AND COMPLEX NUMBERS

## 9.1 Polar Coordinates (page 350)

Circles around the origin are so important that they have their own coordinate system – *polar coordinates*. The center at the origin is sometimes called the "pole." A circle has an equation like $r = 3$. Each point on that circle has two coordinates, say $r = 3$ and $\theta = \frac{\pi}{2}$. This angle locates the point 90° around from the $x$ axis, so it is on the $y$ axis at distance 3.

The connection to $x$ and $y$ is by the equations $x = r\cos\theta$ and $y = r\sin\theta$. Substituting $r = 3$ and $\theta = \frac{\pi}{2}$ as in our example, the point has $x = 3\cos\frac{\pi}{2} = 0$ and $y = 3\sin\frac{\pi}{2} = 3$. The polar coordinates are $(r, \theta) = (3, \frac{\pi}{2})$ and the rectangular coordinates are $(x, y) = (0, 3)$.

1. Find polar coordinates for these points – first with $r \geq 0$ and $0 \leq \theta < 2\pi$, then three other pairs $(r, \theta)$ that give the same point:

   (a) $(x, y) = (\sqrt{3}, 1)$ (b) $(x, y) = (-1, 1)$ (c) $(x, y) = (-3, -4)$

   - (a) $r^2 = x^2 + y^2 = 4$ yields $r = 2$ and $\frac{y}{x} = \frac{1}{\sqrt{3}} = \tan\theta$ leads to $\theta = \frac{\pi}{6}$. The polar coordinates are $(2, \frac{\pi}{6})$. Other representations of the same point are $(2, \frac{\pi}{6} + 2\pi)$ and $(2, \frac{\pi}{6} - 2\pi)$. Allowing $r < 0$ we have $(-2, -\frac{5\pi}{6})$ and $(-2, \frac{7\pi}{6})$. There are an infinite number of possibilities.
   - (b) $r^2 = x^2 + y^2$ yields $r = \sqrt{2}$ and $\frac{y}{x} = \frac{1}{-1} = \tan\theta$. Normally the arctan function gives $\tan^{-1}(-1) = -\frac{\pi}{4}$. But that is a fourth quadrant angle, while the point $(-1, 1)$ is in the second quadrant. The choice $\theta = \frac{3\pi}{4}$ gives the "standard" polar coordinates $(\sqrt{2}, \frac{3\pi}{4})$. Other representations are $(\sqrt{2}, \frac{11\pi}{4})$ and $(\sqrt{2}, -\frac{5\pi}{4})$. Allowing negative $r$ we have $(-\sqrt{2}, -\frac{\pi}{4})$ and $(-\sqrt{2}, \frac{7\pi}{4})$.
   - (c) The point $(-3, -4)$ is in the third quadrant with $r = \sqrt{9 + 16} = 5$. Choose $\theta = \pi + \tan^{-1}(\frac{-4}{-3}) \approx \pi + 0.9 \approx 4$ radians. Other representations of this point are $(5, 2\pi + 4)$ and $(5, 4\pi + 4)$, and $(-5, 0.9)$.

2. Convert $(r, \theta) = (6, -\frac{\pi}{2})$ to rectangular coordinates by $x = r\cos\theta$ and $y = r\sin\theta$.

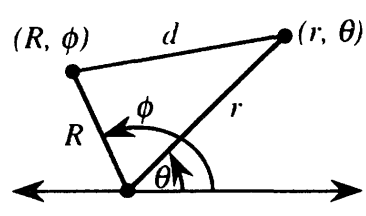
   - The $x$ coordinate is $6\cos(-\frac{\pi}{2}) = 0$. The $y$ coordinate is $6\sin(-\frac{\pi}{2}) = -6$.

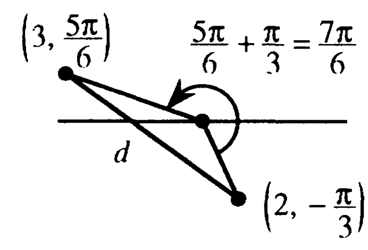
3. The Law of Cosines in trigonometry states that $c^2 = a^2 + b^2 - 2ab\cos C$. Here $a, b$ and $c$ are the side lengths of the triangle and $C$ is the angle opposite side $c$. Use the Law of Cosines to find the distance between the points with polar coordinates $(r, \theta)$ and $(R, \varphi)$.

   *Does it ever happen that $c^2$ is larger than $a^2 + b^2$?*

   - In the figure, the desired distance is labeled $d$. The other sides of the triangle have lengths $R$ and $r$. The angle opposite $d$ is $(\varphi - \theta)$. The Law of Cosines gives $d = \sqrt{R^2 + r^2 - 2Rr\cos(\varphi - \theta)}$.

   Yes, $c^2$ is larger than $a^2 + b^2$ when the angle $C = \varphi - \theta$ is larger than 90°. Its cosine is negative. The next problem is an example. When the angle $C$ is *acute* (smaller than 90°) then the term $-2ab\cos C$ reduces $c^2$ below $a^2 + b^2$.



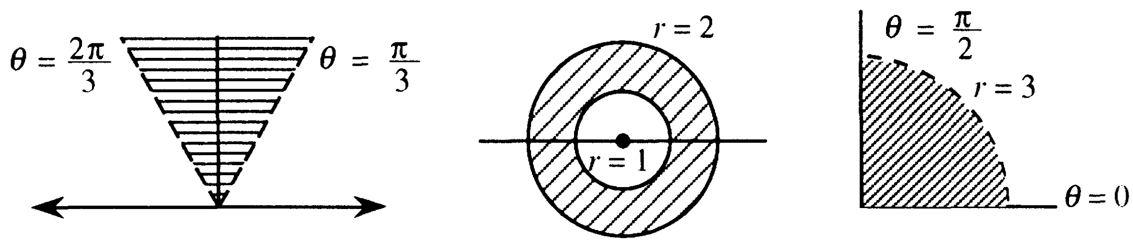



3′. Use the formula in Problem 3 to find the distance between the polar points $(3, \frac{5\pi}{6})$ and $(2, -\frac{\pi}{3})$.

- $d = \sqrt{3^2 + 2^2 - 2 \cdot 3 \cdot 2\cos(\frac{5\pi}{6} - (-\frac{\pi}{3}))} = \sqrt{13 - 12\cos\frac{7\pi}{6}} = \sqrt{13 + 6\sqrt{3}} \approx 4.8.$

4. Sketch the regions that are described in polar coordinates by
   (a) $r \geq 0$ and $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ (b) $1 \leq r \leq 2$ (c) $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq r < 3$.

   - The three regions are drawn. For (a), the dotted lines mean that $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are not included. If $r < 0$ were also allowed, there would be a symmetric region below the axis – a shaded $X$ instead of a shaded $V$.




5. Write the polar equation for the circle centered at $(x, y) = (1, 1)$ with radius $\sqrt{2}$.

   - The rectangular equation is $(x-1)^2 + (y-1)^2 = 2$ or $x^2 - 2x + y^2 - 2y = 0$. Replace $x$ with $r\cos\theta$ and $y$ with $r\sin\theta$. Always replace $x^2 + y^2$ with $r^2$. The equation becomes $r^2 = 2r\cos\theta + 2r\sin\theta$. Divide by $r$ to get $r = 2\ (\cos\theta + \sin\theta)$.

   Note that $r = 0$ when $\theta = -\frac{\pi}{4}$. The circle goes through the origin.

6. Write the polar equations for these lines: (a) $x = 3$ (b) $y = -1$ (c) $x + 2y = 5$.

   - (a) $x = 3$ becomes $r\cos\theta = 3$ or $r = 3\sec\theta$. Remember: $r = 3\cos\theta$ is a circle.
   - (b) $y = -1$ becomes $r\sin\theta = -1$ or $r = -\csc\theta$. But $r = -\sin\theta$ is a circle.
   - (c) $x + 2y = 5$ becomes $r\cos\theta + 2r\sin\theta = 5$. Again $r = \cos\theta + 2\sin\theta$ is a circle.

***Read-throughs and selected even-numbered solutions*** :

Polar coordinates $r$ and $\theta$ correspond to $x = \mathbf{r\cos\theta}$ and $y = \mathbf{r\sin\theta}$. The points with $r > 0$ and $\theta = \pi$ are located **on the negative x axis**. The points with $r = 1$ and $0 \leq \theta \leq \pi$ are located on **a semicircle**. Reversing the sign of $\theta$ moves the point $(x, y)$ to $(\mathbf{x}, -\mathbf{y})$.

Given $x$ and $y$, the polar distance is $r = \sqrt{\mathbf{x^2 + y^2}}$. The tangent of $\theta$ is $\mathbf{y/x}$. The point (6,8) has $r = \mathbf{10}$ and $\theta = \mathbf{\tan^{-1}\frac{8}{6}}$. Another point with the same $\theta$ is $(\mathbf{3}, \mathbf{4})$. Another point with the same $r$ is $(\mathbf{10,0})$. Another point with the same $r$ and $\tan\theta$ is $(-\mathbf{6}, -\mathbf{8})$.

The polar equation $r = \cos\theta$ produces a shifted **circle**. The top point is at $\theta = \pi/\mathbf{4}$, which gives $r = \sqrt{\mathbf{2}}/\mathbf{2}$. When $\theta$ goes from 0 to $2\pi$, we go **two** times around the graph. Rewriting as $r^2 = r\cos\theta$ leads to the $xy$ equation $\mathbf{x^2 + y^2} = \mathbf{x}$. Substituting $r = \cos\theta$ into $x = r\cos\theta$ yields $x = \mathbf{\cos^2}\theta$ and similarly $y = \mathbf{\cos\theta\sin\theta}$. In this form $x$ and $y$ are functions of the **parameter** $\theta$.

**10** $r = 3\pi, \theta = 3\pi$ has rectangular coordinates $\mathbf{x = -3\pi, y = 0}$

**16** (a) $(-1, \frac{\pi}{2})$ is the same point as $(\mathbf{1}, \frac{\mathbf{3\pi}}{\mathbf{2}})$ or $(-1, \frac{5\pi}{2})$ or $\cdots$ (b) $(-1, \frac{3\pi}{4})$ is the same point as $(\mathbf{1}, \frac{\mathbf{7\pi}}{\mathbf{4}})$ or $(-1, -\frac{\pi}{4})$ or $\cdots$ (c) $(1, -\frac{\pi}{2})$ is the same point as $(-1, \frac{\pi}{2})$ or $(\mathbf{1}, \frac{\mathbf{3\pi}}{\mathbf{2}})$ or $\cdots$ (d) $r = 0, \theta = 0$ is the same point as $\mathbf{r} = \mathbf{0}$, $\theta =$ **any angle.**

**18** (a) **False** ($r = 1, \theta = \frac{\pi}{4}$ is a different point from $r = -1, \theta = -\frac{\pi}{4}$) (b) **False** (for fixed $r$ we can add any multiple of $2\pi$ to $\theta$) (c) **True** ($r \sin\theta = 1$ is the horizontal line $y = 1$).

**22** Take the line from (0,0) to $(r_1, \theta_1)$ as the base (its length is $r_1$). The height of the third point $(r_2, \theta_2)$, measured perpendicular to this base, is $r_2$ times $\sin(\theta_2 - \theta_1)$.

**26** From $x = \cos^2\theta$ and $y = \sin\theta\cos\theta$, square and add to find $\mathbf{x^2 + y^2} = \cos^2\theta(\cos^2\theta + \sin^2\theta) = \cos^2\theta = \mathbf{x}$.

**28** Multiply $r = a\cos\theta + b\sin\theta$ by $r$ to find $x^2 + y^2 = ax + by$. Complete squares in $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$ and similarly in $y^2 - by$ to find $(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = (\frac{a}{2})^2 + (\frac{b}{2})^2$. This is a circle centered at $(\frac{a}{2}, \frac{b}{2})$ with radius $r = \sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2} = \frac{1}{2}\sqrt{\mathbf{a^2 + b^2}}$.

## 9.2 Polar Equations and Graphs (page 355)

The polar equation $r = F(\theta)$ is like $y = f(x)$. For each angle $\theta$ the equation tells us the distance $r$ (which is now allowed to be negative). By connecting those points we get a polar curve. Examples are $r = 1$ and $r\cos = \theta$ (circles) and $r = 1 + \cos\theta$ (cardioid) and $r = 1/(1 + e\cos\theta)$ (parabola, hyperbola, or ellipse, depending on $e$). These have nice-looking polar equations – because the origin is a special point for those curves.

*Note* $y = \sin x$ would be a disaster in polar coordinates. Literally it becomes $r\sin\theta = \sin(r\cos\theta)$. This mixes $r$ and $\theta$ together. It is comparable to $x^3 + xy^2 = 1$, which mixes $x$ and $y$. (For mixed equations we need implicit differentiation.) Equations in this section are not mixed, they are $r = F(\theta)$ and sometimes $r^2 = F(\theta)$.

Part of drawing the picture is recognizing the symmetry. One symmetry is "through the pole." If $r$ changes to $-r$, the equation $r^2 = F(\theta)$ stays the same – this curve has *polar symmetry.* But $r = \tan\theta$ also has polar symmetry, because $\tan\theta = \tan(\theta + \pi)$. If we go around by 180°, or $\pi$ radians, we get the same result as changing $r$ to $-r$.

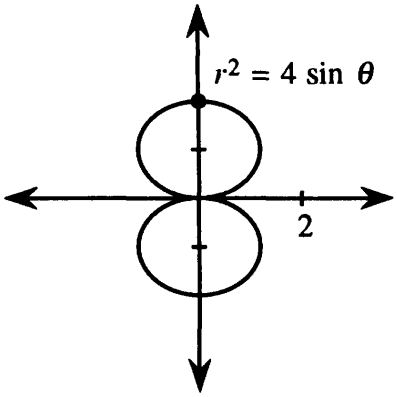
The three basic symmetries are across the $x$ axis, across the $y$ axis, and through the pole. Each symmetry has two main tests. (This is not clear in some texts I consulted.) Since one test could be passed without the other, I think you need to try both tests:

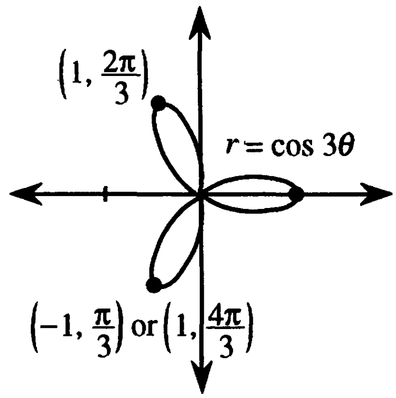
- $x$ axis symmetry: $\theta$ to $-\theta$ (test 1) or $\theta$ to $\pi - \theta$ and $r$ to $-r$ (test 2)
- $y$ axis symmetry: $\theta$ to $\pi - \theta$ (test 1) or $\theta$ to $-\theta$ and $r$ to $-r$ (test 2)
- polar symmetry: $\theta$ to $\pi + \theta$ (test 1) or $r$ to $-r$ (test 2).

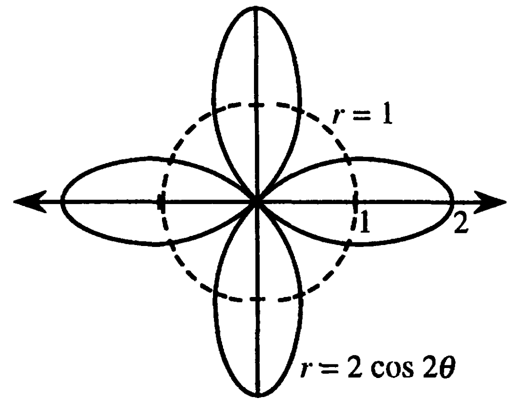
1. Sketch the polar curve $r^2 = 4\sin\theta$ after a check for symmetry.

   - When $r$ is replaced by $-r$, the equation $(-r)^2 = 4\sin\theta$ is the same. This means polar symmetry (through the origin). If $\theta$ is replaced by $(\pi - \theta)$, the equation $r^2 = 4\sin(\pi - \theta) = 4\sin\theta$ is still the same. There is symmetry about the $y$ axis. *Any two symmetries* (out of three) *imply the third.* This graph must be symmetric across the $x$ axis. ($\theta$ to $-\theta$ doesn't show it, because $\sin\theta$ changes. But $r$ to $-r$ and $\theta$ to $\pi - \theta$ leaves $r^2 = 4\sin\theta$ the same.) We can plot the curve in the first quadrant and reflect it to get the complete graph. Here is a table of values for the first quadrant and a sketch of the curve. The two closed parts (not circles) meet at $r = 0$.

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $4\sin\theta$ | $0$ | $2$ | $2\sqrt{2}$ | $2\sqrt{3}$ | $4$ |
| $r = \sqrt{4\sin\theta}$ | $0$ | $\sqrt{2} \approx 1.4$ | $\sqrt{2\sqrt{2}} \approx 1.7$ | $\sqrt{2\sqrt{3}} \approx 1.9$ | $2$ |




2. (This is Problem 9.2.9) Check $r = \cos 3\theta$ for symmetry and sketch its graph.

   - The cosine is even, $\cos(-3\theta) = \cos 3\theta$, so this curve is symmetric across the $x$ axis (where $\theta$ goes to $-\theta$). The other symmetry tests fail. For $\theta$ up to $\frac{\pi}{2}$ we get a loop and a half in the figure. Reflection across the $x$ axis yields the rest. The curve has three petals.

| $\theta$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r\cos 3\theta$ | $1$ | $\frac{\sqrt{2}}{2}$ | $0$ | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | $0$ |

3. Find the eight points where the four petals of $r = 2\cos 2\theta$ cross the circle $r = 1$.

   - Setting $2\cos 2\theta = 1$ leads to four crossing points $(1, \frac{\pi}{6})$, $(1, \frac{7\pi}{6})$, $(1, -\frac{\pi}{6})$, and $(1, -\frac{7\pi}{6})$. The sketch shows *four other crossing points*: $(1, \frac{\pi}{3})$, $(1, \frac{2\pi}{3})$, $(1, \frac{4\pi}{3})$ and $(1, \frac{5\pi}{3})$. These coordinates *do not* satisfy $r = 2\cos 2\theta$. But $r < 0$ yields other names $(-1, \frac{4\pi}{3})$, $(-1, \frac{5\pi}{3})$, $(-1, \frac{\pi}{3})$ and $(-1, \frac{2\pi}{3})$ for these points, that do satisfy the equation.

   In general, you need a sketch to find all intersections.

4. Identify these five curves:

   (a) $r = 5\csc\theta$ (b) $r = 6\sin\theta + 4\cos\theta$ (c) $r = \frac{9}{1+6\cos\theta}$ (d) $r = \frac{4}{2+\cos\theta}$ (e) $r = \frac{1}{3-3\sin\theta}$.

   - (a) $r = \frac{5}{\sin\theta}$ is $r\sin\theta = 5$. This is the *horizontal line* $y = 5$.
   - Multiply equation (b) by $r$ to get $r^2 = 6r\sin\theta + 4r\cos\theta$, or $x^2 + y^2 = 6y + 4x$. Complete squares to $(x-2)^2 + (y-3)^2 = 2^2 + 3^2 = 13$. This is a *circle* centered at (2,3) with radius $\sqrt{13}$.
   - (c) The pattern for conic sections (ellipse, parabola, and hyperbola) is $r = \frac{A}{1+e\cos\theta}$. Our equation has $A = 9$ and $e = 6$. The graph is a *hyperbola* with one focus at (0,0). The directrix is the line $x = \frac{9}{6} = \frac{3}{2}$.
   - (d) $r = \frac{4}{2+\cos\theta}$ doesn't exactly fit $\frac{A}{1+e\cos\theta}$ because of the 2 in the denominator. Factor it out: $\frac{2}{1+\frac{1}{2}\cos\theta}$ is an *ellipse* with $e = \frac{1}{2}$.
   - (e) $r = \frac{1}{3-3\sin\theta}$ is actually a *parabola*. To recognize the standard form, remember that $-\sin\theta = \cos(\frac{\pi}{2}+\theta)$. So $r = \frac{\frac{1}{3}}{1+\cos(\frac{\pi}{2}+\theta)}$. Since $\theta$ is replaced by $(\frac{\pi}{2}+\theta)$, the standard parabola has been rotated.

5. Find the length of the major axis (the distance between vertices) of the hyperbola $r = \frac{A}{1+e\cos\theta}$.

- Figure 9.5c in the text shows the vertices on the $x$ axis: $\theta = 0$ gives $r = \frac{A}{1+e}$ and $\theta = \pi$ gives $r = \frac{A}{1-e}$. (The hyperbola has $A > 0$ and $e > 1$.) Notice that $(\frac{A}{1-e}, \pi)$ is on the *right* of the origin because $r = \frac{A}{1-e}$ is negative. The distance between the vertices is $\frac{A}{e-1} - \frac{A}{e+1} = \frac{2A}{e^2-1}$.

Compare with exercise 9.2.35 for the ellipse. The distance between its vertices is $2a = \frac{2A}{1-e^2}$. The distance between vertices of a parabola ($e = 1$) is $\frac{2A}{0}$ = infinty! One vertex of the parabla is out at infinity.

***Read-throughs and selected even-numbered solutions*** :

The circle of radius 3 around the origin has polar equation $\mathbf{r = 3}$. The 45° line has polar equation $\boldsymbol{\theta} = \pi/4$. Those graphs meet at an angle of **90°**. Multiplying $r = 4\cos\theta$ by $r$ yields the $xy$ equation $\mathbf{x^2 + y^2 = 4x}$. Its graph is a **circle** with center at **(2,0)**. The graph of $r = 4/\cos\theta$ is the line $x = \mathbf{4}$. The equation $r^2 = \cos 2\theta$ is not changed when $\theta \to -\theta$ (symmetric across **the x axis**) and when $\theta \to \pi + \theta$ (or $r \to \mathbf{-r}$). The graph of $r = 1 + \cos\theta$ is a **cardioid**.

The graph of $r = A/(\mathbf{1} + \mathbf{e}\cos\theta)$ is a conic section with one focus at $(\mathbf{0}, \mathbf{0})$. It is an ellipse if $\mathbf{e < 1}$ and a hyperbola if $\mathbf{e > 1}$. The equation $r = 1/(1+\cos\theta)$ leads to $r + x = 1$ which gives a **parabola**. Then $r$ = distance from origin equals $1 - x$ = distance from **directrix** $\mathbf{y = 1}$. The equations $r = 3(1-x)$ and $r = \frac{1}{3}(1-x)$ represent a **hyperbola** and an **ellipse**. Including a shift and rotation, conics are determined by **five** numbers.

**6** $r = \frac{1}{1+2\cos\theta}$ is the hyperbola of Example 7 and Figure 9.5c: $r + 2r\cos\theta = 1$ is $r = 1 - 2x$ or $x^2 + y^2 = 1 - 4x + 4x^2$. The figure should show $r = -1$ and $\theta = \pi$ on the right branch.

**14** $r = 1 - 2\sin 3\theta$ has **y axis symmetry**: change $\theta$ to $\pi - \theta$, then $\sin 3(\pi - \theta) = \sin(\pi - 3\theta) = \sin 3\theta$.

**22** If $\cos\theta = \frac{r^2}{4}$ and $\cos\theta = 1 - r$ then $\frac{r^2}{4} = 1 - r$ and $r^2 + 4r - 4 = 0$. This gives $r = -2 - \sqrt{8}$ and $\mathbf{r = -2 + \sqrt{8}}$. The first $r$ is negative and cannot equal $1 - \cos\theta$. The second gives $\cos\theta = 1 - r = 3 - \sqrt{8}$ and $\theta \approx \mathbf{80°}$ **or** $\theta \approx \mathbf{-80°}$. The curves also meet at **the origin** $\mathbf{r = 0}$ and at the point $\mathbf{r = -2}, \theta = \mathbf{0}$ which is also $\mathbf{r = +2}, \theta = \pi$.

**26** The other 101 petals in $r = \cos 101\theta$ are **duplicates of the first 101**. For example $\theta = \pi$ gives $r = \cos 101\pi = -1$ which is also $\theta = 0, r = +1$. (Note that $\cos 100\pi = +1$ gives a new point.)

**28** (a) Yes, $x$ and $y$ symmetry imply $r$ symmetry. Reflections across the $x$ axis and then the $y$ axis take $(x, y)$ to $(x, -y)$ to $(-x, -y)$ which is reflection through the origin. (b) The point $r = -1, \theta = \frac{3\pi}{2}$ satisfies the equation $r = \cos 2\theta$ and it is the same point as $r = 1, \theta = \frac{\pi}{2}$.

**32** (a) $\theta = \frac{\pi}{2}$ gives $r = 1$; this is $x = 0, y = 1$ (b) The graph crosses the $x$ axis at $\theta = 0$ and $\pi$ where $x = \frac{1}{1+e}$ and $x = \frac{-1}{1-e}$. The center of the graph is halfway between at $x = \frac{1}{2}(\frac{1}{1+e} - \frac{1}{1-e}) = \frac{-e}{1-e^2}$. The second focus is twice as far from the origin at $\frac{\mathbf{-2e}}{\mathbf{1-e^2}}$. (Check: $e = 0$ gives center of circle, $e = 1$ gives second focus of parabola at infinity.)

# 9.3 Slope, Length, and Area for Polar Curves (page 359)

This section does *calculus in polar coordinates*. All the calculations for $y = f(x)$ – its slope $\frac{dy}{dx}$ and area

$\int y\,dx$ and arc length $\int \sqrt{1+(\frac{dy}{dx})^2}\,dx$ – can also be done for polar curves $r = F(\theta)$. But the formulas are a little more complicated! The slope is not $\frac{dF}{d\theta}$ and the area is not $\int F(\theta)d\theta$. These problems give practice with the polar formulas for slope, area, arc length, and surface area of revolution.

1. (This is 9.3.5) Draw the 4-petaled flower $r = \cos 2\theta$ and find the area inside. The petals are along the axes.

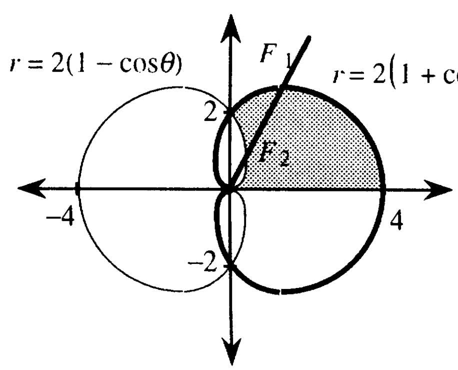
   - We compute the area of one petal and multiply by 4. The right-hand petal lies between the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$. Those are the limits of integration:

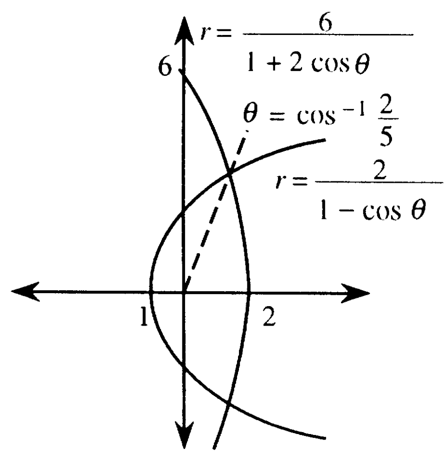
$$\text{Area} = 4\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2 d\theta = \int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)d\theta = \frac{\pi}{2}.$$

2. Find the area inside $r = 2(1+\cos\theta)$ and outside $r = 2(1-\cos\theta)$. Sketch those cardioids.

   - In the figure, half the required area is shaded. *Take advantage of symmetries!* A typical line through the origin is also sketched. Imagine this line sweeping from $\theta = 0$ to $\theta = \frac{\pi}{2}$ – the whole shaded area is covered. The outer radius is $2(1+\cos\theta)$, the inner radius is $2(1-\cos\theta)$. The shaded area is

$$\int_0^{\pi/2} \frac{1}{2}[4(1+\cos\theta)^2 - 4(1-\cos\theta)^2]d\theta = 8\int_0^{\pi/2} \cos\theta\, d\theta = 8. \qquad \text{Total area 16.}$$




3. Set up the area integral(s) between the parabola $r = \frac{2}{1-\cos\theta}$ and the hyperbola $r = \frac{6}{1+2\cos\theta}$.

   - The curves are shown in the sketch. We need to find where they cross. Solving $\frac{6}{1+2\cos\theta} = \frac{2}{1-\cos\theta}$ yields $6(1-\cos\theta) = 2(1+2\cos\theta)$ or $\cos\theta = \frac{2}{5} = .4$. At that angle $r = \frac{6}{1+2(\frac{2}{5})} = \frac{6}{1.8}$.

   Imagine a ray sweeping around the origin from $\theta = 0$ to $\theta = \pi$. From $\theta = 0$ to $\theta = \cos^{-1}.4$, the ray crosses the *hyperbola*. Then it crosses the *parabola*. That is why the area must be computed in two parts. Using symmetry we find only the top half:

$$\text{Half-area} = \int_0^{\cos^{-1}.4} \frac{1}{2}\left(\frac{6}{1+2\cos\theta}\right)^2 d\theta + \int_{\cos^{-1}.4}^{\pi} \frac{1}{2}\left(\frac{2}{1-\cos\theta}\right)^2 d\theta.$$

   Simpson's rule gives the total area (top half doubled) as approximately 12.1.

***Problems* 4 *and* 5 *are about lengths of curves.***

4. Find the distance around the cardioid $r = 1 + \cos\theta$.

   - Length in polar coordinates is $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$. For the cardioid this square root is

$$\sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta} = \sqrt{2 + 2\cos\theta}.$$

Half the curve is traced as $\theta$ goes from 0 to $\pi$. The total length is $\int ds = 2\int_0^\pi \sqrt{2+2\cos\theta}\, d\theta$. Evaluating this integral uses the trick $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$. Thus the cardioid length is

$$2\int_0^\pi \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 4\int_0^\pi \cos\frac{\theta}{2}\, d\theta = 8\sin\frac{\theta}{2}\Big]_0^\pi = 8.$$

5. Find the length of the spiral $r = e^{\theta/2}$ as $\theta$ goes from 0 to $2\pi$.

   - For this curve $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$ is equal to $\sqrt{\frac{1}{4}e^\theta + e^\theta}\, d\theta = \sqrt{\frac{5}{4}e^\theta}\, d\theta = \frac{\sqrt{5}}{2}e^{\theta/2}d\theta$:

$$\text{Length } = \int_0^{2\pi} \frac{\sqrt{5}}{2}e^{\theta/2}d\theta = \sqrt{5}\, e^{\theta/2}\Big]_0^{2\pi} = \sqrt{5}(e^\pi - 1) \approx 49.5.$$

***Problems* 6 *and* 7 *ask for the areas of surfaces of revolution.***

6. Find the surface area when the spiral $r = e^{\theta/2}$ between $\theta = 0$ and $\theta = \pi$ is revolved about the horizontal axis.

   - From Section 8.3 we know that the area is $\int 2\pi y\, ds$. For this curve the previous problem found $ds = \frac{\sqrt{5}}{2}e^{\theta/2}d\theta$. The factor $y$ in the area integral is $r\sin\theta = e^{\theta/2}\sin\theta$. The area is

$$\begin{aligned}\int_0^\pi 2\pi(e^{\theta/2}\sin\theta)\tfrac{\sqrt{5}}{2}e^{\theta/2}d\theta &= \sqrt{5}\pi\int_0^\pi e^\theta\sin\theta d\theta \\ &= \tfrac{\sqrt{5}\pi}{2}e^\theta(\sin\theta - \cos\theta)\Big]_0^\pi = \tfrac{\sqrt{5}\pi}{2}(e^\pi + 1) \approx 84.8.\end{aligned}$$

7. Find the surface area when the curve $r^2 = 4\sin\theta$ is revolved around the $y$ axis.

   - The curve is drawn in Section 9.2 of this guide (Problem 1).
   - If we revolve the piece from $\theta = 0$ to $\theta = \pi/2$, and double that area, we get the total surface area. In the integral $\int_{\theta=0}^{\pi/2} 2\pi x\, ds$ we replace $x$ by $r\cos\theta = 2\sqrt{\sin\theta}\cos\theta$. Also $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta = \sqrt{\frac{\cos^2\theta}{\sin\theta} + 4\sin\theta}\, d\theta$. The integral for surface area is not too easy:

$$4\pi\int_0^{\pi/2} 2\sqrt{\sin\theta}\cos\theta\sqrt{\frac{\cos^2\theta}{\sin\theta} + 4\sin\theta}\, d\theta = 8\pi\int_0^{\pi/2}\cos\theta\sqrt{\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= 8\pi\int_0^{\pi/2}\cos\theta\sqrt{1 + 3\sin^2\theta}\, d\theta = 8\pi\int_0^1 \sqrt{1+3u^2}\, du \text{ (where } u = \sin\theta).$$

A table of integrals gives $8\pi\sqrt{3}(\frac{u}{2}\sqrt{\frac{1}{3} + u^2} + \frac{1}{6}\ln(u + \sqrt{\frac{1}{3} + u^2})]_0^1 = 8\pi\sqrt{3}(\frac{1}{\sqrt{3}} + \frac{1}{6}\ln(2+\sqrt{3})) \approx 34.1$.

8. Find the slope of the three-petal flower $r = \cos 3\theta$ at the tips of the petals.

- The flower is drawn in Section 9.2. The tips are at (1,0), $(-1, \frac{\pi}{3})$, and $(-1, -\frac{\pi}{3})$. Clearly the tangent line at (1,0) is vertical (infinite slope). For the other two slopes, find $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$. From $y = r\sin\theta$ we get $\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$. Similarly $x = r\cos\theta$ gives $\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}$. Substitute $\frac{dr}{d\theta} = -3\sin 3\theta$ for this flower, and set $r = -1$, $\theta = \frac{\pi}{3}$:

$$\frac{dy}{dx} = \frac{r\cos\theta - 3\sin 3\theta\sin\theta}{-r\sin\theta - 3\sin 3\theta\cos\theta} = \frac{(-1)\cos\pi/3 - 3\sin\pi\sin\pi/3}{\sin\pi/3 - 3\sin\pi\cos\pi/3} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$

9. If $F(3) = 0$, show that the graph of $r = F(\theta)$ at $r = 0$, $\theta = 3$ has slope $\tan 3$.

- As an example of this idea, look at the graph of $r = \cos 3\theta$ (Section 9.1 of this guide). At $\theta = \pi/6$, $\theta = \pi/2$, and $\theta = -\pi/6$ we find $r = 0$. The rays out from the origin at those three angles are tangent to the graph. In other words the slope of $r = \cos 3\theta$ at $(0, \pi/6)$ is $\tan(\pi/6)$, the slope at $(0, \pi/2)$ is $\tan(\pi/2)$ and the slope at $(0, -\pi/6)$ is $\tan(-\pi/6)$.
- To prove the general statement, write $\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta dr/d\theta}{-r\sin\theta + \cos\theta dr/d\theta}$ as in Problem 8. With $r = F(\theta)$ and $F(3) = 0$, substitute $\theta = 3$, $r = 0$, and $dr/d\theta = F'(3)$. The slope at $\theta = 3$ is $\frac{dy}{dx} = \frac{\sin(3)F'(3)}{\cos(3)F'(3)} = \tan(3)$.

***Read-throughs and selected even-numbered solutions*** :

A circular wedge with angle $\Delta\theta$ is a fraction $\mathbf{\Delta\theta/2\pi}$ of a whole circle. If the radius is $r$, the wedge area is $\mathbf{\frac{1}{2}r^2\Delta\theta}$. Then the area inside $r = F(\theta)$ is $\int \mathbf{\frac{1}{2}r^2 d\theta} = \int \mathbf{\frac{1}{2}(F(\theta))^2 d\theta}$. The area inside $r = \theta^2$ from 0 to $\pi$ is $\mathbf{\pi^5/10}$. That spiral meets the circle $r = 1$ at $\theta = \mathbf{1}$. The area inside the circle and outside the spiral is $\mathbf{\frac{1}{2} - \frac{1}{10}}$. A chopped wedge of angle $\Delta\theta$ between $r_1$ and $r_2$ has area $\mathbf{\frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta}$.

The curve $r = F(\theta)$ has $x = r\cos\theta = \mathbf{F(\theta)\cos\theta}$ and $y = \mathbf{F(\theta)\sin\theta}$. The slope $dy/dx$ is $dy/d\theta$ divided by $\mathbf{dx/d\theta}$. For length $(ds)^2 = (dx)^2 + (dy)^2 = \mathbf{(dr)^2 + (rd\theta)^2}$. The length of the spiral $r = \theta$ to $\theta = \pi$ is $\int \mathbf{\sqrt{1+\theta^2}d\theta}$. The surface area when $r = \theta$ is revolved around the $x$ axis is $\int 2\pi y\, ds = \int \mathbf{2\pi\theta\sin\theta\sqrt{1+\theta^2}d\theta}$. The volume of that solid is $\int \pi y^2 dx = \int \mathbf{\pi\theta^2\sin^2\theta(\cos\theta - \theta\sin\theta)d\theta}$.

**4** The inner loop is where $r < 0$ or $\cos\theta < -\frac{1}{2}$ or $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$. Its area is $\int \frac{r^2}{2}d\theta = \int \frac{1}{2}(1+4\cos\theta+4\cos^2\theta)d\theta = [\frac{\theta}{2} + 2\sin\theta + \theta + \cos\theta\,\sin\theta]_{2\pi/3}^{4\pi/3} = \frac{\pi}{3} - 2(\sqrt{3}) + \frac{2\pi}{3} + \frac{1}{2}\sqrt{3} = \pi - \mathbf{\frac{3}{2}\sqrt{3}}$.

**16** The spiral $r = e^{-\theta}$ starts at $r = 1$ and returns to the $x$ axis at $r = e^{-2\pi}$. Then it goes inside itself (no new area). So area $= \int_0^{2\pi} \frac{1}{2}e^{-2\theta}d\theta = [-\frac{1}{4}e^{-2\theta}]_0^{2\pi} = \mathbf{\frac{1}{4}(1 - e^{-4\pi})}$.

**20** Simplify $\frac{\tan\phi - \tan\theta}{1+\tan\phi\,\tan\theta} = \frac{\frac{F+\tan\theta F'}{-\tan\theta F+F'} - \tan\theta}{1+\frac{F+\tan\theta F'}{-\tan\theta F+F'}\tan\theta} = \frac{F+\tan\theta F' - \tan\theta(-\tan\theta F+F')}{-\tan\theta F+F'+\tan\theta(F+\tan\theta F')} = \frac{(1+\tan^2\theta)F}{(1+\tan^2\theta)F'} = \mathbf{\frac{F}{F'}}$.

**22** $r = 1 - \cos\theta$ is the mirror image of Figure 9.4c across the $y$ axis. By Problem 20, $\tan\psi = \frac{F}{F'} = \frac{1-\cos\theta}{\sin\theta}$. This is $\frac{\frac{1}{2}\sin^2\frac{\theta}{2}}{\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$. So $\psi = \frac{\theta}{2}$ (check at $\theta = \pi$ where $\psi = \frac{\pi}{2}$).

**24** By Problem 18 $\frac{dy}{dx} = \frac{\cos\theta + \tan\theta(-\sin\theta)}{-\cos\theta\,\tan\theta - \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta(-2\sin\theta)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{6}$. At that point $x = r\,\cos\,\theta = \cos^2\frac{\pi}{6} = (\frac{\sqrt{3}}{2})^2$ and $y = r\,\sin\theta = \cos\frac{\pi}{6}\,\sin\frac{\pi}{6} = \frac{1}{2}(\frac{\sqrt{3}}{2})$. The tangent line is $\mathbf{y - \frac{\sqrt{3}}{4} = -\frac{1}{\sqrt{3}}(x - \frac{3}{4})}$.

**26** $r = \sec\theta$ has $\frac{dr}{d\theta} = \sec\theta\tan\theta$ and $\frac{ds}{d\theta} = \sqrt{\sec^2\theta + \sec^2\theta\,\tan^2\theta} = \sqrt{\sec^4\theta} = \sec^2\theta$. Then arc length $= \int_0^{\pi/4}\sec^2\theta\,d\theta = \tan\frac{\pi}{4} = \mathbf{1}$. Note: $r = \sec\theta$ is the line $r\,\cos\theta = 1$ or $x = 1$ from $y = 0$ up to $y = 1$.

**32** $r = 1 + \cos\,\theta$ has $\frac{ds}{d\theta} = \sqrt{(1 + 2\,\cos\,\theta + \cos^2\theta) + \sin^2\theta} = \sqrt{2 + 2\,\cos\theta}$. Also $y = r\,\sin\,\theta = (1+\cos\,\theta)\sin\theta$. Surface area $\int 2\pi\,y\,ds = 2\pi\sqrt{2}\int_0^\pi(1+\cos\,\theta)^{3/2}\sin\,\theta\,d\theta = [2\pi\sqrt{2}(-\frac{2}{5})(1+\cos\,\theta)^{5/2}]_0^\pi = \frac{32\pi}{5}$.

**40** The parameter $\theta$ along the ellipse $x = 4\,\cos\,\theta, y = 3\,\sin\,\theta$ is *not* the angle from the origin. For example at $\theta = \frac{\pi}{4}$ the point $(x, y)$ is *not* on the 45° line. So the area formula $\int \frac{1}{2}r^2 d\theta$ does not apply. The correct area is $12\pi$.

# 9.4 Complex Numbers (page 364)

There are two important forms for every complex number: the *rectangular form* $x+iy$ and the *polar form* $re^{i\theta}$. Converting from one to the other is like changing between rectangular and polar coordinates. In one direction use $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. In the other direction (definitely easier) use $x = r\cos\theta$ and $y = r\sin\theta$. Problem 1 goes to polar and Problem 2 goes to rectangular.

1. Convert these complex numbers to polar form: (a) $3+4i$ (b) $-5-12i$ (c) $i\sqrt{3}-1$.

- (a) $r = \sqrt{3^2+4^2} = 5$ and $\theta = \tan^{-1}\frac{4}{3} \approx .93$. Therefore $3+4i \approx 5e^{.93i}$.
- (b) $-5-12i$ lies in the third quadrant of the complex plane, so $\theta = \pi + \arctan^{-1}\frac{-12}{-5} \approx \pi + 1.17 \approx 4.3$. The distance from the origin is $r = \sqrt{(-5)^2+(-12)^2} = 13$. Thus $-5-12i \approx 13e^{4.3i}$.
- (c) $i\sqrt{3}-1$ is not exactly in standard form: rewrite as $-1+i\sqrt{3}$. Then $x = -1$ and $y = \sqrt{3}$ and $r = \sqrt{1+3} = 2$. This complex number is in the second quadrant of the complex plane, since $x < 0$ and $y > 0$. The angle is $\theta = \frac{2\pi}{3}$. Then $-1+i\sqrt{3} = 2e^{2\pi/3}$.

We chose the standard polar form, with $r > 0$ and $0 \le \theta < 2\pi$. Other polar forms are allowed. The answer for (c) could also be $2e^{(2\pi+2\pi/3)i}$ or $2e^{-4\pi i/3}$.

2. Convert these complex numbers to rectangular form: (a) $6e^{i\pi/4}$ (b) $e^{-7\pi/6}$ (c) $3e^{\pi/3}$

- (a) The point $z = 6e^{i\pi/4}$ is 6 units out along the ray $\theta = \pi/4$. Since $x = 6\cos\frac{\pi}{4} = 3\sqrt{2}$ and $y = 6\sin\frac{\pi}{4} = 3\sqrt{2}$, the rectangular form is $3\sqrt{2}+3\sqrt{2}i$.
- (b) We have $r = 1$. The number is $\cos(-\frac{7\pi}{6}) + i\sin(-\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.
- (c) *There is no $i$ in the exponent*! $3e^{\pi/3}$ is just a plain real number (approximately 8.5). Its rectangular form is $3e^{\pi/3} + 0i$.

3. For each pair of numbers find $z_1+z_2$ and $z_1-z_2$ and $z_1z_2$ and $z_1/z_2$:

(a) $z_1 = 4-3i$ and $z_2 = 12+5i$ (b) $z_1 = 3e^{i\pi/6}$ and $z_2 = 2e^{i7\pi/4}$.

- (a) Add $z_1+z_2 = 4-3i+12+5i = 16+2i$. Subtract $(4-3i)-(12+5i) = -8-8i$. Multiply:

$$(4-3i)(12+5i) = 48-36i+20i-15i^2 = 63-16i$$

To divide by $12+5i$, multiply top and bottom by its complex conjugate $12-5i$.
*Then the bottom is real*:

$$\frac{4-3i}{12+5i}\cdot\frac{12-5i}{12-5i} = \frac{33-56i}{12^2+5^2} = \frac{33}{169} - \frac{56}{169}i.$$

- You could choose to multiply in polar form. First convert $4-3i$ to $re^{i\theta}$ with $r = 5$ and $\tan\theta = -\frac{3}{4}$. Also $12+5i$ has $r = 13$ and $\tan\theta = \frac{5}{12}$. Multiply the $r$'s to get $5\cdot 13 = 65$. Add the $\theta$'s. This is hard without a calculator that knows $\tan^{-1}(-\frac{3}{4})$ and $\tan^{-1}(\frac{5}{12})$. Our answer is $\theta_1+\theta_2 \approx -.249$.

So multiplication gives $65e^{-.249i}$ which is close to the first answer $63 - 16i$. Probably a trig identity would give $\tan^{-1}(-\frac{3}{4}) + \tan^{-1}(\frac{5}{12}) = \tan^{-1}(-\frac{16}{63})$.

For division in polar form, divide $r$'s and subtract angles: $\frac{5}{13}e^{i(\theta_1-\theta_2)} \approx \frac{5}{13}e^{-i}$. This is $\frac{z_1}{z_2} = \frac{5}{13}\cos(-1) + \frac{5}{13}i\sin(-1) \approx .2 - .3i \approx \frac{33}{169} - \frac{56}{169}i$.

- (b) Numbers in polar form are not easy to add. Convert to rectangular form:

$$3e^{i\pi/6} \text{ equals } 3\cos\frac{\pi}{6} + 3i\sin\frac{\pi}{6} = \frac{3\sqrt{3}}{2} + \frac{3i}{2}. \text{ Also } 2e^{i7\pi/4} \text{ equals } 2\cos\frac{7\pi}{4} + 2i\sin\frac{7\pi}{4} = \sqrt{2} - i\sqrt{2}.$$

The sum is $(\frac{3\sqrt{3}}{2} + \sqrt{2}) + (\frac{3}{2} - \sqrt{2})i$. The difference is $(\frac{3\sqrt{3}}{2} - \sqrt{2}) + (\frac{3}{2} + \sqrt{2})i$.

Multiply and divide in polar form whenever possible. ***Multiply*** $r$'s ***and add*** $\theta$'s:

$$z_1z_2 = (3\cdot 2)e^{i(\frac{\pi}{6}+\frac{7\pi}{4})} = 6e^{\frac{23\pi i}{12}} \text{ and } \frac{z_1}{z_2} = \frac{3}{2}e^{i(\frac{\pi}{6}-\frac{7\pi}{4})} = \frac{3}{2}e^{-19\pi i/24}.$$

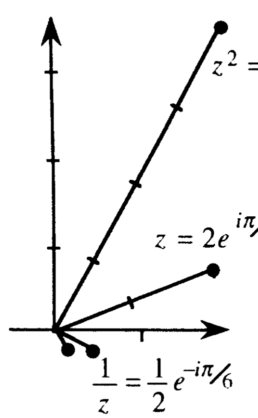
3. Find $(2 - 2\sqrt{3}i)^{10}$ in polar and rectangular form.

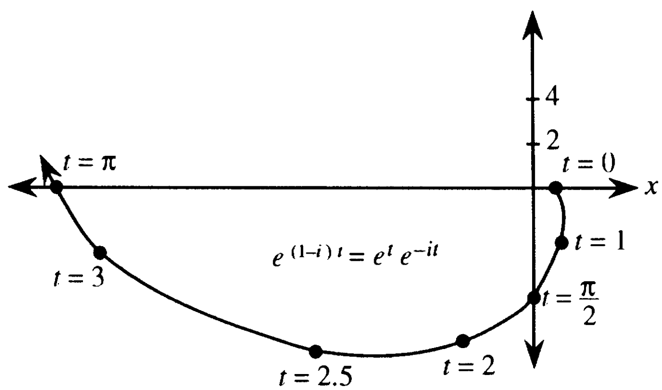
- DeMoivre's Theorem is based on the polar form: $2 - 2\sqrt{3}i = 4e^{-i\pi/3}$. The tenth power is $(4e^{-i\pi/3})^{10} = 4^{10}e^{-10\pi/3}$. In rectangular form this is

$$4^{10}(\cos\frac{-10\pi}{3} + i\sin\frac{-10\pi}{3}) = 4^{10}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 2^{20} - (\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -2^{19} + 2^{19}i\sqrt{3}.$$

4. (This is 9.4.3) Plot $z = 2e^{i\pi/6}$ and its reciprocal $\frac{1}{z} = \frac{1}{2}e^{-i\pi/6}$ and their squares.

- The squares are $(2e^{i\pi/6})^2 = 4e^{i\pi/3}$ and $(\frac{1}{2}e^{-i\pi/6})^2 = \frac{1}{4}e^{-i\pi/3}$. The points $z$, $\frac{1}{z}$, $z^2$, $\frac{1}{z^2}$ are plotted.




5. (This is 9.4.25) For $c = 1 - i$, sketch the path of $y = e^{ct}$ as $t$ increases from 0.

- The moving point $e^{ct}$ is $e^{(1-i)t} = e^te^{-it} = e^t(\cos(-t) + i\sin(-t))$. The table gives $x$ and $y$:

| $t$ | 0 | .5 | 1.0 | $\pi/2$ | 2.0 | 2.5 | 3.0 | $\pi$ |
|---|---|---|---|---|---|---|---|---|
| $x = e^t\cos(-t)$ | 1 | 1.4 | 1.5 | 0 | −3.1 | −9.8 | −19.9 | 23.1 |
| $y = e^t\sin(-t)$ | 0 | −.8 | −2.3 | −4.8 | −6.7 | −7.3 | −2.8 | 0 |

The sketch shows how $e^{ct}$ spirals rapidly outwards from $e^0 = 1$.

6. For the differential equation $y'' + 4y' + 3y = 0$, find all solutions of the form $y = e^{ct}$.

   - The derivatives of $y = e^{ct}$ are $y' = ce^{ct}$ and $y'' = c^2e^{ct}$. The equation asks for $c^2e^{ct} + 4ce^{ct} + 3e^{ct} = 0$. This means that $e^{ct}(c^2 + 4c + 3) = 0$. Factor $c^2 + 4c + 3$ into $(c + 3)(c + 1)$. This is zero for $c = -3$ and $c = -1$. The pure exponential solutions are $y = e^{-3t}$ and $y = e^{-t}$. Any combination like $2e^{-3t} + 7e^{-t}$ also solves the differential equation.

7. Construct two real solutions of $y'' + 2y' + 5y = 0$. Start with solutions of the form $y = e^{ct}$.

   - Substitute $y'' = c^2e^{ct}$ and $y' = ce^{ct}$ and $y = e^{ct}$. This leads to $c^2 + 2c + 5 = 0$ or $c = -1 \pm 2i$. The pure (but complex) exponential solutions are $y = e^{(-1+2i)t}$ and $y = e^{(-1-2i)t}$. The first one is $y = e^{-t}(\cos 2t + i \sin 2t)$. The real part is $x = e^{-t} \cos 2t$; the imaginary part is $y = e^{-t} \sin 2t$. (*Note:* The imaginary part is without the $i$.) Each of these is a real solution, as may be checked by substitution into $y'' + 2y' + 5y = 0$.

The other exponential is $y = e^{(-1-2i)t} = e^{-t}(\cos(-2t) + i \sin(-2t))$. Its real and imaginary parts are the *same real solutions* – except for the minus sign in $\sin(-2t) = -\sin 2t$.

***Read-throughs and selected even-numbered solutions*** :

The complex number $3 + 4i$ has real part **3** and imaginary part **4**. Its absolute value is $r = \mathbf{5}$ and its complex conjugate is **3 − 4i**. Its position in the complex plane is at **(3,4)**. Its polar form is $r \cos\theta + ir \sin\theta = \mathbf{re^{i\theta}}$ **(or $\mathbf{5e^{i\theta}}$)**. Its square is **−7 − 14i**. Its $n$th power is $\mathbf{r^n}e^{in\theta}$.

The sum of $1 + i$ and $1 - i$ is **2**. The product of $1 + i$ and $1 - i$ is **2**. In polar form this is $\sqrt{2}e^{i\pi/4}$ times $\mathbf{\sqrt{2}e^{-i\pi/4}}$. The quotient $(1+i)/(1-i)$ equals the imaginary number i. The number $(1+i)^8$ equals **16**. An eighth root of 1 is $w = \mathbf{(1 + i)/\sqrt{2}}$. The other eighth roots are $\mathbf{w^2, w^3, \cdots, w^7, w^8 = 1}$.

To solve $d^8y/dt^8 = y$, look for a solution of the form $y = \mathbf{e^{ct}}$. Substituting and canceling $e^{ct}$ leads to the equation $\mathbf{c^8 = 1}$. There are **eight** choices for $c$, one of which is $(-1 + i)/\sqrt{2}$. With that choice $| e^{ct} | = \mathbf{e^{-t/\sqrt{2}}}$. The real solutions are Re $e^{ct} = \mathbf{e^{-t/\sqrt{2}} \cos \frac{t}{\sqrt{2}}}$ and Im $e^{ct} = \mathbf{e^{-t/\sqrt{2}} \sin \frac{t}{\sqrt{2}}}$.

**10** $e^{ix} = i$ yields $\mathbf{x = \frac{\pi}{2}}$ (note that $\frac{i\pi}{2}$ becomes $\ln i$); $e^{ix} = e^{-1}$ yields $\mathbf{x = i}$, second solutions are $\frac{\pi}{2} + 2\pi$ and $i + 2\pi$.

**14** The roots of $c^2 - 4c + 5 = 0$ must multiply to give **5**. Check: The roots are $\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$. Their product is $(2 + i)(2 - i) = 4 - i^2 = 5$.

**18** The fourth roots of $re^{i\theta}$ are $r^{1/4}$ times $e^{i\theta/4}, e^{i(\theta+2\pi)/4}, e^{i(\theta+4\pi)/4}, e^{i(\theta+6\pi)/4}$. Multiply $(r^{1/4})^4$ to get $r$. Add angles to get $(4\theta + 12\pi)/4 = \theta + 3\pi$. The product of the 4 roots is $re^{i(\theta+3\pi)} = -re^{i\theta}$.

**28** $\frac{dy}{dt} = iy$ leads to $y = e^{it} = \cos t + i \sin t$. Matching real and imaginary parts of $\frac{d}{dt}(\cos t + i \sin t) = i(\cos t + i \sin t)$ yields $\frac{d}{dt} \cos t = -\sin t$ and $\frac{d}{dt} \sin t = \cos t$.

**34** Problem 30 yields $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh x$; similarly $\sin ix = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{i}{2i}(e^{-x} - e^{x}) = i \sinh x$. With $x = 1$ the cosine of $i$ equals $\frac{1}{2}(e^{-1} + e^{1}) =$ **3.086**. The cosine of $i$ is larger than 1!

# 9 Chapter Review Problems

### *Review Problems*

**R1** Express the point $(r, \theta)$ in rectangular coordinates. Express the point $(a, b)$ in polar coordinates. Express the point $(r, \theta)$ with three other pairs of polar coordinates.

**R2** As $\theta$ goes from 0 to $2\pi$, how often do you cover the graph of $r = \cos\theta$? $r = \cos 2\theta$? $r = \cos 3\theta$?

**R3** Give an example of a polar equation for each of the conic sections, including circles.

**R4** How do you find the area between two polar curves $r = F(\theta)$ and $r = G(\theta)$ if $0 < F < G$?

**R5** Write the polar form for $ds$. How is this used for surface areas of revolution?

**R6** What is the polar formula for slope? Is it $dr/d\theta$ or $dy/dx$?

**R7** Multiply $(a+ib)(c+id)$ and divide $(a+ib)/(c+id)$.

**R8** Sketch the eighth roots of 1 in the complex plane. How about the roots of $-1$?

**R9** Starting with $y = e^{ct}$, find two real solutions to $y'' + 25y = 0$.

**R10** How do you test the symmetry of a polar graph? Find the symmetries of

(a) $r = 2\cos\theta + 1$ (b) $= 8\sin\theta$ (c) $r = \frac{6}{1-\cos\theta}$ (d) $r = \sin 2\theta$ (e) $r = 1 + 2\sin\theta$

### *Drill Problems*

**D1** Show that the area inside $r^2 = \sin 2\theta$ and outside $r = \frac{\sqrt{2}}{2}$ is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

**D2** Find the area inside both curves $r = 2 - \cos\theta$ and $r = 3\cos\theta$.

**D3** Show that the area enclosed by $r = 2\cos 3\theta$ is $\pi$.

**D4** Show that the length of $r = 4\sin^3\frac{\theta}{3}$ between $\theta = 0$ and $\theta = \pi$ is $2\pi - \frac{3}{2}\sqrt{3}$.

**D5** Confirm that the length of the spiral $r = 3\theta^2$ from $\theta = 0$ to $\theta = \frac{5}{3}$ is $\frac{7}{3}$.

**D6** Find the slope of $r = \sin 3\theta$ at $\theta = \frac{\pi}{6}$.

**D7** Find the slope of the tangent line to $r = \tan\theta$ at $(1, \frac{\pi}{2})$.

**D8** Show that the slope of $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{6}$ is $\frac{2}{\sqrt{3}}$.

**D9** The curve $r^2 = \cos 2\theta$ from $(1, -\frac{\pi}{4})$ to $(1, \frac{\pi}{4})$ is revolved around the $y$ axis. Show that the surface area is $2\sqrt{2}\pi$.

**D10** Sketch the parabola $r = 4/(1 + \cos\theta)$ to see its focus and vertex.

**D11** Find the center of the ellipse whose polar equation is $r = \frac{6}{2-\cos\theta}$. What is the eccentricity $e$?

**D12** The asymptotes of the hyperbola $r = \frac{6}{1+3\cos\theta}$ are the rays where $1 + 3\cos\theta = 0$. Find their slopes.

**D13** Find all the sixth roots (two real, four complex) of 64.

**D14** Find four roots of the equation $z^4 - 2z^2 + 4 = 0$.

**D15** Add, subtract, multiply, and divide $1 + \sqrt{3}\,i$ and $1 - \sqrt{3}\,i$.

**D16** Add, subtract, multiply, and divide $e^{i\pi/4}$ and $e^{-i\pi/4}$.

**D17** Find all solutions of the form $y = e^{ct}$ for $y'' - y' - 2y = 0$ and $y''' - 2y' - 3y = 0$.

**D18** Construct real solutions of $y'' - 4y' + 13y = 0$ from the real and imaginary parts of $y = e^{ct}$.

**D19** Use a calculator or an integral to estimate the length of $r = 1 + \sin\theta$ (near 2.5?).

***Graph Problems*** (intended to be drawn by hand)

**G1** $r^2 = \sin 2\theta$

**G2** $r = 6\sin\theta$

**G3** $r = \sin 4\theta$

**G4** $r = 5\sec\theta$

**G5** $r = e^{\theta/2}$

**G6** $r = 2 - 3\cos\theta$

**G7** $r = \frac{6}{1+2\cos\theta}$

**G8** $r = \frac{1}{1-\sin\theta}$