CHAPTER 16

Mathematics after Calculus

I would like this book to do more than help you pass calculus. (I hope it does that too.) After calculus you will have choices—***Which mathematics course to take next***?—and these pages aim to serve as a guide. Part of the answer depends on where you are going—toward engineering or management or teaching or science or another career where mathematics plays a part. The rest of the answer depends on where the courses are going. This chapter can be a useful reference, to give a clearer idea than course titles can do:

Linear Algebra   Differential Equations   Discrete Mathematics

Advanced Calculus (with Fourier Series)   Numerical Methods   Statistics

Pure mathematics is often divided into analysis and algebra and geometry. Those parts come together in the "mathematical way of thinking"—a mixture of logic and ideas. It is a deep and creative subject—here we make a start.

Two main courses after calculus are ***linear algebra*** and ***differential equations***. I hope you can take both. To help you later, Sections 16.1 and 16.2 organize them by examples. First a few words to compare and contrast those two subjects.

Linear algebra is about *systems* of equations. There are n variables to solve for. A change in one affects the others. They can be prices or velocities or currents or concentrations—outputs from any model with interconnected parts.

Linear algebra makes only one assumption—*the model must be linear*. A change in one variable produces proportional changes in all variables. Practically every subject begins that way. (When it becomes nonlinear, we solve by a sequence of linear equations. Linear programming is nonlinear because we require $x \geqslant 0$.) Elsewhere I wrote that "Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier." I recommend taking it.

A differential equation is *continuous* (from calculus), where a matrix equation is *discrete* (from algebra). The rate dy/dt is determined by the present state y—which changes by following that rule. Section 16.2 solves $y' = cy + s(t)$ for economics and life sciences, and $y'' + by' + cy = f(t)$ for physics and engineering. Please keep it and refer to it.

A third key direction is ***discrete mathematics***. Matrices are a part, networks and algorithms are a bigger part. Derivatives are not a part—this is closer to algebra. It is needed in computer science. Some people have a knack for counting the ways a computer can send ten messages in parallel—and for finding the fastest way.

Typical question: ***Can*** 25 ***states be matched with*** 25 ***neighbors, so one state in each pair has an even number of letters***? New York can pair with New Jersey, Texas with Oklahoma, California with Arizona. We need rules for Hawaii and Alaska. This matching question doesn't sound mathematical, but it is.

Section 16.3 selects four topics from discrete mathematics, so you can decide if you want more.

Go back for a moment to calculus and differential equations. A completely realistic problem is seldom easy, but we can solve models. (Developing a good model is a skill in itself.) One method of solution involves complex numbers:

$$\begin{array}{llll} \text{any function} & u(x+iy) & \text{solves} & u_{xx}+u_{yy}=0 \quad \text{(Laplace equation)} \\ \text{any function} & e^{ik(x+ct)} & \text{solves} & u_{tt}-c^2u_{xx}=0 \quad \text{(wave equation).} \end{array}$$

From those building blocks we assemble solutions. For the wave equation, a signal starts at $t=0$. It is a combination of pure oscillations e^{ikx}. The coefficients in that combination make up the ***Fourier transform***—to tell how much of each frequency is in the signal. A lot of engineers and scientists would rather know those Fourier coefficients than $f(x)$.

A Fourier series breaks the signal into $\Sigma\, a_k \cos kx$ or $\Sigma\, b_k \sin kx$ or $\Sigma\, c_k e^{ikx}$. These sums can be infinite (like power series). Instead of values of $f(x)$, or derivatives at the basepoint, the function is described by a_k, b_k, c_k. Everything is computed by the "***Fast Fourier Transform***." This is the greatest algorithm since Newton's method.

A radio signal is near one frequency. A step function has many frequencies. A delta function has every frequency in the same amount: $\delta(x)=\Sigma \cos kx$. Channel 4 can't broadcast a perfect step function. You wouldn't want to hear a delta function.

We mentioned ***computing***. For nonlinear equations this means Newton's method. For $Ax=b$ it means elimination—***algorithms take the place of formulas***. Exact solutions are gone—speed and accuracy and stability become essential. It seems right to make scientific computing a part of applied mathematics, and teach the algorithms with the theory. My text *Introduction to Applied Mathematics* is one step in this direction, trying to present advanced calculus as it is actually used.

We cannot discuss applications and forget ***statistics***. Our society produces oceans of data—somebody has to draw conclusions. To decide if a new drug works, and if oil spills are common or rare, and how often to have a checkup, we can't just guess. I am astounded that the connection between smoking and health was hidden for centuries. It was in the data! Eventually the statisticians uncovered it. Professionals can find patterns, and the rest of us can understand (with a little mathematics) what has been found.

One purpose in studying mathematics is to know more about your own life. Calculus lights up a key idea: ***Functions***. Shapes and populations and heart signals and profits and growth rates, all are given by functions. They change in time. They have integrals and derivatives. To understand and use them is a challenge—mathematics takes effort. A lot of people have contributed, in whatever way they could—as you and I are doing. We may not be Newton or Leibniz or Gauss or Einstein, but we can share some part of what they created.