CHAPTER 9

Polar Coordinates and Complex Numbers

9.1 Polar Coordinates

Up to now, points have been located by their x and y coordinates. But if you were a flight controller, and a plane appeared on the screen, you would not give its position that way. Instead of x and y, you would read off the ***direction*** of the plane and its ***distance***. The direction is given by an angle θ. The distance is given by a positive number r. Those are the ***polar coordinates*** of the point, where x and y are the *rectangular coordinates*.

The angle θ is measured from the horizontal. Suppose the distance is 2 and the direction is $30°$ or $\pi/6$ (degrees preferred by flight controllers, radians by mathematicians). A pilot looking along the x axis would give the plane's direction as "11 o'clock." This totally destroys our system of units, by measuring direction in hours. But the angle and the distance locate the plane.

How far to a landing strip at $r = 1$ and $\theta = -\pi/2$? For that question polar coordinates are not good. They are perfect for distance from the origin (which equals r), but for most other distances I would switch to x and y. It is extremely simple to determine x and y from r and θ, and we will do it constantly. The most used formulas in this chapter come from Figure 9.1—where the right triangle has angle θ and hypotenuse r. *The sides of that triangle are x and y*:

$$x = r\cos\theta \quad \textbf{\textit{and}} \quad y = r\sin\theta. \tag{1}$$

The point at $r = 2, \theta = \pi/6$ has $x = 2\cos(\pi/6)$ and $y = 2\sin(\pi/6)$. The cosine of $\pi/6$ is $\sqrt{3}/2$ and the sine is $\frac{1}{2}$. So $x = \sqrt{3}$ and $y = 1$. Polar coordinates convert easily to xy coordinates—now we go the other way.

Always $x^2 + y^2 = r^2$. In this example $(\sqrt{3})^2 + (1)^2 = (2)^2$. Pythagoras produces r from x and y. The direction θ is also available, but the formula is not so beautiful:

$$r = \sqrt{x^2 + y^2} \quad \textbf{\textit{and}} \quad \tan\theta = \frac{y}{x} \quad \textbf{\textit{and}}\ (\textbf{\textit{almost}}) \quad \theta = \tan^{-1}\frac{y}{x}. \tag{2}$$

Our point has $y/x = 1/\sqrt{3}$. One angle with this tangent is $\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$.

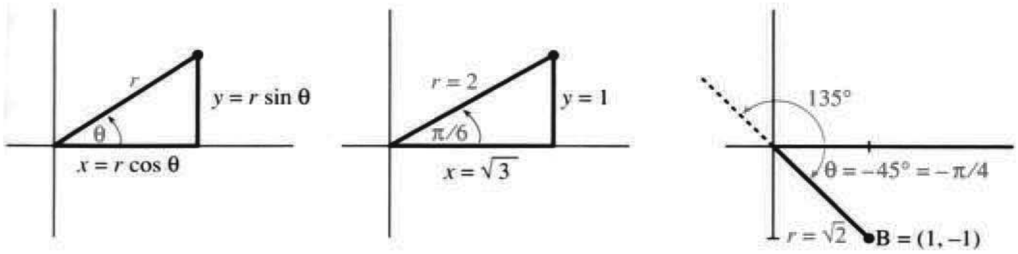


Fig. 9.1 Polar coordinates r, θ and rectangular coordinates $x = r\cos\theta$, $y = r\sin\theta$.

EXAMPLE 1 Point B in Figure 9.1c is at a *negative angle* $\theta = -\pi/4$. The x coordinate $r\cos(-\pi/4)$ is the same as $r\cos\pi/4$ (the cosine is even). But the y coordinate $r\sin(-\pi/4)$ is negative. Computing r and θ from $x = 1$ and $y = 1$, the distance is $r = \sqrt{1+1}$ and $\tan\theta$ is $-1/1$.

Warning To any angle θ we can add or subtract 2π—which goes a full 360° circle and keeps the same direction. Thus $-\pi/4$ or $-45°$ is the same angle as $7\pi/4$ or 315°. So is $15\pi/4$ or 675°.

If we add or subtract 180°, the tangent doesn't change. The point $(1,-1)$ is on the $-45°$ line at $r = \sqrt{2}$. The point $(-1,1)$ is on the 135° line also with $r = \sqrt{2}$. Both have $\tan\theta = -1$. We had to write "almost" in equation (2), because a point has many θ's and two points have the same r and $\tan\theta$.

Even worse, we could say that $B = (1,-1)$ is on the 135° line but at a ***negative distance*** $r = -\sqrt{2}$. A negative r carries the point *backward* along the 135° line, which is forward to B. In giving the position of B, I would always keep $r > 0$. But in drawing the graph of a polar equation, $r < 0$ is allowed. We move now to those graphs.

THE CIRCLE $r = \cos\theta$

The basis for Chapters 1–8 was $y = f(x)$. The key to this chapter is $r = F(\theta)$. That is a relation between the polar coordinates, and the points satisfying an equation like $r = \cos\theta$ produce a ***polar graph***.

It is not obvious why $r = \cos\theta$ gives a circle. The equations $r = \cos 2\theta$ and $r = \cos^2\theta$ and $r = 1 + \cos\theta$ produce entirely different graphs—not circles. The direct approach is to take $\theta = 0°, 30°, 60°, \ldots$ and go out the distance $r = \cos\theta$ on each ray. The points are marked in Figure 9.2a, and connected into a curve. It seems to be a circle of radius $\frac{1}{2}$, with its center at the point $(\frac{1}{2}, 0)$. We have to be able to show mathematically that $r = \cos\theta$ represents a ***shifted circle***.

One point must be mentioned. ***The angles from*** 0 ***to*** π ***give the whole circle***. The number $r = \cos\theta$ becomes negative after $\pi/2$, and we go backwards along each ray.

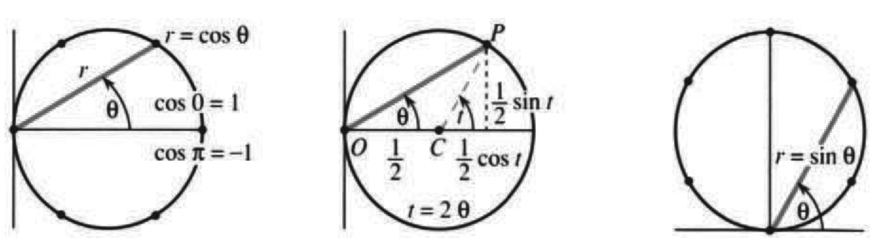


Fig. 9.2 The circle $r = \cos\theta$ and the switch to x and y. The circle $r = \sin\theta$.

At $\theta=\pi$ (to the *left* of the origin) the cosine is -1. Going backwards brings us to the same point as $\theta=0$ and $r=+1$—which completes the circle.

When θ continues from π to 2π we go around again. The polar equation gives the circle *twice*. (Or more times, when θ continues past 2π.) If you don't like negative r's and multiple circles, restrict θ to the range from $-\pi/2$ to $\pi/2$. We still have to see why the graph of $r=\cos\theta$ is a circle.

***Method* 5** Multiply by r and convert to rectangular coordinates x and y:

$$r=\cos\theta \quad\Rightarrow\quad r^2=r\cos\theta \quad\Rightarrow\quad x^2+y^2=x. \tag{3}$$

This is a circle because of x^2+y^2. From rewriting as $(x-\frac{1}{2})^2+y^2=(\frac{1}{2})^2$ we recognize its center and radius. Center at $x=\frac{1}{2}$ and $y=0$; radius $\frac{1}{2}$. Done.

***Method* 6** Write x and y *separately* as functions of θ. Then θ is a "*parameter*":

$$x=r\cos\theta=\cos^2\theta \qquad\text{and}\qquad y=r\sin\theta=\sin\theta\cos\theta. \tag{4}$$

These are not ***polar*** equations but ***parametric*** equations. The parameter θ is the angle, but it could be the time—the curve would be the same. Chapter 12 studies parametric equations in detail—here we stay with the circle.

To find the circle, square x and y and add. This produces $x^2+y^2=x$ in Problem 26. But here we do something new: ***Start with the circle and find equation*** (4). In case you don't reach Chapter 12, the idea is this. Add the vectors OC to the center and CP out the radius:

The point P in Figure 9.2 *has* $(x,y)=OC+CP=(\frac{1}{2},0)+(\frac{1}{2}\cos t,\frac{1}{2}\sin t)$.

The parameter t is the angle at the center of the circle. The equations are $x=\frac{1}{2}+\frac{1}{2}\cos t$ and $y=\frac{1}{2}\sin t$. A trigonometric person sees a double angle and sets $t=2\theta$. The result is equation (4) for the circle:

$$x=\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta=\cos^2\theta \qquad\text{and}\qquad y=\tfrac{1}{2}\sin 2\theta=\sin\theta\cos\theta. \tag{5}$$

This step rediscovers a basic theorem of geometry: *The angle t at the center is twice the angle θ at the circumference*. End of quick introduction to parameters.

A second circle is $r=\sin\theta$, drawn in Figure 9.2c. A third circle is $r=\cos\theta+\sin\theta$, not drawn. Problem 27 asks you to find its xy equation and its radius. All calculations go back to $x=r\cos\theta$ and $y=r\sin\theta$—the basic facts of polar coordinates! The last exercise shows a parametric equation with beautiful graphs, because it may be possible to draw them now. Then the next section concentrates on $r=F(\theta)$—and goes far beyond circles.

9.1 EXERCISES

Read-through questions

Polar coordinates r and θ correspond to $x=$ __a__ and $y=$ __b__. The points with $r>0$ and $\theta=\pi$ are located __c__. The points with $r=1$ and $0\leqslant\theta\leqslant\pi$ are located __d__. Reversing the sign of θ moves the point (x,y) to __e__.

Given x and y, the polar distance is $r=$ __f__. The tangent of θ is __g__. The point $(6,8)$ has $r=$ __h__ and $\theta=$ __i__. Another point with the same θ is __j__. Another point with the same r is __k__. Another point with the same r and $\tan\theta$ is __l__.

The polar equation $r=\cos\theta$ produces a shifted __m__. The top point is at $\theta=$ __n__, which gives $r=$ __o__. When θ goes from 0 to 2π, we go __p__ times around the graph. Rewriting as $r^2=r\cos\theta$ leads to the xy equation __q__. Substituting $r=\cos\theta$ into $x=r\cos\theta$ yields $x=$ __r__ and similarly $y=$ __s__. In this form x and y are functions of __t__ θ.

Find the polar coordinates $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$ of these points.

1 $(x,y) = (0,1)$

2 $(x,y) = (-4,0)$

3 $(x,y) = (\sqrt{2},\sqrt{2})$

4 $(x,y) = (-1,\sqrt{3})$

5 $(x,y) = (-1,-1)$

6 $(x,y) = (3,4)$

Find rectangular coordinates (x,y) from polar coordinates.

7 $(r,\theta) = (2,\pi/2)$

8 $(r,\theta) = (1,3\pi/2)$

9 $(r,\theta) = (\sqrt{20},\pi/4)$

10 $(r,\theta) = (3\pi,3\pi)$

11 $(r,\theta) = (2,-\pi/6)$

12 $(r,\theta) = (2,5\pi/6)$

13 What is the distance from $(x,y) = (\sqrt{3},1)$ to $(1,-\sqrt{3})$?

14 How far is the point $r = 3, \theta = \pi/2$ from $r = 4, \theta = \pi$?

15 How far is $(x,y) = (r\cos\theta, r\sin\theta)$ from $(X,Y) = (R\cos\phi, R\sin\phi)$? Simplify $(x-X)^2 + (y-Y)^2$ by using $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

16 Find a second set of polar coordinates (a different r or θ) for the points

$$(r,\theta) = (-1,\pi/2), \quad (-1,3\pi/4), \quad (1,-\pi/2), \quad (0,0).$$

17 Using polar coordinates describe (a) the half-plane $x > 0$; (b) the half-plane $y \leqslant 0$; (c) the ring with $x^2 + y^2$ between 4 and 5; (d) the wedge $x \geqslant |y|$.

18 True or false, with a reason or an example:

(a) Changing to $-r$ and $-\theta$ produces the same point.

(b) Each point has only one r and θ, when $r < 0$ is not allowed.

(c) The graph of $r = 1/\sin\theta$ is a straight line.

19 From x and θ find y and r.

20 Which other point has the same r and $\tan\theta$ as $x = \sqrt{3}$, $y = 1$ in Figure 9.1b ?

21 Convert from rectangular to polar equations:

(a) $y = x$ (b) $x + y = 1$ (c) $x^2 + y^2 = x + y$

22 Show that the triangle with vertices at $(0,0), (r_1,\theta_1)$, and (r_2,θ_2) has area $A = \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1)$. Find the base and height assuming $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$.

Problems 23–28 are about polar equations that give circles.

23 Convert $r = \sin\theta$ into an xy equation. Multiply first by r.

24 Graph $r = \sin\theta$ at $\theta = 0°, 30°, 60°, \ldots, 360°$. These thirteen values of θ give ______ different points on the graph. What range of θ's goes once around the circle ?

25 Substitute $r = \sin\theta$ into $x = r\cos\theta$ and $y = r\sin\theta$ to find x and y in terms of the parameter θ. Then compute $x^2 + y^2$ to reach the xy equation.

26 From the parametric equations $x = \cos^2\theta$ and $y = \sin\theta\cos\theta$ in (4), recover the xy equation. Square, add, eliminate θ.

27 (a) Multiply $r = \cos\theta + \sin\theta$ by r to convert into an xy equation. (b) Rewrite the equation as $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = R^2$ to find the radius R. (c) Draw the graph.

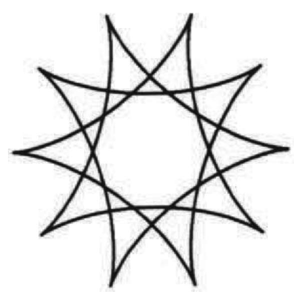
28 Find the radius of $r = a\cos\theta + b\sin\theta$. (Multiply by r.)

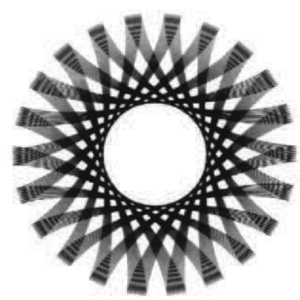
29 Convert $x + y = 1$ into an $r\theta$ equation and solve for r. Then substitute this r into $x = r\cos\theta$ and $y = r\sin\theta$ to find parametric equations for the line.

30 The equations $x = \cos^2\theta$ and $y = \sin^2\theta$ also lead to $x + y = 1$—but they are different from the answer to Problem 29. Explanation: θ is no longer the polar angle and we should have written t. Find a point $x = \cos^2\theta, y = \sin^2\theta$ that is *not* at the angle θ.

31 Convert $r = \cos^2\theta$ into an xy equation (of sixth degree!)

32 If you have a graphics package for parametric curves, graph some *hypocycloids*. The equations are $x = (1-b)\cos t + b\cos(1-b)t/b$, $y = (1-b)\sin t - b\sin(1-b)t/b$. The figure shows $b = \frac{3}{10}$ and part of $b = .31831$.





9.2 Polar Equations and Graphs

The most important equation in polar coordinates, by far, is $r=1$. The angle θ does not even appear. The equation looks too easy, but that is the point! The graph is a circle around the origin (the unit circle). Compare with the line $x=1$. More important, compare the simplicity of $r=1$ with the complexity of $y=\pm\sqrt{1-x^2}$. Circles are so common in applications that they created the need for polar coordinates.

This section studies polar curves $r=F(\theta)$. The cardioid is a sentimental favorite—maybe parabolas are more practical. The cardioid is $r=1+\cos\theta$, the parabola is $r=1/(1+\cos\theta)$. Section 12.2 adds cycloids and astroids. A graphics package can draw them and so can we.

Together with the circles $r=\text{constant}$ go the straight lines $\theta=\text{constant}$. The equation $\theta=\pi/4$ is a ray out from the origin, at that fixed angle. If we allow $r<0$, as we do in drawing graphs, the one-directional ray changes to a full line. Important: ***The circles are perpendicular to the rays***. We have "orthogonal coordinates"—more interesting than the $x-y$ grid of perpendicular lines. In principle x could be mixed with θ (non-orthogonal), but in practice that never happens.

Other curves are attractive in polar coordinates—we look first at five examples. Sometimes we switch back to $x=r\cos\theta$ and $y=r\sin\theta$, to recognize the graph.

EXAMPLE 1 The graph of $r=1/\cos\theta$ is the ***straight line*** $x=1$ (because $r\cos\theta=1$).

EXAMPLE 2 The graph of $r=\cos 2\theta$ is the ***four-petal flower*** in Figure 9.3.

The points at $\theta=30°$ and $-30°$ and $150°$ and $-150°$ are marked on the flower. They all have $r=\cos 2\theta=\frac{1}{2}$. ***There are three important symmetries—across the x axis, across the y axis, and through the origin***. This four-petal curve has them all. So does the vertical flower $r=\sin 2\theta$—but surprisingly, the tests it passes are different.

(*Across the x axis*: *y to $-y$*) There are two ways to cross. First, change θ to $-\theta$. The equation $r=\cos 2\theta$ stays the same. Second, change θ to $\pi-\theta$ and also r to $-r$. The equation $r=\sin 2\theta$ stays the same. Both flowers have x axis symmetry.

(*Across the y axis*: *x to $-x$*) There are two ways to cross. First, change θ to $\pi-\theta$. The equation $r=\cos 2\theta$ stays the same. Second, change θ to $-\theta$ and r to $-r$. Now $r=\sin 2\theta$ stays the same (the sine is odd). Both curves have y axis symmetry.

(*Through the origin*) Now we change r to $-r$ ***or*** θ to $\theta+\pi$. The flower equations pass the second test only: $\cos 2(\theta+\pi)=\cos 2\theta$ and $\sin 2(\theta+\pi)=\sin 2\theta$. Every equation $r^2=F(\theta)$ passes the first test, since $(-r)^2=r^2$.

The circle $r=\cos\theta$ has x axis symmetry, but not y or r. The spiral $r=\theta^3$ has y axis symmetry, because $-r=(-\theta)^3$ is the same equation.

Question What happens if you change r to $-r$ and also change θ to $\theta+\pi$?
Answer *Nothing*—because (r,θ) and $(-r,\theta+\pi)$ are always the same point.

EXAMPLE 3 The graph of $r=\theta$ is a ***spiral of Archimedes***—or maybe two spirals.

The spiral adds new points as θ increases past 2π. Our other examples are "periodic"—$\theta=2\pi$ gives the same point as $\theta=0$. A periodic curve repeats itself. The spiral moves out by 2π each time it comes around. If we allow negative angles and negative $r=\theta$, a second spiral appears.

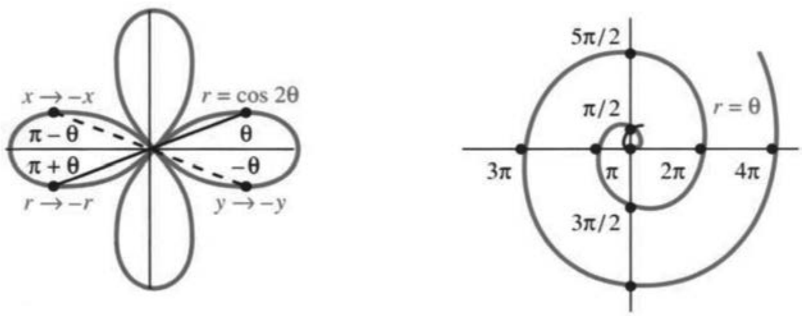


Fig. 9.3 The four-petal flower $r = \cos 2\theta$ and the spiral $r = \theta$ ($r > 0$ in red).

EXAMPLE 4 The graph of $r = 1 + \cos\theta$ is a ***cardioid***. It is drawn in Figure 9.4c.

The cardioid has no simple xy equation. Still the curve is very attractive. It has a cusp at the origin and it is heart-shaped (hence its name). To draw it, plot $r = 1 + \cos\theta$ at 30° intervals and connect the points. For this curve r is never negative, since $\cos\theta$ never goes below -1.

It is a curious fact that the electrical vector in your heart almost traces out a cardioid. See Section 11.1 about electrocardiograms. If it is a perfect cardioid you are in a little trouble.

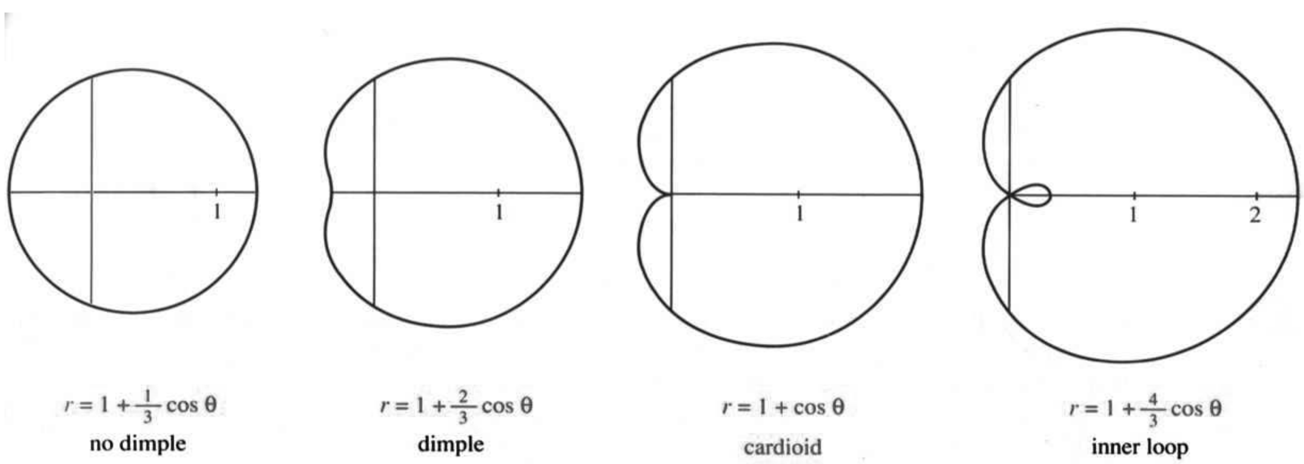


Fig. 9.4 Limaçons $r = 1 + b\cos\theta$, including a cardioid and Mars seen from Earth.

EXAMPLE 5 The graph of $r = 1 + b\cos\theta$ is a ***limaçon*** (a cardioid when $b = 1$).

Limaçon (soft c) is a French word for snail—not so well known as escargot but just as inedible. (*I am only referring to the shell. Excusez-moi*!) Figure 9.4 shows how a dimple appears as b increases. Then an inner loop appears beyond $b = 1$ (the cardioid at $b = 1$ is giving birth to a loop). For large b the curve looks more like two circles. The limiting case is a double circle, when the inner loop is the same as the outer loop. Remember that $r = \cos\theta$ goes around the circle twice.

We could magnify the limaçon by a factor c, changing to $r = c(1 + b\cos\theta)$. We could rotate 180° to $r = 1 - b\cos\theta$. But the real interest is whether these figures arise in applications, and Donald Saari showed me a nice example.

Mars seen from Earth The Earth goes around the Sun and so does Mars. Roughly speaking Mars is $1\frac{1}{2}$ times as far out, and completes its orbit in two Earth years.

We take the orbits as circles: $r = 2$ for Earth and $r = 3$ for Mars. Those equations tell *where* but not *when*. With time as a parameter, the coordinates of Earth and Mars are given at every instant t:

$$x_{\mathrm{E}} = 2\cos 2\pi t,\ y_{\mathrm{E}} = 2\sin 2\pi t \qquad \text{and} \qquad x_{\mathrm{M}} = 3\cos \pi t,\ y_{\mathrm{M}} = 3\sin \pi t.$$

At $t = 1$ year, the Earth completes a circle (angle $= 2\pi$) and Mars is halfway.

Now the key step. Subtract to find the position of Mars *relative to Earth*:

$$x_{\mathrm{M-E}} = 3\cos \pi t - 2\cos 2\pi t \qquad \text{and} \qquad y_{\mathrm{M-E}} = 3\sin \pi t - 2\sin 2\pi t.$$

Replacing $\cos 2\pi t$ by $2\cos^2 \pi t - 1$ and $\sin 2\pi t$ by $2\sin \pi t\cos \pi t$, this is

$$x_{\mathrm{M-E}} = (3 - 4\cos \pi t)\cos \pi t + 2 \qquad \text{and} \qquad y_{\mathrm{M-E}} = (3 - 4\cos \pi t)\sin \pi t.$$

Seen from the Earth, Mars does a loop in the sky! There are two t's for which $3 - 4\cos \pi t = 0$ (or $\cos \pi t = \frac{3}{4}$). At both times, Mars is two units from Earth ($x_{\mathrm{M-E}} = 2$ and $y_{\mathrm{M-E}} = 0$). When we move the origin to that point, the 2 is subtracted away—the $\mathrm{M-E}$ coordinates become $x = r\cos \pi t$ and $y = r\sin \pi t$ with $r = 3 - 4\cos \pi t$. That is a limaçon with a loop, like Figure 9.4d.

Note added in proof I didn't realize that a 3-to-2 ratio is also responsible for heating up two spots on opposite sides of Mercury. From the newspaper of June 13, 1990:

> "Astronomers today reported the first observations showing that Mercury has two extremely hot spots. That is because Mercury, the planet closest to the Sun, turns on its axis once every 59.6 days, which is a day on Mercury. It goes around the sun every 88 days, a Mercurian year. With this 3-to-2 ratio between spin and revolution, ***the Sun appears to stop in the sky and move backward, describing a loop*** over each of the hot spots."

CONIC SECTIONS IN POLAR COORDINATES

The exercises include other polar curves, like lemniscates and 200-petal flowers. But get serious. The most important curves are the ***ellipse*** and ***parabola*** and ***hyperbola***. In Section 3.5 their equations involved $1, x, y, x^2, xy, y^2$. With one focus at the origin, their polar equations are even better.

9A The graph of $r = A/(1 + e\cos\theta)$ is a conic section with "eccentricity" e:

circle if $e = 0$ ellipse if $0 < e < 1$ parabola if $e = 1$ hyperbola if $e > 1$.

EXAMPLE 6 ($e = 1$) The graph of $r = 1/(1 + \cos\theta)$ is a parabola. This equation is $r + r\cos\theta = 1$ or $r = 1 - x$. Squaring both sides gives $x^2 + y^2 = 1 - 2x + x^2$. Canceling x^2 leaves $y^2 = 1 - 2x$, the parabola in Figure 9.5b.

The amplifying factor A blows up all curves, with no change in shape.

EXAMPLE 7 ($e = 2$) The same steps lead from $r(1 + 2\cos\theta) = 1$ to $r = 1 - 2x$. Squaring gives $x^2 + y^2 = 1 - 4x + 4x^2$ and the x^2 terms do not cancel. Instead we have $y^2 - 3x^2 = 1 - 4x$. This is the hyperbola in Figure 9.5c, with a focus at $(0, 0)$.

The hyperbola $y^2 - 3x^2 = 1$ (without the $-4x$) has its *center* at $(0, 0)$.

EXAMPLE 8 ($e=\frac{1}{2}$) The same steps lead from $r(1+\frac{1}{2}\cos\theta)=1$ to $r=1-\frac{1}{2}x$. Squaring gives the ellipse $x^2+y^2=1-x+\frac{1}{4}x^2$. Polar equations look at conics in a new way, which happens to match the sun and planets perfectly. *The sun at* $(0,0)$ *is not the center of the system, but a focus.*

Finally $e=0$ gives the circle $r=1$. Center of circle = both foci = $(0,0)$.

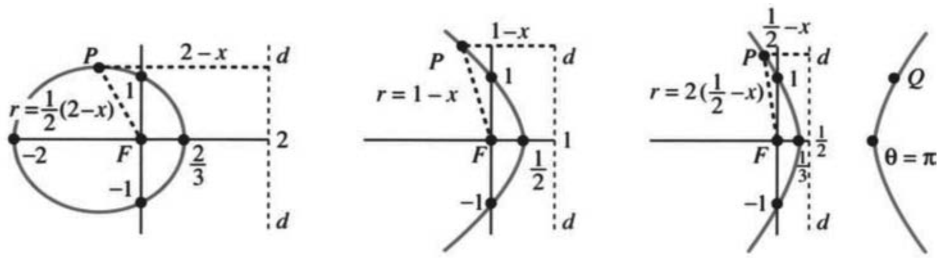


Fig. 9.5 $r=1/(1+e\cos\theta)$ is an ellipse for $e=\frac{1}{2}$, a parabola for $e=1$, a hyperbola for $e=2$.

The directrix The figure shows the line d (the "directrix") for each curve. All points P on the curve satisfy $r=|PF|=e|Pd|$. ***The distance to the focus is e times the distance to the directrix***. (e is still the eccentricity, nothing to do with exponentials.) A geometer would start from this property $r=e|Pd|$ and construct the curve. We derive the property from the equation:

$$r=\frac{A}{1+e\cos\theta} \quad\Rightarrow\quad r+ex=A \quad\Rightarrow\quad r=e\left(\frac{A}{e}-x\right). \qquad (1)$$

The directrix is the line at $x=A/e$. That last equation is exactly $|PF|=e|Pd|$.

Notice how two numbers determine these curves. Here the numbers are A and e. In Section 3.5 they were a and b. (The ellipse was $x^2/a^2+y^2/b^2=1$.) Using A and e we go smoothly from ellipses through parabolas (at $e=1$) and on to hyperbolas. With three more numbers we can move the focus to any point and rotate the curve through any angle. ***Conics are determined by five numbers***.

9.2 EXERCISES

Read-through questions

The circle of radius 3 around the origin has polar equation __a__. The 45° line has polar equation __b__. Those graphs meet at an angle of __c__. Multiplying $r=4\cos\theta$ by r yields the xy equation __d__. Its graph is a __e__ with center at __f__. The graph of $r=4/\cos\theta$ is the line $x=$ __g__. The equation $r^2=\cos 2\theta$ is not changed when $\theta\to-\theta$ (symmetric across __h__) and when $\theta\to\pi+\theta$ (or $r\to$ __i__). The graph of $r=1+\cos\theta$ is a __j__.

The graph of $r=A/($__k__$)$ is a conic section with one focus at __l__. It is an ellipse if __m__ and a hyperbola if __n__. The equation $r=1/(1+\cos\theta)$ leads to $r+x=1$ which gives a __o__. Then $r=$ distance from origin equals $1-x=$ distance from __p__. The equations $r=3(1-x)$ and $r=\frac{1}{3}(1-x)$ represent a __q__ and an __r__. Including a shift and rotation, conics are determined by __s__ numbers.

Convert to xy coordinates to draw and identify these curves.

1 $r\sin\theta=1$

2 $r(\cos\theta-\sin\theta)=2$

3 $r=2\cos\theta$

4 $r=-2\sin\theta$

5 $r=1/(2+\cos\theta)$

6 $r=1/(1+2\cos\theta)$

In 7–14 sketch the curve and check for x, y, and r symmetry.

7 $r^2=4\cos 2\theta$ (lemniscate)

8 $r^2=4\sin 2\theta$ (lemniscate)

9 $r=\cos 3\theta$ (three petals)

10 $r^2 = 10 + 6\cos 4\theta$

11 $r = e^\theta$ (logarithmic spiral)

12 $r = 1/\theta$ (hyperbolic spiral)

13 $r = \tan\theta$

14 $r = 1 - 2\sin 3\theta$ (rose inside rose)

15 Convert $r = 6\sin\theta + 8\cos\theta$ to the xy equation of a circle (what radius, what center?).

***16** Squaring and adding in the Mars-Earth equation gives $x^2_{\mathrm{M-E}} + y^2_{\mathrm{M-E}} = 13 - 12\cos\pi t$. The graph of $r^2 = 13 - 12\cos\theta$ is not at all like Figure 9.4d. What went wrong?

In 17–23 find the points where the two curves meet.

17 $r = 2\cos\theta$ and $r = 1 + \cos\theta$
Warning: You might set $2\cos\theta = 1 + \cos\theta$ to find $\cos\theta = 1$. But the graphs have another meeting point—they reach it at different θ's. Draw graphs to find all meeting points.

18 $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$

19 $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$

20 $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

21 $r = 2$ and $r = 4\sin 2\theta$

22 $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$

23 $r\sin\theta = 1$ and $r\cos(\theta - \pi/4) = \sqrt{2}$ (straight lines)

24 When is there a dimple in $r = 1 + b\cos\theta$? From $x = (1 + b\cos\theta)\cos\theta$ find $dx/d\theta$ and $d^2x/d\theta^2$ at $\theta = \pi$. When that second derivative is negative the limaçon has a dimple.

25 How many petals for $r = \cos 5\theta$? For $r = \cos\theta$ there was one, for $r = \cos 2\theta$ there were four.

26 Explain why $r = \cos 100\,\theta$ has 200 petals but $r = \cos 101\,\theta$ only has 101. The other 101 petals are ______. What about $r = \cos\frac{1}{2}\theta$?

27 Find an xy equation for the cardioid $r = 1 + \cos\theta$.

28 (a) The flower $r = \cos 2\theta$ is symmetric across the x and y axes. Does that make it symmetric about the origin? (Do two symmetries imply the third, so $-r = \cos 2\theta$ produces the same curve?)
(b) How can $r = 1$, $\theta = \pi/2$ lie on the curve but fail to satisfy the equation?

29 Find an xy equation for the flower $r = \cos 2\theta$.

30 Find equations for curves with these properties:
(a) Symmetric about the origin but not the x axis
(b) Symmetric across the 45° line but not symmetric in x or y or r
(c) Symmetric in x and y and r (like the flower) but changed when $x \leftrightarrow y$ (not symmetric across the 45° line).

Problems 31–37 are about conic sections—especially ellipses.

31 Find the top point of the ellipse in Figure 9.5a, by maximizing $y = r\sin\theta = \sin\theta/(1 + \frac{1}{2}\cos\theta)$.

32 (a) Show that all conics $r = 1/(1 + e\cos\theta)$ go through $x = 0, y = 1$.
(b) Find the second focus of the ellipse and hyperbola. For the parabola ($e = 1$) where is the second focus?

33 The point Q in Figure 9.5c has $y = 1$. By symmetry find x and then r (negative!). Check that $x^2 + y^2 = r^2$ and $|QF| = 2|Qd|$.

34 The equations $r = A/(1 + e\cos\theta)$ and $r = 1/(C + D\cos\theta)$ are the same if $C =$ ______ and $D =$ ______. For the mirror image across the y axis replace θ by ______. This gives $r = 1/(C - D\cos\theta)$ as in Figure 12.10 for a planet around the sun.

35 The ellipse $r = A/(1 + e\cos\theta)$ has length $2a$ on the x axis. Add r at $\theta = 0$ to r at $\theta = \pi$ to prove that $a = A/(1 - e^2)$. The Earth's orbit has $a = 92{,}600{,}000$ miles = one astronomical unit (AU).

36 The maximum height b occurs when $y = r\sin\theta = A\sin\theta/(1 + e\cos\theta)$ has $dy/d\theta = 0$. Show that $b = y_{\max} = A/\sqrt{1 - e^2}$.

37 Combine a and b from Problems $35 - 36$ to find $c = \sqrt{a^2 - b^2} = Ae/(1 - e^2)$. *Then the eccentricity e is c/a.* Halley's comet is an ellipse with $a = 18.1$ AU and $b = 4.6$ AU so $e =$ ______.

Comets have large eccentricity, **planets have much smaller** e: **Mercury** .21, **Venus** .01, **Earth** .02, **Mars** .09, **Jupiter** .05, **Saturn** .05, **Uranus** .05, **Neptune** .01, **Pluto** .25, **Kohoutek** .9999.

38 If you have a computer with software to do polar graphs, start with these:

1. Flowers $r = A + \cos n\theta$ for $n = \frac{1}{2}, 3, 7, 8$; $A = 0, 1, 2$

2. Petals $r = (\cos m\theta + 4\cos n\theta)/\cos\theta$, $(m, n) = (5, 3)$, $(3, 5)$, $(9, 1)$, $(2, 3)$

3. Logarithmic spiral $r = e^{\theta/2\pi}$

4. Nephroid $r = 1 + 2\sin\frac{1}{2}\theta$ from the bottom of a teacup

5. Dr. Fay's butterfly $r = e^{\cos\theta} - 2\cos 4\theta + \sin^5(\theta/12)$

Then create and name your own curve.

9.3 Slope, Length, and Area for Polar Curves

The previous sections introduced polar coordinates and polar equations and polar graphs. There was no calculus! We now tackle the problems of ***area*** (integral calculus) and ***slope*** (differential calculus), when the equation is $r = F(\theta)$. The use of F instead of f is a reminder that the slope is *not* $dF/d\theta$ and the area is *not* $\int F(\theta)d\theta$.

Start with area. The region is always divided into small pieces—what is their shape? Look between the angles θ and $\theta + \Delta\theta$ in Figure 9.6a. Inside the curve is a narrow wedge—almost a triangle, with $\Delta\theta$ as its small angle. If the radius is constant

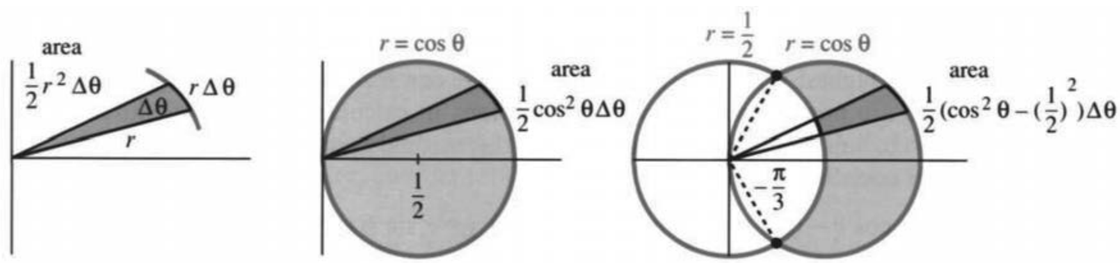


Fig. 9.6 Area of a wedge and a circle and an intersection of circles.

the wedge is a sector of a circle. It is a piece of pie cut at the extremely narrow angle $\Delta\theta$. The area of that piece is a fraction (the angle $\Delta\theta$ divided by the whole angle 2π) of the whole area πr^2 of the circle:

$$\textbf{\textit{area of wedge}} = \frac{\Delta\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\Delta\theta = \frac{1}{2}[F(\theta)]^2\,\Delta\theta. \tag{1}$$

We admit that the exact shape is not circular. The true radius $F(\theta)$ varies with θ—but in a narrow angle that variation is small. When we add up the wedges and let $\Delta\theta$ approach zero, the area becomes an integral.

9B The area inside the polar curve $r = F(\theta)$ is the limit of $\sum \frac{1}{2}r^2\Delta\theta = \sum \frac{1}{2}F^2\Delta\theta$:

$$\textit{area} = \int \frac{1}{2}r^2\,d\theta = \int \frac{1}{2}[F(\theta)]^2\,d\theta. \tag{2}$$

EXAMPLE 1 Find the area inside the circle $r = \cos\theta$ of radius $\frac{1}{2}$ (Figure 9.6).

$$\textit{area} = \int_0^{2\pi} \frac{1}{2}\cos^2\theta\,d\theta = \left.\frac{\cos\theta\sin\theta + \theta}{4}\right]_0^{2\pi} = \frac{2\pi}{4}.$$

That is wrong! The correct area of a circle of radius $\frac{1}{2}$ is $\pi/4$. The mistake is that we went *twice* around the circle as θ increased to 2π. Integrating from θ to π gives $\pi/4$.

EXAMPLE 2 Find the area between the circles $r = \cos\theta$ and $r = \frac{1}{2}$.

The circles cross at the points where $r = \cos\theta$ agrees with $r = \frac{1}{2}$. Figure 9.6 shows these points at $\pm 60°$, or $\theta = \pm\pi/3$. Those are the limits of integration, where $\cos\theta = \frac{1}{2}$.

The integral adds up the difference between two wedges,one out to $r = \cos\theta$ and a smaller one with $r = \frac{1}{2}$:

$$area = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left[(\cos\theta)^2 - \left(\frac{1}{2}\right)^2\right] d\theta. \tag{3}$$

Note that chopped wedges have area $\frac{1}{2}(F_1^2 - F_2^2)\Delta\theta$ ***and not*** $\frac{1}{2}(F_1 - F_2)^2\Delta\theta$.

EXAMPLE 3 Find the area between the cardioid $r = 1 + \cos\theta$ and the circle $r = 1$.

$$area = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - 1^2\right] d\theta \quad \left(\text{limits } \theta = \pm\frac{\pi}{2} \text{ where } 1 + \cos\theta = 1\right)$$

SLOPE OF A POLAR CURVE

Where is the highest point on the cardioid $r = 1 + \cos\theta$? What is the slope at $\theta = \pi/4$? Those are not the most important questions in calculus, but still we should know how to answer them. I will describe the method quickly, by switching to rectangular coordinates:

$$x = r\cos\theta = (1+\cos\theta)\cos\theta \qquad \text{and} \qquad y = r\sin\theta = (1+\cos\theta)\sin\theta.$$

For the highest point, maximize y by setting its derivative to zero:

$$dy/d\theta = (1+\cos\theta)(\cos\theta) + (-\sin\theta)(\sin\theta) = 0. \tag{3}$$

Thus $\cos\theta + \cos 2\theta = 0$, which happens at 60°. The height is $y = (1 + \frac{1}{2})(\sqrt{3}/2)$.

For the slope, use the chain rule $dy/d\theta = (dy/dx)(dx/d\theta)$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1+\cos\theta)(\cos\theta) + (-\sin\theta)(\sin\theta)}{(1+\cos\theta)(-\sin\theta) + (-\sin\theta)\cos\theta}. \tag{4}$$

Equations (3) and (4) avoid the awkward (or impossible) step of eliminating θ. Instead of trying to find y as a function of x, *we keep x and y as functions of* θ. At $\theta = \pi/4$, the ratio in equation (4) yields $dy/dx = -1/(1+\sqrt{2})$.

Problem 18 finds a general formula for the slope, using $dy/dx = (dy/d\theta)/(dx/d\theta)$. Problem 20 finds a more elegant formula, by looking at the question differently.

LENGTH OF A POLAR CURVE

The length integral always starts with $ds = \sqrt{(dx)^2 + (dy)^2}$. A polar curve has $x = r\cos\theta = F(\theta)\cos\theta$ and $y = F(\theta)\sin\theta$. Now take derivatives by the product rule:

$$dx = (F'(\theta)\cos\theta - F(\theta)\sin\theta)d\theta \qquad \text{and} \qquad dy = (F'(\theta)\sin\theta + F(\theta)\cos\theta)d\theta.$$

Squaring and adding (note $\cos^2\theta + \sin^2\theta$) gives the element of length ds:

$$ds = \sqrt{[F'(\theta)]^2 + [F(\theta)]^2}\,d\theta. \qquad (5)$$

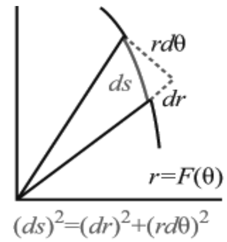


Fig. 9.7

The figure shows $(ds)^2 = (dr)^2 + (rd\theta)^2$, the same formula with different letters. The total arc length is $\int ds$.

The area of a surface of revolution is $\int 2\pi y\,ds$ (around the x axis) or $\int 2\pi x\,ds$ (around the y axis). ***Write x, y, and ds in terms of θ and $d\theta$.*** Then integrate.

EXAMPLE 4 The circle $r = \cos\theta$ has $ds = \sqrt{1}\,d\theta$. So its length is π (not 2π!!—don't go around twice). Revolved around the y axis the circle yields a doughnut with no hole. Since $x = r\cos\theta = \cos^2\theta$, the surface area of the doughnut is

$$\int 2\pi x\,ds = \int_0^\pi 2\pi\cos^2\theta\,d\theta = \pi^2.$$

EXAMPLE 5 The length of $r = 1+\cos\theta$ is, by symmetry, double the integral from 0 to π:

$$\text{length of cardioid} = 2\int_0^\pi \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\,d\theta$$
$$= 2\int_0^\pi \sqrt{2+2\cos\theta}\,d\theta = 4\int_0^\pi \cos\frac{\theta}{2}\,d\theta = 8.$$

We substituted $4\cos^2\frac{1}{2}\theta$ for $2+2\cos\theta$ in the square root. It is possible to skip symmetry and integrate from 0 to 2π—but that needs the absolute value $|\cos\frac{1}{2}\theta|$ to maintain a positive square root.

EXAMPLE 6 The logarithmic spiral $r = e^{-\theta}$ has $ds = \sqrt{e^{-2\theta} + e^{-2\theta}}\,d\theta$. It spirals to zero as θ goes to infinity, and the total length is finite:

$$\int ds = \int_0^\infty \sqrt{2}\,e^{-\theta}\,d\theta = -\sqrt{2}\,e^{-\theta}\Big]_0^\infty = \sqrt{2}.$$

Revolve this spiral for a mathematical seashell with area $\int_0^\infty (2\pi e^{-\theta}\cos\theta)\sqrt{2}\,e^{-\theta}\,d\theta$.

9.3 EXERCISES

Read-through questions

A circular wedge with angle $\Delta\theta$ is a fraction __a__ of a whole circle. If the radius is r, the wedge area is __b__. Then the area inside $r = F(\theta)$ is $\int$ __c__. The area inside $r = \theta^2$ from 0 to π is __d__. That spiral meets the circle $r = 1$ at $\theta =$ __e__. The area inside the circle and outside the spiral is __f__. A chopped wedge of angle $\Delta\theta$ between r_1 and r_2 has area __g__.

The curve $r = F(\theta)$ has $x = r\cos\theta =$ __h__ and $y =$ __i__. The slope dy/dx is $dy/d\theta$ divided by __j__. For length $(ds)^2 = (dx)^2 + (dy)^2 =$ __k__. The length of the spiral $r = \theta$ to $\theta = \pi$ is $\int$ __l__ (not to compute integrals). The surface area when $r = \theta$ is revolved around the x axis is $\int 2\pi y\,ds = \int$ __m__. The volume of that solid is $\int \pi y^2\,dx = \int$ __n__.

In 1–6 draw the curve and find the area inside.

1 $r = 1+\cos\theta$

2 $r = \sin\theta + \cos\theta$ from 0 to π

3 $r = 2+\cos\theta$

4 $r = 1+2\cos\theta$ (inner loop only)

5 $r = \cos 2\theta$ (one petal only)

6 $r = \cos 3\theta$ (one petal only)

Find the area between the curves in 7–12 after locating their intersections (draw them first).

7 circle $r = \cos\theta$ and circle $r = \sin\theta$

8 spiral $r = \theta$ and y axis (first arch)

9 outside cardioid $r = 1 + \cos\theta$ inside circle $r = 3\cos\theta$

10 lemniscate $r^2 = 4\cos 2\theta$ outside $r = \sqrt{2}$

11 circle $r = 8\cos\theta$ beyond line $r\cos\theta = 4$

12 circle $r = 10$ beyond line $r\cos\theta = 6$

13 Locate the mistake and find the correct area of the lemniscate $r^2 = \cos 2\theta$: area $= \int_0^\pi \frac{1}{2} r^2\, d\theta = \int_0^\pi \frac{1}{2}\cos 2\theta\, d\theta = 0$.

14 Find the area between the two circles in Example 2.

15 Compute the area between the cardioid and circle in Example 3.

16 Find the complete area (carefully) between the spiral $r = e^{-\theta}(\theta \geqslant 0)$ and the origin.

17 At what θ's does the cardioid $r = 1 + \cos\theta$ have infinite slope? Which points are furthest to the left (minimum x)?

18 Apply the chain rule $dy/dx = (dy/d\theta)/(dx/d\theta)$ to $x = F(\theta)\cos\theta$, $y = F(\theta)\sin\theta$. Simplify to reach

$$\frac{dy}{dx} = \frac{F + \tan\theta\, dF/d\theta}{-F\tan\theta + dF/d\theta}.$$

19 The groove in a record is nearly a spiral $r = c\theta$:

$$\text{length } = \int\sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_6^{14}\sqrt{r^2 + c^2}\, dr/c.$$

Take $c = .002$ to give 636 turns between the outer radius 14 cm and the inner radius 6 cm ($14 - 6$ equals $.002(636)2\pi$).

(a) Omit c^2 and just integrate $r\, dr/c$.

(b) Compute the length integral. Tables and calculators allowed. You will never trust integrals again.

20 Show that the angle ψ between the ray from the origin and the tangent line has $\tan\psi = F/(dF/d\theta)$.
Hint: If the tangent line is at an angle ϕ with the horizontal, then $\tan\phi$ is the slope dy/dx in Problem 18. Therefore

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\, \tan\theta}.$$

Substitute for $\tan\phi$ and simplify like mad.

21 The circle $r = F(\theta) = 4\sin\theta$ has $\psi = \theta$. Draw a figure including θ, ϕ, ψ and check $\tan\psi$.

22 Draw the cardioid $r = 1 - \cos\theta$, noticing the minus sign. Include the angles θ, ϕ, ψ and show that $\psi = \theta/2$.

23 The first limaçon in Figure 9.4 looks like a circle centered at $(\frac{1}{3}, 0)$. Prove that it isn't.

24 Find the equation of the tangent line to the circle $r = \cos\theta$ at $\theta = \pi/6$.

In 25–28 compute the length of the curve.

25 $r = \theta$ (θ from 0 to 2π)

26 $r = \sec\theta$ (θ from 0 to $\pi/4$)

27 $r = \sin^3(\theta/3)$ (θ from 0 to 3π)

28 $r = \theta^2$ (θ from 0 to π)

29 The narrow wedge in Figure 9.6 is almost a triangle. It was treated as a circular sector but triangles are more familiar. Why is the area approximately $\frac{1}{2} r^2 \Delta\theta$?

30 In Example 4 revolve the circle around the x axis and find the surface area. *We really only revolve a semicircle*.

31 Compute the seashell area $2\pi\sqrt{2}\int_0^\infty e^{-2\theta}\cos\theta\, d\theta$ using two integrations by parts.

32 Find the surface area when the cardioid $r = 1 + \cos\theta$ is revolved around the x axis.

33 Find the surface area when the lemniscate $r^2 = \cos 2\theta$ is revolved around the x axis. What is θ after one petal?

34 When $y = f(x)$ is revolved around the x axis, the volume is $\int \pi y^2 dx$. When the circle $r = \cos\theta$ is revolved, switch to a θ-integral from 0 to $\pi/2$ and check the volume of a sphere.

35 Find the volume when the cardioid $r = 1 + \cos\theta$ is rotated around the x axis.

36 Find the surface area and volume when the graph of $r = 1/\cos\theta$ is rotated around the y axis ($0 \leqslant \theta \leqslant \pi/4$).

37 Show that the spirals $r = \theta$ and $r = 1/\theta$ are perpendicular when they meet at $\theta = 1$.

38 Draw three circles of radius 1 that touch each other and find the area of the curved triangle between them.

39 Draw the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. In polar coordinates its right side is $r =$ ______. Find the area from $\int \frac{1}{2} r^2 d\theta$.

40 (Unravel the paradox) The area of the ellipse $x = 4\cos\theta$, $y = 3\sin\theta$ is $\pi \cdot 4 \cdot 3 = 12\pi$. But the integral of $\frac{1}{2} r^2 d\theta$ is

$$\int_0^{2\pi} \frac{1}{2}(16\cos^2\theta + 9\sin^2\theta)\, d\theta = 12\frac{1}{2}\pi.$$

9.4 Complex Numbers

Real numbers are sufficient for most of calculus. Starting from x^2+4, its integral $\frac{1}{3}x^3+4x+C$ is also real. If we are given x^3-1, its derivative $3x^2$ is real. *But the* ***roots*** (*or* ***zeros***) *of those polynomials are complex numbers*:

$$x^2+4=0 \quad \textbf{\textit{and}} \quad x^3-1=0 \quad \textbf{\textit{have complex solutions}}.$$

We expect two square roots of -4. There are three cube roots of 1. Complex numbers are unavoidable, in order to find n roots for each polynomial of degree n.

This section explains how to work with complex numbers. You will see their relation to polar coordinates. At the end, we use them to solve differential equations.

Start with the imaginary number i. Everybody knows that $x^2=-1$ has no real solution. When you square a real number, the result is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition, subtraction, multiplication, and division, with one difference: ***Whenever*** i^2 ***appears it is replaced by*** -1. In particular $-i$ times $-i$ gives $+i^2=-1$. In other words, $-i$ is also a square root of -1. There are two solutions (i and $-i$) to the equation $x^2+1=0$.

Finding cube roots of 1 will stretch us further. We need complex numbers—real plus imaginary.

9B A ***complex number*** (say $1+3i$) is the sum of a real number (1) and a pure imaginary number ($3i$). Addition keeps those parts separate; multiplication uses $i^2=-1$:

$$\text{Addition}: \quad (1+3i)+(1+3i)=1+1+i(3+3)=2+6i$$
$$\text{Multiplication}: \quad (1+3i)(1+3i)=1+3i+3i+9i^2=-8+6i.$$

Adding $1+3i$ to $5-i$ is easy $(6+2i)$. Multiplying is longer, but you see the rules:

$$(1+3i)(5-i)=5+15i-i-3i^2=8+14i.$$

The point is this: We don't have to imagine any more new numbers. After accepting i, the rest is straightforward. A real number is just a complex number with no imaginary part! When $1+3i$ combines with its "***complex conjugate***" $1-3i$—adding or multiplying—the answer is real:

$$\begin{aligned}(1+3i)+(1-3i)&=2 \quad \textbf{\textit{(real)}}\\ (1+3i)(1-3i)&=1-3i+3i-9i^2=10. \quad \textbf{\textit{(real)}}\end{aligned} \tag{1}$$

The complex conjugate offers a way to do division, by making the denominator real:

$$\frac{1}{1+3i}=\frac{1}{1+3i}\frac{1-3i}{1-3i}=\frac{1-3i}{10} \qquad \text{and} \qquad \frac{1}{x+iy}=\frac{1}{x+iy}\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}.$$

9C The complex number $x+iy$ has real part x and imaginary part y. Its complex conjugate is $x-iy$. The product $(x+iy)(x-iy)$ equals $x^2+y^2=r^2$. The ***absolute value*** (or modulus) is $r=|x+iy|=\sqrt{x^2+y^2}$.

THE COMPLEX PLANE

Complex numbers correspond to points in a plane. The number $1+3i$ corresponds to the point $(1, 3)$. Similarly $x+iy$ is paired with (x, y)—which is x units along the "real axis" and y units up the "imaginary axis." The ordinary plane turns into the ***complex plane***. The absolute value r is the same as the polar coordinate r (Figure 9.8a).

The figure shows two more copies of the complex plane. The one in the middle is for addition and subtraction. It uses rectangular coordinates. The one on the right is for multiplication and division and squaring. It uses polar coordinates. In squaring a complex number, r is squared and θ is doubled—as the right figure and equation (3) both show.

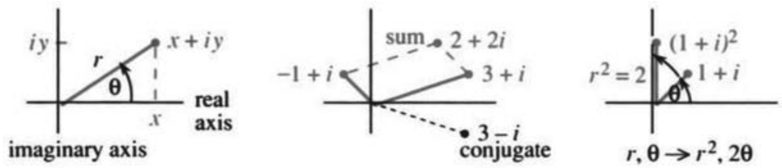


Fig. 9.8 The complex plane shows x, y, r, θ. Add with x and y, multiply with r and θ.

Adding complex numbers is like adding vectors (Chapter 11). The real parts give $3-1$ and the imaginary parts give $1+1$. The vector sum $(2, 2)$ corresponds to the complex sum $2+2i$. The complex conjugate $3-i$ is the mirror image across the real axis (i reversed to $-i$). The connection to r and θ is the same as before (you see it in the triangle):

$$x = r\cos\theta \quad \textbf{\textit{and}} \quad y = r\sin\theta \quad \textbf{\textit{so that}} \quad x+iy = r(\cos\theta + i\sin\theta). \tag{2}$$

In the third figure, $1+i$ has $r=\sqrt{2}$ and $\theta = \pi/4$. The polar form is $\sqrt{2}\cos\pi/4 + \sqrt{2}i\sin\pi/4$. When this number is squared, its 45° angle becomes 90°. The square is $(1+i)^2 = 1+2i-1 = 2i$. Its polar form is $2\cos\pi/2 + 2i\sin\pi/2$.

> **9D** Multiplication adds angles, division subtracts angles, and squaring doubles angles. The absolute values are multiplied, divided, and squared:
>
> $$(r\cos\theta + ir\sin\theta)^2 = r^2\cos 2\theta + ir^2\sin 2\theta. \tag{3}$$

For nth powers we reach r^n and $n\theta$. For square roots, r goes to $\sqrt{r}$ and θ goes to $\frac{1}{2}\theta$. The number -1 is at 180°, so its square root i is at 90°.

To see why θ is doubled in equation (3), factor out r^2 and multiply as usual:

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

The right side is $\cos 2\theta + i\sin 2\theta$. The double-angle formulas from trigonometry match the squaring of complex numbers. The cube would be $\cos 3\theta + i\sin 3\theta$ (because 2θ and θ add to 3θ, and r is still 1). The nth power is in ***de Moivre's formula***:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{4}$$

With $n=-1$ we get $\cos(-\theta)+i\,\sin(-\theta)$—which is $\cos\theta - i\,\sin\theta$, the complex conjugate:

$$\frac{1}{\cos\theta+i\,\sin\theta}=\frac{1}{\cos\theta+i\,\sin\theta}\,\frac{\cos\theta-i\,\sin\theta}{\cos\theta-i\,\sin\theta}=\frac{\cos\theta-i\,\sin\theta}{1}. \qquad (5)$$

We are almost touching ***Euler's formula***, the key to all numbers on the unit circle:

$$\textit{Euler's formula}: \qquad \cos\theta+i\,\sin\theta=e^{i\theta}. \qquad (6)$$

Squaring both sides gives $(e^{i\theta})(e^{i\theta})=e^{2i\theta}$. That is equation (3). The -1 power is $1/e^{i\theta}=e^{-i\theta}$. That is equation (5). Multiplying any $e^{i\theta}$ by $e^{i\phi}$ produces $e^{i(\theta+\phi)}$. The special case $\phi=\theta$ gives the square, and the special case $\phi=-\theta$ gives $e^{i\theta}e^{-i\theta}=1$.

Euler's formula appeared in Section 6.7, by changing x to $i\theta$ in the series for e^x:

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots \quad \text{becomes} \quad e^{i\theta}=1+i\theta-\frac{\theta^2}{2}-i\frac{\theta^3}{6}+\cdots.$$

A highlight of Chapter 10 is to recognize two new series on the right. The real terms $1-\frac{1}{2}\theta^2+\cdots$ add up to $\cos\theta$. The imaginary part $\theta-\frac{1}{6}\theta^3+\cdots$ adds up to $\sin\theta$. Therefore $e^{i\theta}$ equals $\cos\theta+i\,\sin\theta$. It is fantastic that the most important periodic functions in all of mathematics come together in this graceful way.

We learn from Euler (pronounced *oiler*) that $e^{2\pi i}=1$. The cosine of 2π is 1, the sine is zero. If you substitute $x=2\pi i$ into the infinite series, somehow everything cancels except the 1—this is almost a miracle. From the viewpoint of angles, $\theta=2\pi$ carries us around a full circle and back to $e^{2\pi i}=1$.

Multiplying Euler's formula by r, we have a third way to write a complex number:

$$\textit{Every complex number is} \qquad x+iy=r\,\cos\theta+i\,r\,\sin\theta=re^{i\theta}. \qquad (7)$$

EXAMPLE 1 $2e^{i\theta}$ times $3e^{i\theta}$ equals $6e^{2i\theta}$. For $\theta=\pi/2$, $2i$ times $3i$ is -6.

EXAMPLE 2 Find w^2 and w^4 and w^8 and w^{25} when $w=e^{i\pi/4}$.

Solution $e^{i\pi/4}$ is $1/\sqrt{2}+i/\sqrt{2}$. Note that $r^2=\frac{1}{2}+\frac{1}{2}=1$. Now watch angles:

$$w^2=e^{i\pi/2}=i \quad w^4=e^{i\pi}=-1 \quad w^8=1 \quad w^{25}=w^8w^8w^8w=w.$$

Figure 9.9 shows the eight powers of w. ***They are the eighth roots of*** 1.

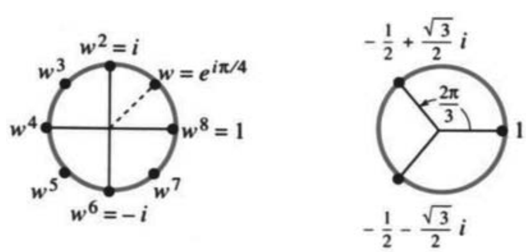


Fig. 9.9 The eight powers of w and the cube roots of 1.

EXAMPLE 3 $(x^2+4=0)$ The square roots of -4 are $2i$ and $-2i$. Instead of $(i)(i)=-1$ we have $(2i)(2i)=-4$. If Euler insists, we write $2i$ and $-2i$ as $2e^{i\pi/2}$ and $2e^{i3\pi/2}$.

EXAMPLE 4 (The cube roots of 1) In rectangular coordinates we have to solve $(x+iy)^3=1$, which is not easy. In polar coordinates this same equation is $r^3e^{3i\theta}=1$. Immediately $r=1$. The angle θ can be $2\pi/3$ or $4\pi/3$ or $6\pi/3$—***the cube roots in the figure are evenly spaced***:

$$(e^{2\pi i/3})^3=e^{2\pi i}=1 \quad (e^{4\pi i/3})^3=e^{4\pi i}=1 \quad (e^{6\pi i/3})^3=e^{6\pi i}=1.$$

You see why the angle $8\pi/3$ gives nothing new. It completes a full circle back to $2\pi/3$.

The nth ***roots of*** 1 ***are*** $e^{2\pi i/n}$, $e^{4\pi i/n}$, ..., 1. ***There are*** n ***of them. They lie at angles*** $2\pi/n$, $4\pi/n$, ..., 2π ***around the unit circle.***

SOLUTION OF DIFFERENTIAL EQUATIONS

The algebra of complex numbers is now applied to the calculus of complex functions. The complex number is c, the complex function is e^{ct}. It will solve the equations $y''=-4y$ and $y'''=y$, by connecting them to $c^2=-4$ and $c^3=1$. Chapter 16 does the same for all linear differential equations with constant coefficients—this is an optional preview.

Please memorize the one key idea: ***Substitute*** $y=e^{ct}$ ***into the differential equation and solve for*** c. Each derivative brings a factor c, so $y'=ce^{ct}$ and $y''=c^2e^{ct}$:

$$d^2y/dt^2=-4y \text{ leads to } c^2e^{ct}=-4e^{ct}, \text{ which gives } c^2=-4. \tag{8}$$

For this differential equation, c must be a square root of -4. We know the candidates ($c=2i$ and $c=-2i$). The equation has two "pure exponential solutions" e^{ct}:

$$y=e^{2it} \qquad \text{and} \qquad y=e^{-2it}. \tag{9}$$

Their combinations $y=Ae^{2it}+Be^{-2it}$ give all solutions. In Chapter 16 we will choose the two numbers A and B to match two initial conditions at $t=0$.

The solution $y=e^{2it}=\cos 2t+i\,\sin 2t$ is complex. The differential equation is real. For real y's, ***take the real and imaginary parts of the complex solutions***:

$$y_{\text{real}}=\cos 2t \text{ and } y_{\text{imaginary}}=\sin 2t. \tag{10}$$

These are the "pure oscillatory solutions." When $y=e^{2it}$ travels around the unit circle, its imaginary part $\sin 2t$ moves up and down. (It is like the ball and its shadow in Section 1.4, but twice as fast because of $2t$.) The real part $\cos 2t$ goes backward and forward. By the chain rule, *the second derivative of* $\cos 2t$ *is* $-4\cos 2t$. Thus $d^2y/dt^2=-4y$ and we have real solutions.

EXAMPLE 5 Find three solutions and then three *real* solutions to $d^3y/dt^3=y$.

Key step: ***Substitute*** $y=e^{ct}$. The result is $c^3e^{ct}=e^{ct}$. Thus $c^3=1$ and c is a cube root of 1. The candidate $c=1$ gives $y=e^t$ (our first solution). The next c is complex:

$$c=e^{2\pi i/3}=-\frac{1}{2}+i\,\frac{\sqrt{3}}{2} \quad \textbf{\textit{yields}} \quad y=e^{ct}=e^{-t/2}e^{i\sqrt{3}t/2}. \tag{11}$$

The real part of the exponent leads to the absolute value $|y|=e^{-t/2}$. It decreases as t gets larger, so y moves toward zero. At the same time, the factor $e^{i\sqrt{3}t/2}$ goes around the unit circle. Therefore y spirals in to zero (Figure 9.10). So does its complex conjugate, which is the third exponential. Changing i to $-i$ in (11) gives the third cube root of 1 and the third solution $e^{-t/2}e^{-i\sqrt{3}t/2}$.

The first real solution is $y = e^t$. The others are the two parts of the spiral:

$$y_{\text{real}} = e^{-t/2}\cos\sqrt{3}t/2 \qquad \text{and} \qquad y_{\text{imaginary}} = e^{-t/2}\sin\sqrt{3}t/2. \qquad (12)$$

That is $r\cos\theta$ and $r\sin\theta$. It is the ultimate use (until Chapter 16) of polar coordinates and complex numbers. We might have discovered $\cos 2t$ and $\sin 2t$ without help, for $y'' = -4y$. I don't think these solutions to $y''' = y$ would have been found.

EXAMPLE 6 Find four solutions to $d^4y/dt^4 = y$ by substituting $y = e^{ct}$.

Four derivatives lead to $c^4 = 1$. Therefore c is i or -1 or $-i$ or 1. The solutions are $y = e^{it}$, e^{-t}, e^{-it}, and e^t. If we want real solutions, e^{it} and e^{-it} combine into $\cos t$ and $\sin t$. In all cases $y'''' = y$.

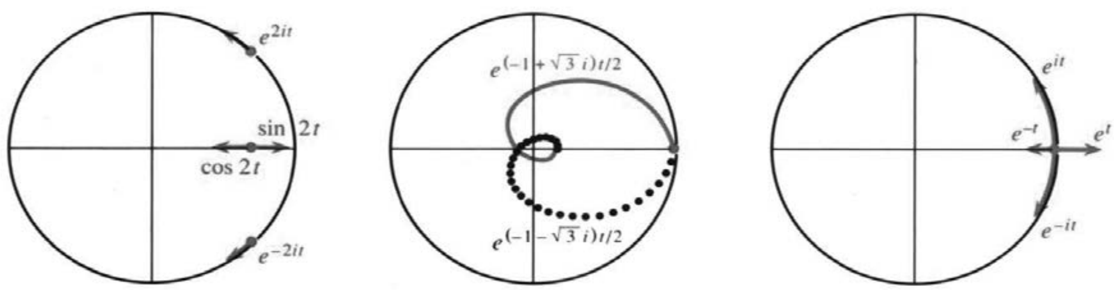


Fig. 9.10 Solutions move in the complex plane: $y'' = -4y$ and $y''' = y$ and $y'''' = y$.

9.4 EXERCISES

Read-through questions

The complex number $3+4i$ has real part __a__ and imaginary part __b__. Its absolute value is $r =$ __c__ and its complex conjugate is __d__. Its position in the complex plane is at (__e__). Its polar form is $r\cos\theta + ir\sin\theta =$ __f__ $e^{i\theta}$. Its square is __g__ $+ i$ __h__. Its nth power is __i__ $e^{in\theta}$.

The sum of $1+i$ and $1-i$ is __j__. The product of $1+i$ and $1-i$ is __k__. In polar form this is $\sqrt{2}e^{i\pi/4}$ times __l__. The quotient $(1+i)/(1-i)$ equals the imaginary number __m__. The number $(1+i)^8$ equals __n__. An eighth root of 1 is $w =$ __o__. The other eighth roots are __p__.

To solve $d^8y/dt^8 = y$, look for a solution of the form $y =$ __q__. Substituting and canceling e^{ct} leads to the equation __r__. There are __s__ choices for c, one of which is $(-1+i)/\sqrt{2}$. With that choice $|e^{ct}| =$ __t__. The real solutions are Re $e^{ct} =$ __u__ and Im $e^{ct} =$ __v__.

In 1–6 plot each number in the complex plane.

1 $2+i$ and its complex conjugate $2-i$ and their sum and product

2 $1+i$ and its square $(1+i)^2$ and its reciprocal $1/(1+i)$

3 $2e^{i\pi/6}$ and its reciprocal $\frac{1}{2}e^{-i\pi/6}$ and their squares

4 The sixth roots of 1 (six of them)

5 $\cos 3\pi/4 + i\sin 3\pi/4$ and its square and cube

6 $4e^{i\pi/3}$ and its square roots

7 For complex numbers $c = x+iy = re^{i\theta}$ and their conjugates $\bar{c} = x - iy = re^{-i\theta}$, find all possible locations in the complex plane of (1) $c+\bar{c}$ (2) $c-\bar{c}$ (3) $c\bar{c}$ (4) $c/\bar{c}$.

8 Find x and y for the complex numbers $x+iy$ at angles $\theta = 45°, 90°, 135°$ on the unit circle. Verify directly that the square of the first is the second and the cube of the first is the third.

9 If $c = 2+i$ and $d = 4+3i$ find cd and c/d. Verify that the absolute value $|cd|$ equals $|c|$ times $|d|$, and $|c/d|$ equals $|c|$ divided by $|d|$.

10 Find a solution x to $e^{ix} = i$ and a solution to $e^{ix} = 1/e$. Then find a second solution.

Find the sum and product of the numbers in 11–14.

11 $e^{i\theta}$ and $e^{-i\theta}$, also $e^{2\pi i/3}$ and $e^{4\pi i/3}$

12 $e^{i\theta}$ and $e^{i\phi}$, also $e^{\pi i/4}$ and $e^{-\pi i/4}$

13 The sixth roots of 1 (add and multiply all six)

14 The two roots of $c^2 - 4c + 5 = 0$

15 If $c = re^{i\theta}$ is not zero, what are c^4 and c^{-1} and c^{-4} ?

16 Multiply out $(\cos\theta + i \sin\theta)^3 = e^{i3\theta}$, to find the real part $\cos 3\theta$ and the imaginary part $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

17 Plot the three cube roots of a typical number $re^{i\theta}$. Show why they add to zero. One cube root is $r^{1/3}e^{i\theta/3}$.

18 Prove that the four fourth roots of $re^{i\theta}$ multiply to give $-re^{i\theta}$.

In 19–22, find all solutions of the form $y = e^{ct}$.

19 $y'' + y = 0$

20 $y''' + y = 0$

21 $y''' - y' = 0$

22 $y'' + 6y' + 5y = 0$

Construct two real solutions from the real and imaginary parts of e^{ct} (first find c):

23 $y'' + 49y = 0$

24 $y'' - 2y' + 2y = 0$

Sketch the path of $y = e^{ct}$ as t increases from zero, and mark $y = e^c$:

25 $c = 1 - i$

26 $c = -1 + i$

27 $c = \pi i/4$

28 What is the solution of $dy/dt = iy$ starting from $y_0 = 1$? For this solution, matching real parts and imaginary parts of $dy/dt = iy$ gives ______ and ______.

29 In Figure 9.10b, at what time t does the spiral cross the real axis at the far left? What does y equal at that time?

30 Show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and find a similar formula for $\sin\theta$.

31 *True or false*, with an example to show why:

(a) If $c_1 + c_2$ is real, the c's are complex conjugates.

(b) If $|c_1| = 2$ and $|c_2| = 4$ then c_1c_2 has absolute value 8.

(c) If $|c_1| = 1$ and $|c_2| = 1$ then $|c_1 + c_2|$ is (at least 1) (at most 2) (equal to 2).

(d) If e^{ct} approaches zero as $t \to \infty$, then (c is negative) (the real part of c is negative) ($|c|$ is less than 1).

32 The polar form of $re^{i\theta}$ times $Re^{i\phi}$ is ______. The rectangular form is ______. Circle the terms that give $rR\cos(\theta + \phi)$.

33 The complex number $1/(re^{i\theta})$ has polar form ______ and rectangular form ______ and square roots ______.

34 Show that $\cos ix = \cosh x$ and $\sin ix = i \sinh x$. What is the cosine of i ?

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Resource: Calculus
Gilbert Strang

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