CHAPTER 6

Exponentials and Logarithms

This chapter is devoted to exponentials like 2^x and 10^x and above all e^x. The goal is to understand them, differentiate them, integrate them, solve equations with them, and invert them (to reach the logarithm). The overwhelming importance of e^x makes this a crucial chapter in pure and applied mathematics.

In the traditional order of calculus books, e^x waits until other applications of the integral are complete. I would like to explain why it is placed earlier here. I believe that the equation $dy/dx = y$ has to be emphasized above techniques of integration. The laws of nature are expressed by ***differential equations***, and at the center is e^x. Its applications are to life sciences and physical sciences and economics and engineering (and more—wherever change is influenced by the present state). The model produces a differential equation and I want to show what calculus can do.

The key is always $b^{m+n} = (b^m)(b^n)$. Section 6.1 applies that rule in three ways:

1. to understand the ***logarithm*** as the ***exponent***;
2. to draw ***graphs*** on ordinary and semilog and log-log paper;
3. to find ***derivatives***. The slope of b^x will use $b^{x+\Delta x} = (b^x)(b^{\Delta x})$.

6.1 An Overview

There is a good chance you have met logarithms. They turn multiplication into addition, which is a lot simpler. They are the basis for slide rules (not so important) and for graphs on log paper (very important). Logarithms are mirror images of exponentials—and those I know you have met.

Start with exponentials. The numbers 10 and 10^2 and 10^3 are basic to the decimal system. For completeness I also include 10^0, which is "ten to the zeroth power" or 1. ***The logarithms of those numbers are the exponents***. The logarithms of 1 and 10 and 100 and 1000 are 0 and 1 and 2 and 3. These are logarithms "*to base* 10," because the powers are powers of 10.

Question When the base changes from 10 to b, what is the logarithm of 1 ?
Answer Since $b^0 = 1$, $\log_b 1$ is always ***zero***. To base b, ***the logarithm of*** b^n ***is*** n.
Negative powers are also needed. The number 10^x is positive, but its exponent x can be negative. The first examples are $1/10$ and $1/100$, which are the same as 10^{-1} and 10^{-2}. *The logarithms are the exponents* -1 *and* -2:

$$1000 = 10^3 \quad \text{and} \quad \log 1000 = 3$$
$$1/1000 = 10^{-3} \quad \text{and} \quad \log 1/1000 = -3.$$

Multiplying 1000 times $1/1000$ gives $1 = 10^0$. Adding logarithms gives $3 + (-3) = 0$. Always 10^m times 10^n equals 10^{m+n}. In particular 10^3 times 10^2 produces five tens:

$$(10)(10)(10) \text{ times } (10)(10) \text{ equals } (10)(10)(10)(10)(10) = 10^5.$$

The law for b^m times b^n extends to all exponents, as in $10^{4.6}$ times 10^{π}. Furthermore the law applies to all bases (we restrict the base to $b > 0$ and $b \neq 1$). In every case ***multiplication of numbers is addition of exponents***.

> **6A** b^m times b^n equals b^{m+n}, so logarithms (exponents) add
> b^m divided by b^n equals b^{m-n}, so logarithms (exponents) subtract
>
> $$\log_b(yz) = \log_b y + \log_b z \quad \text{and} \quad \log_b(y/z) = \log_b y - \log_b z. \quad (1)$$

Historical note In the days of slide rules, 1.2 and 1.3 were multiplied by sliding one edge across to 1.2 and reading the answer under 1.3. A slide rule made in Germany would give the third digit in 1.56. Its photograph shows the numbers on a log scale. The distance from 1 to 2 equals the distance from 2 to 4 and from 4 to 8. By sliding the edges, you add distances and multiply numbers.

Division goes the other way. *Notice how* $1000/10 = 100$ *matches* $3 - 1 = 2$. To divide 1.56 by 1.3, look back along line D for the answer 1.2.

The second figure, though smaller, is the important one. ***When*** x ***increases by*** 1, 2^x ***is multiplied by*** 2. *Adding to* x *multiplies* y. This rule easily gives $y = 1, 2, 4, 8$, but look ahead to calculus—which doesn't stay with whole numbers.

Calculus will add Δx. Then y is multiplied by $2^{\Delta x}$. This number is near 1. If $\Delta x = \frac{1}{10}$ then $2^{\Delta x} \approx 1.07$—the tenth root of 2. ***To find the slope, we have to consider*** $(2^{\Delta x} - 1)/\Delta x$. The limit is near $(1.07 - 1)/\frac{1}{10} = .7$, but the exact number will take time.

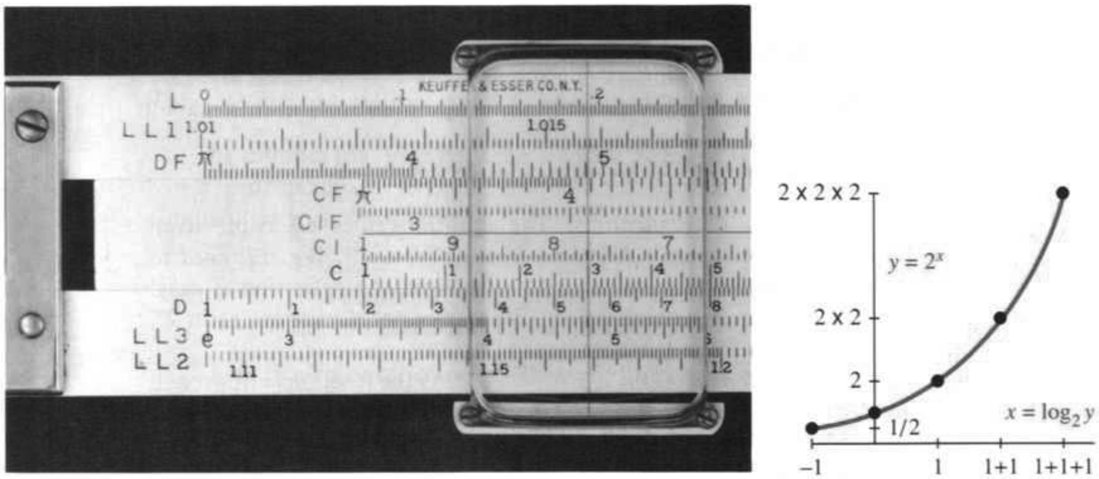


Fig. 6.1 An ancient relic (the slide rule). When exponents x add, powers 2^x multiply.

Base Change Bases other than 10 and exponents other than $1, 2, 3, \ldots$ are needed for applications. The population of the world x years from now is predicted to grow by a factor close to 1.02^x. Certainly x does not need to be a whole number of years. And certainly the base 1.02 should not be 10 (or we are in real trouble). This prediction will be refined as we study the differential equations for growth. It can be rewritten to base 10 if that is preferred (*but look at the exponent*):

$$1.02^x \quad \textit{\textbf{is the same as}} \quad 10^{(\log 1.02)x}.$$

When the base changes from 1.02 to 10, the exponent is multiplied—as we now see.

For practice, start with base b and change to base a. The logarithm to base a will be written "log ." Everything comes from the rule that logarithm = exponent:

$$\textit{\textbf{base change for numbers}}: \quad b = a^{\log b}.$$

Now raise both sides to the power x. You see the change in the exponent:

$$\textit{\textbf{base change for exponentials}}: \quad b^x = a^{(\log b)x}.$$

Finally set $y = b^x$. Its logarithm to base b is x. Its logarithm to base a is the exponent on the right hand side: $\log_a y = (\log_a b)x$. Now replace x by $\log_b y$:

$$\textit{\textbf{base change for logarithms}}: \quad \log_a y = (\log_a b)(\log_b y).$$

We absolutely need this ability to change the base. An example with $a = 2$ is

$$b = 8 = 2^3 \qquad 8^2 = (2^3)^2 = 2^6 \qquad \log_2 64 = 3 \cdot 2 = (\log_2 8)(\log_8 64).$$

The rule behind base changes is $(a^m)^x = a^{mx}$. When the mth power is raised to the xth power, *the exponents multiply*. The square of the cube is the sixth power:

$$(a)(a)(a) \text{ times } (a)(a)(a) \text{ equals } (a)(a)(a)(a)(a)(a): \quad (a^3)^2 = a^6.$$

Another base will soon be more important than 10—here are the rules for base changes:

6B To change *numbers*, *powers*, and *logarithms* from base b to base a, use

$$b = a^{\log_a b} \qquad b^x = a^{(\log_a b)x} \qquad \log_a y = (\log_a b)(\log_b y) \tag{2}$$

The first is the definition. The second is the xth power of the first. The third is the logarithm of the second (remember y is b^x). An important case is $y = a$:

$$\log_a a = (\log_a b)(\log_b a) = 1 \text{ so } \log_a b = 1/\log_b a. \tag{3}$$

EXAMPLE $8 = 2^3$ means $8^{1/3} = 2$. Then $(\log_2 8)(\log_8 2) = (3)(1/3) = 1$.

This completes the algebra of logarithms. The addition rules **6A** came from $(b^m)(b^n) = b^{m+n}$. The multiplication rule **6B** came from $(a^m)^x = a^{mx}$. *We still need to define b^x and a^x for all real numbers x.* When x is a fraction, the definition is easy. The square root of a^8 is $a^4(m = 8$ times $x = 1/2)$. When x is not a fraction, as in 2^π, the graph suggests one way to fill in the hole.

We could define 2^π as the limit of $2^3, 2^{31/10}, 2^{314/100}, \ldots$. As the fractions r approach π, the powers 2^r approach 2^π. This makes $y = 2^x$ into a continuous function, with the desired properties $(2^m)(2^n) = 2^{m+n}$ and $(2^m)^x = 2^{mx}$—whether m and n and x are integers or not. But the ε's and δ's of continuity are not attractive, and we eventually choose (in Section 6.4) a smoother approach based on integrals.

GRAPHS OF b^x AND $\log_b y$

It is time to draw graphs. In principle one graph should do the job for both functions, because $y = b^x$ means the same as $x = \log_b y$. ***These are inverse functions***. What one function does, its inverse undoes. The logarithm of $g(x) = b^x$ is x:

$$g^{-1}(g(x)) = \log_b(b^x) = x. \tag{4}$$

In the opposite direction, the exponential of the logarithm of y is y:

$$g(g^{-1}(y)) = b^{(\log_b y)} = y. \tag{5}$$

This holds for every base b, and it is valuable to see $b = 2$ and $b = 4$ on the same graph. Figure 6.2a shows $y = 2^x$ and $y = 4^x$. Their mirror images in the 45° line give the logarithms to base 2 and base 4, which are in the right graph.

When x is negative, $y = b^x$ is still positive. If the first graph is extended to the left, it stays above the x axis. ***Sketch it in with your pencil***. Also extend the second graph down, to be the mirror image. Don't cross the vertical axis.

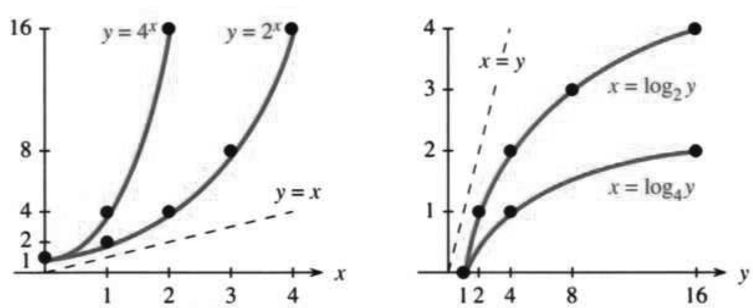


Fig. 6.2 Exponentials and mirror images (logarithms). Different scales for x and y.

There are interesting relations within the left figure. All exponentials start at 1, because b^0 is always 1. At the height $y=16$, one graph is above $x=2$ (because $4^2=16$). The other graph is above $x=4$ (because $2^4=16$). ***Why does*** 4^x ***in one graph equal*** 2^{2x} ***in the other***? This is the base change for powers, since $4=2^2$.

The figure on the right shows the mirror image—the logarithm. All logarithms start from zero at $y=1$. The graphs go down to $-\infty$ at $y=0$. (Roughly speaking $2^{-\infty}$ is zero.) Again x in one graph corresponds to $2x$ in the other (base change for logarithms). Both logarithms climb slowly, since the exponentials climb so fast.

The number $\log_2 10$ is between 3 and 4, because 10 is between 2^3 and 2^4. The slope of 2^x is proportional to 2^x—which never happened for x^n. But there are two practical difficulties with those graphs:

1. 2^x and 4^x increase too fast. The curves turn virtually straight up.
2. The most important fact about Ab^x is the value of b—and the base doesn't stand out in the graph.

There is also another point. In many problems we don't know the function $y=f(x)$. We are looking for it! All we have are measured values of y (with errors mixed in). When the values are plotted on a graph, we want to discover $f(x)$.

Fortunately there is a solution. ***Scale the*** y ***axis differently***. On ordinary graphs, each unit upward adds a fixed amount to y. *On a log scale each unit* ***multiplies*** y *by a fixed amount*. The step from $y=1$ to $y=2$ is the same length as the step from 3 to 6 or 10 to 20.

On a log scale, $y=11$ is not halfway between 10 and 12. And $y=0$ is not there at all. Each step down divides by a fixed amount—*we never reach zero*. This is completely satisfactory for Ab^x, which also never reaches zero.

Figure 6.3 is on ***semilog paper*** (also known as *log-linear*), with an ordinary x axis. ***The graph of*** $y=Ab^x$ ***is a straight line***. To see why, take logarithms of that equation:

$$\log y = \log A + x \log b. \tag{6}$$

The relation between x ***and*** $\log y$ ***is linear***. It is really $\log y$ that is plotted, so the graph is straight. The markings on the y axis allow you to enter y without looking up its logarithm—you get an ordinary graph of $\log y$ against x.

Figure 6.3 shows two examples. One graph is an exact plot of $y=2\cdot 10^x$. It goes upward with slope 1, because a unit across has the same length as multiplication by 10 going up. 10^x ***has slope*** 1 ***and*** $10^{(\log b)x}$ (which is b^x) ***will have slope*** $\log b$. The crucial number $\log b$ can be measured directly as the slope.

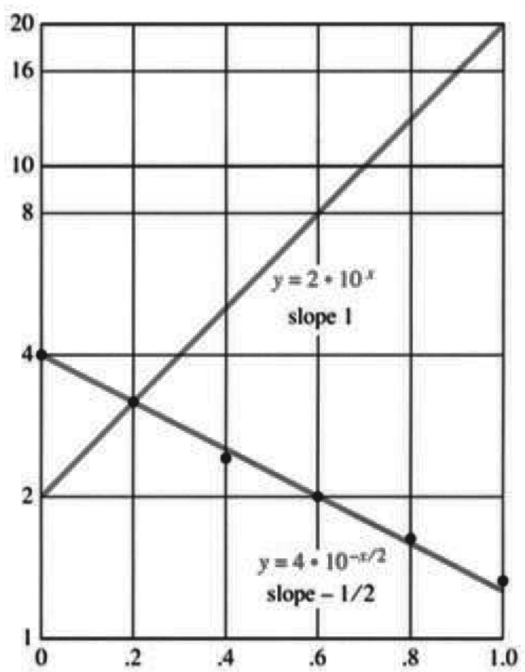
The second graph in Figure 6.3 is more typical of actual practice, in which we start with measurements and look for $f(x)$. Here are the data points:

$x=0.0$	0.2	0.4	0.6	0.8	1.0
$y=4.0$	3.2	2.4	2.0	1.6	1.3

We don't know in advance whether these values fit the model $y=Ab^x$. The graph is strong evidence that they do. The points lie close to a line with negative slope—indicating $\log b<0$ and $b<1$. The slope down is half of the earlier slope up, so the model is consistent with

$$y = A\cdot 10^{-x/2} \quad \text{or} \quad \log y = \log A - \tfrac{1}{2}x. \tag{7}$$

When x reaches 2, y drops by a factor of 10. At $x=0$ we see $A\approx 4$.



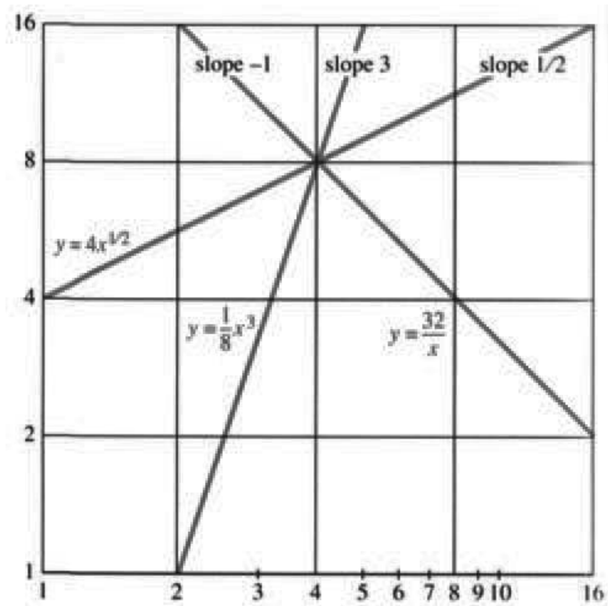


Fig. 6.3 $2 \cdot 10^x$ and $4 \cdot 10^{-x/2}$ on semilog paper.

Fig. 6.4 Graphs of Ax^k on log-log paper.

Another model—a ***power*** $y = Ax^k$ instead of an exponential—also stands out with logarithmic scaling. This time we use ***log-log paper***, with both axes scaled. The logarithm of $y = Ax^k$ gives a linear relation between $\log y$ and $\log x$:

$$\log y = \log A + k \log x. \tag{8}$$

The exponent k ***becomes the slope on log-log paper***. The base b makes no difference. We just measure the slope, and a straight line is a lot more attractive than a power curve.

The graphs in Figure 6.4 have slopes 3 and $\frac{1}{2}$ and -1. They represent Ax^3 and $A\sqrt{x}$ and A/x. To find the A's, look at one point on the line. At $x = 4$ the height is 8, so adjust the A's to make this happen: The functions are $x^3/8$ and $4\sqrt{x}$ and $32/x$. On semilog paper those graphs would not be straight!

You can buy log paper or create it with computer graphics.

THE DERIVATIVES OF $y = b^x$ AND $x = \log_b y$

This is a calculus book. *We have to ask about slopes*. The algebra of exponents is done, the rules are set, and on log paper the graphs are straight. Now come limits.

The central question is the derivative. ***What is*** dy/dx ***when*** $y = b^x$? ***What is*** dx/dy ***when*** x ***is the logarithm*** $\log_b y$? Those questions are closely related, because b^x and $\log_b y$ are inverse functions. If one slope can be found, the other is known from $dx/dy = 1/(dy/dx)$. The problem is to find one of them, and the exponential comes first.

You will now see that those questions have quick (and beautiful) answers, *except for a mysterious constant*. There is a multiplying factor c which needs more time.

I think it is worth separating out the part that can be done immediately, leaving c in dy/dx and $1/c$ in dx/dy. Then Section 6.2 discovers c by studying the special number called e (but $c \neq e$).

> **6C** The derivative of b^x is a multiple cb^x. The number c depends on the base b.

The product and power and chain rules do not yield this derivative. We are pushed all the way back to the original definition, the limit of $\Delta y/\Delta x$:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim_{h\to 0}\frac{b^{x+h}-b^x}{h}. \tag{9}$$

Key idea: Split b^{x+h} into b^x times b^h. Then the crucial quantity b^x factors out. More than that, b^x comes *outside the limit* because it does not depend on h. The remaining limit, inside the brackets, is the number c that we don't yet know:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{b^x b^h - b^x}{h} = b^x\left[\lim_{h\to 0}\frac{b^h-1}{h}\right] = cb^x. \tag{10}$$

This equation is central to the whole chapter: dy/dx ***equals*** cb^x ***which equals*** cy. *The rate of change of* y *is proportional to* y. The slope increases in the same way that b^x increases (except for the factor c). A typical example is money in a bank, where interest is proportional to the principal. The rich get richer, and the poor get slightly richer. We will come back to compound interest, and identify b and c.

The inverse function is $x = \log_b y$. Now the unknown factor is $1/c$:

> **6D** The slope of $\log_b y$ is $1/cy$ with the same c (depending on b).

Proof If $dy/dx = cb^x$ then $dx/dy = 1/cb^x = 1/cy$. (11)

That proof was like a Russian toast, powerful but too quick! We go more carefully:

$$\begin{aligned} f(b^x) &= x && \text{(logarithm of exponential)}\\ f'(b^x)(cb^x) &= 1 && (x \text{ derivative by chain rule})\\ f'(b^x) &= 1/cb^x && (\text{divide by } cb^x)\\ f'(y) &= 1/cy && (\text{identify } b^x \text{ as } y) \end{aligned}$$

The logarithm gives another way to find c. From its slope we can discover $1/c$. *This is the way that finally works* (next section).

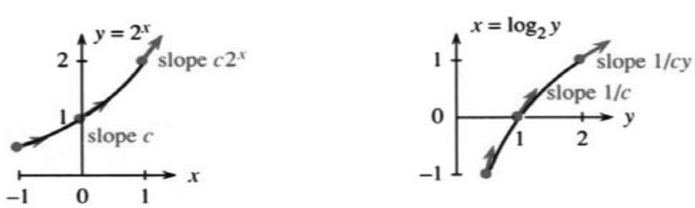


Fig. 6.5 The slope of 2^x is about $.7 \cdot 2^x$. The slope of $\log_2 y$ is about $1/.7y$.

Final remark It is extremely satisfying to meet an $f(y)$ whose derivative is $1/cy$. At last the "-1 power" has an antiderivative. Remember that $\int x^n\,dx = x^{n+1}/(n+$

1) is a failure when $n=-1$. The derivative of x^0 (a constant) does not produce x^{-1} ***We had no integral for x^{-1}, and the logarithm fills that gap***. If y is replaced by x or t (all dummy variables) then

$$\frac{d}{dx}\log_b x = \frac{1}{cx} \quad \text{and} \quad \frac{d}{dt}\log_b t = \frac{1}{ct}. \tag{12}$$

The base b can be chosen so that $c=1$. Then the derivative is $1/x$. This final touch comes from the magic choice $b=e$—the highlight of Section 6.2.

6.1 EXERCISES

Read-through questions

In $10^4 = 10,000$, the exponent 4 is the ___a___ of $10,000$. The base is $b=$___b___. The logarithm of 10^m times 10^n is ___c___. The logarithm of $10^m/10^n$ is ___d___. The logarithm of $10,000^x$ is ___e___. If $y=b^x$ then $x=$___f___. Here x is any number, and y is always ___g___.

A base change gives $b=a^{\_\_h\_\_}$ and $b^x=a^{\_\_i\_\_}$. Then 8^5 is 2^{15}. In other words $\log_2 y$ is ___j___ times $\log_8 y$. When $y=2$ it follows that $\log_2 8$ times $\log_8 2$ equals ___k___.

On ordinary paper the graph of $y=$___l___ is a straight line. Its slope is ___m___. On semilog paper the graph of $y=$___n___ is a straight line. Its slope is ___o___. On log-log paper the graph of $y=$___p___ is a straight line. Its slope is ___q___.

The slope of $y=b^x$ is $dy/dx=$___r___, where c depends on b. The number c is the limit as $h\to 0$ of ___s___. Since $x=\log_b y$ is the inverse, $(dx/dy)(dy/dx)=$___t___. Knowing $dy/dx=cb^x$ yields $dx/dy=$___u___. Substituting b^x for y, the slope of $\log_b y$ is ___v___. With a change of letters, the slope of $\log_b x$ is ___w___.

Problems 1–10 use the rules for logarithms.

1 Find these logarithms (or exponents):

(a) $\log_2 32$ (b) $\log_2(1/32)$ (c) $\log_{32}(1/32)$
(d) $\log_{32} 2$ (e) $\log_{10}(10\sqrt{10})$ (f) $\log_2(\log_2 16)$

2 Without a calculator find the values of

(a) $3^{\log_3 5}$ (b) $3^{2\log_3 5}$
(c) $\log_{10} 5+\log_{10} 2$ (d) $(\log_3 b)(\log_b 9)$
(e) $10^5 10^{-4} 10^3$ (f) $\log_2 56-\log_2 7$

3 Sketch $y=2^{-x}$ and $y=\frac{1}{2}(4^x)$ from -1 to 1 on the same graph. Put their mirror images $x=-\log_2 y$ and $x=\log_4 2y$ on a second graph.

4 Following Figure 6.2 sketch the graphs of $y=\left(\frac{1}{2}\right)^x$ and $x=\log_{1/2} y$. What are $\log_{1/2} 2$ and $\log_{1/2} 4$?

5 Compute without a computer:

(a) $\log_2 3+\log_2 \frac{2}{3}$ (b) $\log_2\left(\frac{1}{2}\right)^{10}$
(c) $\log_{10} 100^{40}$ (d) $(\log_{10} e)(\log_e 10)$
(e) $2^{2^3}/(2^2)^3$ (f) $\log_e(1/e)$

6 Solve the following equations for x:

(a) $\log_{10}(10^x)=7$ (b) $\log 4x-\log 4=\log 3$
(c) $\log_x 10=2$ (d) $\log_2(1/x)=2$
(e) $\log x+\log x=\log 8$ (f) $\log_x(x^x)=5$

7 The logarithm of $y=x^n$ is $\log_b y=$ ______.

***8** Prove that $(\log_b a)(\log_d c)=(\log_d a)(\log_b c)$.

9 2^{10} is close to 10^3 (1024 versus 1000). If they were equal then $\log_2 10$ would be ______. Also $\log_{10} 2$ would be ______ instead of 0.301.

10 The number 2^{1000} has approximately how many (decimal) digits?

Questions 11–19 are about the graphs of $y=b^x$ and $x=\log_b y$.

11 By hand draw the axes for semilog paper and the graphs of $y=1.1^x$ and $y=10(1.1)^x$.

12 Display a set of axes on which the graph of $y=\log_{10} x$ is a straight line. What other equations give straight lines on those axes?

13 When noise is measured in *decibels*, amplifying by a factor A increases the decibel level by $10\log A$. If a whisper is 20db and a shout is 70db then $10\log A=50$ and $A=$ ______.

14 Draw semilog graphs of $y=10^{1-x}$ and $y=\frac{1}{2}\left(\sqrt{10}\right)^x$.

15 The Richter scale measures earthquakes by $\log_{10}(I/I_0)=R$. What is R for the standard earthquake of intensity I_0? If the 1989 San Francisco earthquake measured $R=7$, how did its intensity I compare to I_0? The 1906 San Francisco quake had $R=8.3$. The record quake was four times as intense with $R=$ ______.

16 The frequency of A above middle C is $440/\text{second}$. The frequency of the next higher A is ______. Since $2^{7/12}\approx 1.5$, the note with frequency $660/\sec$ is ______.

17 Draw your own semilog paper and plot the data

$$y=7,\ 11,\ 16,\ 28,\ 44 \quad \text{at} \quad x=0,\ 1/2,\ 1,\ 3/2,\ 2.$$

Estimate A and b in $y=Ab^x$.

18 Sketch log-log graphs of $y=x^2$ and $y=\sqrt{x}$.

19 On log-log paper, printed or homemade, plot $y = 4, 11, 21, 32, 45$ at $x = 1, 2, 3, 4, 5$. Estimate A and k in $y = Ax^k$.

Questions 20–29 are about the derivative $dy/dx = cb^x$.

20 $g(x) = b^x$ has slope $g' = cg$. Apply the chain rule to $g(f(y)) = y$ to prove that $df/dy = 1/cy$.

21 If the slope of $\log x$ is $1/cx$, find the slopes of $\log(2x)$ and $\log(x^2)$ and $\log(2^x)$.

22 What is the equation (including c) for the tangent line to $y = 10^x$ at $x = 0$? Find also the equation at $x = 1$.

23 What is the equation for the tangent line to $x = \log_{10} y$ at $y = 1$? Find also the equation at $y = 10$.

24 With $b = 10$, the slope of 10^x is $c10^x$. Use a calculator for small h to estimate $c = \lim\ (10^h - 1)/h$.

25 The unknown constant in the slope of $y = (.1)^x$ is $L = \lim\ (.1^h - 1)/h$. (a) Estimate L by choosing a small h. (b) Change h to $-h$ to show that $L = -c$ from Problem 24.

26 Find a base b for which $(b^h - 1)/h \approx 1$. Use $h = 1/4$ by hand or $h = 1/10$ and $1/100$ by calculator.

27 Find the second derivative of $y = b^x$ and also of $x = \log_b y$.

28 Show that $C = \lim\ (100^h - 1)/h$ is twice as large as $c = \lim\ (10^h - 1)/h$. (Replace the last h's by $2h$.)

29 In 28, the limit for $b = 100$ is twice as large as for $b = 10$. So c probably involves the ______ of b.

6.2 The Exponential e^x

The last section discussed b^x and $\log_b y$. The base b was arbitrary—it could be 2 or 6 or 9.3 or any positive number except 1. But in practice, only a few bases are used. I have never met a logarithm to base 6 or 9.3. Realistically there are two leading candidates for b, and 10 is one of them. This section is about the other one, which is an extremely remarkable number. This number is not seen in arithmetic or algebra or geometry, where it looks totally clumsy and out of place. In calculus it comes into its own.

The number is e. That symbol was chosen by Euler (initially in a fit of selfishness, but he was a wonderful mathematician). It is the base of the ***natural logarithm***. It also controls the exponential e^x, which is much more important than $\ln x$. Euler also chose π to stand for perimeter—anyway, our first goal is to find e.

Remember that the derivatives of b^x and $\log_b y$ include a constant c that depends on b. Equations (10) and (11) in the previous section were

$$\frac{d}{dx}\,b^x = cb^x \qquad \text{and} \qquad \frac{d}{dy}\,\log_b y = \frac{1}{cy}. \tag{1}$$

At $x=0$, the graph of b^x starts from $b^0=1$. The slope is c. At $y=1$, the graph of $\log_b y$ starts from $\log_b 1 = 0$. The logarithm has slope $1/c$. ***With the right choice of the base*** b ***those slopes will equal*** 1 (because c will equal 1).

For $y=2^x$ the slope c is near .7. We already tried $\Delta x = .1$ and found $\Delta y \approx .07$. The base has to be larger than 2, for a starting slope of $c=1$.

We begin with a direct computation of the slope of $\log_b y$ at $y=1$:

$$\frac{1}{c} = \text{slope at } 1 = \lim_{h\to 0}\frac{1}{h}\Big[\log_b(1+h) - \log_b 1\Big] = \lim_{h\to 0}\log_b\Big[(1+h)^{1/h}\Big]. \tag{2}$$

Always $\log_b 1 = 0$. The fraction in the middle is $\log_b(1+h)$ times the number $1/h$. This number can go up into the exponent, and it did.

The quantity $(1+h)^{1/h}$ is unusual, to put it mildly. As $h\to 0$, the number $1+h$ is approaching 1. At the same time, $1/h$ is approaching infinity. ***In the limit we have*** 1^∞. But that expression is meaningless (like $0/0$). Everything depends on the balance between "nearly 1" and "nearly ∞." This balance produces the extraordinary number e:

DEFINITION ***The number*** e ***is equal to*** $\lim_{h\to 0}(1+h)^{1/h}$. ***Equivalently*** $e = \lim_{h\to 0}\left(1+\frac{1}{n}\right)^n$.

Before computing e, look again at the slope $1/c$. At the end of equation (2) is the logarithm of e:

$$1/c = \log_b e. \tag{3}$$

When the base is $b=e$, ***the slope is*** $\log_e e = 1$. ***That base*** e ***has*** $c=1$ ***as desired***:

$$\textit{The derivative of } e^x \textit{ is } 1\cdot e^x \textit{ and the derivative of } \log_e y \textit{ is } \frac{1}{1\cdot y}. \tag{4}$$

This is why the base e is all-important in calculus. It makes $c=1$.

To compute the actual number e from $(1+h)^{1/h}$, choose $h = 1, 1/10, 1/100, \ldots$. Then the exponents $1/h$ are $n = 1, 10, 100, \ldots$. (All limits and derivatives will become official in Section 6.4.) The table shows $(1+h)^{1/h}$ approaching e as $h\to 0$ and $n\to\infty$:

n	$h = \frac{1}{n}$	$1+h = 1+\frac{1}{n}$	$(1+h)^{1/h} = \left(1+\frac{1}{n}\right)^n$
1	1.0	2.0	2.0
2	0.5	1.5	2.25
10	0.1	1.1	2.593742
100	0.01	1.01	2.704814
1000	0.001	1.001	2.716924
10000	0.0001	1.0001	2.718146

The last column is converging to e (not quickly). There is an infinite series that converges much faster. We know $125,000$ digits of e (and a billion digits of π). There are no definite patterns, although you might think so from the first sixteen digits:

$$e = 2.7\ \ 1828\ \ 1828\ \ 45\ \ 90\ \ 45 \cdots \quad (\text{and } 1/e \approx .37).$$

The powers of e produce $y = e^x$. At $x = 2.3$ and 5, we are close to $y = 10$ and 150.

The logarithm is the inverse function. The logarithms of 150 and 10, to the base e, are close to $x = 5$ and $x = 2.3$. There is a special name for this logarithm—the ***natural logarithm***. There is also a special notation "ln" to show that the base is e:

$\ln y$ ***means the same as*** $\log_e y$. ***The natural logarithm is the exponent in*** $e^x = y$.

The notation $\ln y$ (or $\ln x$—it is the function that matters, not the variable) is standard in calculus courses. After calculus, the base is generally assumed to be e. In most of science and engineering, the natural logarithm is the automatic choice. The symbol "exp (x)" means e^x, and the truth is that the symbol "log x" generally means $\ln x$. Base e is understood even without the letters ln. But in any case of doubt—on a calculator key for example—the symbol "$\ln x$" emphasizes that the base is e.

THE DERIVATIVES OF e^x AND $\ln x$

Come back to derivatives and slopes. The derivative of b^x is cb^x, and the derivative of $\log_b y$ is $1/cy$. ***If*** $b = e$ ***then*** $c = 1$. For all bases, equation (3) is $1/c = \log_b e$. This gives c—the slope of b^x at $x = 0$:

6E ***The number*** c ***is*** $1/\log_b e = \log_e b$. ***Thus*** c ***equals*** $\ln b$. (5)

$c = \ln b$ ***is the mysterious constant that was not available earlier***. The slope of 2^x is $\ln 2$ times 2^x. The slope of e^x is $\ln e$ times e^x (but $\ln e = 1$). We have the derivatives on which this chapter depends:

6F The derivatives of e^x and $\ln y$ are e^x and $1/y$. For other bases

$$\frac{d}{dx} b^x = (\ln b) b^x \qquad \text{and} \qquad \frac{d}{dy} \log_b y = \frac{1}{(\ln b) y}. \tag{6}$$

To make clear that those derivatives come from the functions (and not at all from the dummy variables), we rewrite them using t and x:

$$\frac{d}{dt} e^t = e^t \qquad \text{and} \qquad \frac{d}{dx} \ln x = \frac{1}{x}. \tag{7}$$

Remark on slopes at $x=0$: It would be satisfying to see directly that the slope of 2^x is below 1, and the slope of 4^x is above 1. Quick proof: e is between 2 and 4. But the idea is to see the slopes graphically. This is a small puzzle, which is fun to solve but can be skipped.

2^x rises from 1 at $x=0$ to 2 at $x=1$. On that interval its average slope is 1. Its slope at the beginning is *smaller* than average, so it must be less than 1—as desired. On the other hand 4^x rises from $\frac{1}{2}$ at $x=-\frac{1}{2}$ to 1 at $x=0$. Again the average slope is $\frac{1}{2}\big/\frac{1}{2}=1$. Since $x=0$ comes at the *end* of this new interval, the slope of 4^x at that point exceeds 1. Somewhere between 2^x and 4^x is e^x, which starts out with slope 1.

This is the graphical approach to e. There is also the infinite series, and a fifth definition through integrals which is written here for the record:

1. e is the number such that e^x has slope 1 at $x=0$
2. e is the base for which $\ln y=\log_e y$ has slope 1 at $y=1$
3. e is the limit of $\left(1+\dfrac{1}{n}\right)^n$ as $n\to\infty$
4. $e=\dfrac{1}{0!}+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots=1+1+\dfrac{1}{2}+\dfrac{1}{6}+\cdots$
5. the area $\int_1^e x^{-1}dx$ equals 1.

The connections between **1**, **2**, and **3** have been made. The slopes are 1 when e is the limit of $(1+1/n)^n$. Multiplying this out wlll lead to **4**, the infinite series in Section 6.6. The official definition of $\ln x$ comes from $\int dx/x$, and then **5** says that $\ln e=1$. This approach to e (Section 6.4) seems less intuitive than the others.

Figure 6.6b shows the graph of e^{-x}. It is the mirror image of e^x across the vertical axis. Their product is $e^xe^{-x}=1$. Where e^x grows exponentially, e^{-x} decays exponentially—or it grows as x approaches $-\infty$. ***Their growth and decay are faster than any power of*** x. Exponential growth is more rapid than polynomial growth, so that e^x/x^n goes to infinity (Problem 59). It is the fact that e^x has slope e^x which keeps the function climbing so fast.

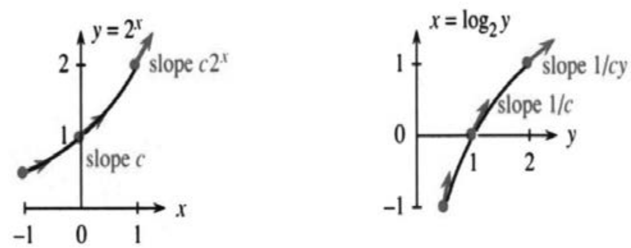


Fig. 6.6 e^x grows between 2^x and 4^x. Decay of e^{-x}, faster decay of $e^{-x^2/2}$.

The other curve is $y=e^{-x^2/2}$. This is the famous "*bell-shaped curve*" of probability theory. After dividing by $\sqrt{2\pi}$, it gives the ***normal distribution***, which applies to so many averages and so many experiments. The Gallup Poll will be an example in Section 8.4. The curve is symmetric around its mean value $x=0$, since changing x to $-x$ has no effect on x^2.

About two thirds of the area under this curve is between $x=-1$ and $x=1$. If you pick points at random below the graph, $2/3$ of all samples are expected in that interval. The points $x=-2$ and $x=2$ are "two standard deviations" from the center, enclosing 95% of the area. There is only a 5% chance of landing beyond. The decay is even faster than an ordinary exponential, because $\frac{1}{2}x^2$ has replaced x.

THE DERIVATIVES OF e^{cx} AND $e^{u(x)}$

The slope of e^x is e^x. This opens up a whole world of functions that calculus can deal with. The chain rule gives the slope of e^{3x} and $e^{\sin x}$ and every $e^{u(x)}$:

6G The derivative of $e^{u(x)}$ is $e^{u(x)}$ times du/dx. (8)

Special case $u = cx$: The derivative of e^{cx} is ce^{cx}. (9)

EXAMPLE 1 The derivative of e^{3x} is $3e^{3x}$ (here $c = 3$). The derivative of $e^{\sin x}$ is $e^{\sin x}\cos x$ (here $u = \sin x$). The derivative of $f(u(x))$ is df/du times du/dx. Here $f = e^u$ so $df/du = e^u$. ***The chain rule demands that second factor*** du/dx.

EXAMPLE 2 $e^{(\ln 2)x}$ is the same as 2^x. Its derivative is $\ln 2$ times 2^x. The chain rule rediscovers our constant $c = \ln 2$. In the slope of b^x it rediscovers the factor $c = \ln b$.

Generally e^{cx} is preferred to the original b^x. The derivative just brings down the constant c. ***It is better to agree on e as the base***, and put all complications (like $c = \ln b$) up in the exponent. The second derivative of e^{cx} is c^2e^{cx}.

EXAMPLE 3 The derivative of $e^{-x^2/2}$ is $-xe^{-x^2/2}$ (here $u = -x^2/2$ so $du/dx = -x$).

EXAMPLE 4 The second derivative of $f = e^{-x^2/2}$, by the chain rule and product rule, is

$$f'' = (-1)\cdot e^{-x^2/2} + (-x)^2e^{-x^2/2} = (x^2-1)e^{-x^2/2}. \qquad (10)$$

Notice how ***the exponential survives***. With every derivative it is multiplied by more factors, but it is still there to dominate growth or decay. The *points of inflection*, where the bell-shaped curve has $f'' = 0$ in equation (10), are $x = 1$ and $x = -1$.

EXAMPLE 5 $(u = n\ln x)$. Since $e^{n\ln x}$ is x^n in disguise, its slope must be nx^{n-1}:

$$\text{slope} = e^{n\ln x}\frac{d}{dx}(n\ln x) = x^n\left(\frac{n}{x}\right) = nx^{n-1}. \qquad (11)$$

This slope is correct for all n, integer or not. Chapter 2 produced $3x^2$ and $4x^3$ from the binomial theorem. Now nx^{n-1} comes from ln and exp and the chain rule.

EXAMPLE 6 An extreme case is $x^x = (e^{\ln x})^x$. Here $u = x\ln x$ and we need du/dx:

$$\frac{d}{dx}(x^x) = e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right) = x^x(\ln x + 1).$$

INTEGRALS OF e^{cx} AND $e^u\,du/dx$

The integral of e^x is e^x. *The integral of e^{cx} is not e^{cx}*. The derivative multiplies by c so the integral divides by c. ***The integral of e^{cx} is*** e^{cx}/c (plus a constant).

EXAMPLES $\displaystyle\int e^{2x}dx = \frac{1}{2}e^{2x} + C \qquad \int b^x dx = \frac{b^x}{\ln b} + C$

$$\int e^{3(x+1)}dx = \frac{1}{3}e^{3(x+1)} + C \qquad \int e^{-x^2/2}dx \to \text{failure}$$

The first one has $c = 2$. The second has $c = \ln b$—remember again that $b^x = e^{(\ln b)x}$. The integral divides by $\ln b$. In the third one, $e^{3(x+1)}$ is e^{3x} times the number e^3

and that number is carried along. Or more likely we see $e^{3(x+1)}$ as e^u. The missing $du/dx = 3$ is fixed by dividing by 3. The last example fails because du/dx is not there. ***We cannot integrate without*** du/dx:

6H ***The indefinite integral*** $\int e^u \frac{du}{dx}\,dx$ ***equals*** $e^{u(x)} + C$.

Here are three examples with du/dx and one without it:

$$\int e^{\sin x} \cos\, x\, dx = e^{\sin x} + C \qquad \int x e^{x^2/2}\, dx = e^{x^2/2} + C$$
$$\int \frac{e^{\sqrt{x}} dx}{\sqrt{x}} = 2e^{\sqrt{x}} + C \qquad \int \frac{e^x\, dx}{(1+e^x)^2} = \frac{-1}{1+e^x} + C$$

The first is a pure $e^u du$. So is the second. The third has $u = \sqrt{x}$ and $du/dx = 1/2\sqrt{x}$,
so only the factor 2 had to be fixed. The fourth example does not belong with the others. It is the integral of du/u^2, not the integral of $e^u du$. I don't know any way to tell you which substitution is best—except that *the complicated part is* $1 + e^x$ *and it is natural to substitute* u. If it works, good.

Without an extra e^x for du/dx, the integral $\int dx/(1+e^x)^2$ looks bad. But $u = 1 + e^x$ is still worth trying. It has $du = e^x dx = (u-1)dx$:

$$\int \frac{dx}{(1+e^x)^2} = \int \frac{du}{(u-1)u^2} = \int du \left(\frac{1}{u-1} - \frac{1}{u} - \frac{1}{u^2} \right). \qquad (12)$$

That last step is "*partial fractions*." The integral splits into simpler pieces (explained in Section 7.4) and we integrate each piece. Here are three other integrals:

$$\int e^{1/x} dx \qquad \int e^x (4+e^x) dx \qquad \int e^{-x} (4+e^x) dx$$

The first can change to $-\int e^u du/u^2$, which is not much better. (It is just as impossible.) The second is actually $\int u\, du$, but I prefer a split: $\int 4e^x$ and $\int e^{2x}$ are safer to do separately. The third is $\int (4e^{-x} + 1) dx$, which also separates. The exercises offer practice in reaching $e^u du/dx$—ready to be integrated.

Warning about definite integrals When the lower limit is $x = 0$, there is a natural tendency to expect $f(0) = 0$—in which case the lower limit contributes nothing. For a power $f = x^3$ that is true. For an exponential $f = e^{3x}$ it is definitely *not true*, because $f(0) = 1$:

$$\int_0^1 e^{3x} dx = \frac{1}{3} e^{3x} \Big]_0^1 = \frac{1}{3} (e^3 - 1) \qquad \int_0^1 x e^{x^2} dx = \frac{1}{2} e^{x^2} \Big]_0^1 = \frac{1}{2} (e - 1).$$

6.2 EXERCISES

Read-through questions

The number e is approximately __a__. It is the limit of $(1+h)$ to the power __b__. This gives 1.01^{100} when $h=$ __c__. An equivalent form is $e = \lim($ __d__ $)^n$.

When the base is $b=e$, the constant c in Section 6.1 is __e__. Therefore the derivative of $y=e^x$ is $dy/dx=$ __f__. The derivative of $x=\log_e y$ is $dx/dy=$ __g__. The slopes at $x=0$ and $y=1$ are both __h__. The notation for $\log_e y$ is __i__, which is the __j__ logarithm of y.

The constant c in the slope of b^x is $c=$ __k__. The function b^x can be rewritten as __l__. Its derivative is __m__. The derivative of $e^{u(x)}$ is __n__. The derivative of $e^{\sin x}$ is __o__. The derivative of e^{cx} brings down a factor __p__.

The integral of e^x is __q__. The integral of e^{cx} is __r__. The integral of $e^{u(x)}du/dx$ is __s__. In general the integral of $e^{u(x)}$ by itself is __t__ to find.

Find the derivatives of the functions in 1–18.

1 $7e^{7x}$

2 $-7e^{-7x}$

3 $(e^x)^8$

4 $(x^{-x})^{-8}$

5 3^x

6 $e^{x \ln 3}$

7 $(2/3)^x$

8 4^{4x}

9 $1/(1+e^x)$

10 $e^{1/(1+x)}$

11 $e^{\ln x} + x^{\ln e}$

12 $xe^{1/x}$

13 $xe^x - e^x$

14 $x^2e^x - 2xe^x + 2e^x$

15 $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

16 $e^{\ln(x^2)} + \ln(e^{x^2})$

17 $e^{\sin x} + \sin e^x$

18 $x^{-1/x}$ (which is e—)

19 The difference between e and $(1+1/n)^n$ is approximately Ce/n. Subtract the calculated values for $n=10, 100, 1000$ from 2.7183 to discover the number C.

20 By algebra or a calculator find the limits of $(1+1/n)^{2n}$ and $(1+1/n)^{\sqrt{n}}$.

21 The limit of $(11/10)^{10}, (101/100)^{100}, \ldots$ is e. So the limit of $(10/11)^{10}, (100/101)^{100}, \ldots$ is ______. So the limit of $(10/11)^{11}, (100/101)^{101}, \ldots$ is ______. The last sequence is $(1-1/n)^n$.

22 Compare the number of correct decimals of e for $(1.001)^{1000}$ and $(1.0001)^{10000}$ and if possible $(1.00001)^{100000}$. Which power n would give all the decimals in 2.71828?

23 The function $y=e^x$ solves $dy/dx=y$. Approximate this equation by $\Delta Y/\Delta x = Y$, which is $Y(x+h)-Y(x)=hY(x)$. With $h=\frac{1}{10}$ find $Y(h)$ after one step starting from $Y(0)=1$. What is $Y(1)$ after ten steps?

24 The function that solves $dy/dx=-y$ starting from $y=1$ at $x=0$ is ______. Approximate by $Y(x+h)-Y(x)=-hY(x)$. If $h=\frac{1}{4}$ what is $Y(h)$ after one step and what is $Y(1)$ after four steps?

25 Invent three functions f, g, h such that for $x>10$
$$(1+1/x)^x < f(x) < e^x < g(x) < e^{2x} < h(x) < x^x.$$

26 Graph e^x and $\sqrt{e^x}$ at $x=-2, -1, 0, 1, 2$. Another form of $\sqrt{e^x}$ is ______.

Find antiderivatives for the functions in 27–36.

27 $e^{3x} + e^{7x}$

28 $(e^{3x})(e^{7x})$

29 $1^x + 2^x + 3^x$

30 2^{-x}

31 $(2e)^x + 2e^x$

32 $(1/e^x) + (1/x^e)$

33 $xe^{x^2} + xe^{-x^2}$

34 $(\sin x)e^{\cos x} + (\cos x)e^{\sin x}$

35 $\sqrt{e^x} + (e^x)^2$

36 xe^x (trial and error)

37 Compare e^{-x} with e^{-x^2}. Which one decreases faster near $x=0$? Where do the graphs meet again? When is the ratio of e^{-x^2} to e^{-x} less than $1/100$?

38 Compare e^x with x^x: Where do the graphs meet? What are their slopes at that point? Divide x^x by e^x and show that the ratio approaches infinity.

39 Find the tangent line to $y=e^x$ at $x=a$. From which point on the graph does the tangent line pass through the origin?

40 By comparing slopes, prove that if $x>0$ then

(a) $e^x > 1+x$ (b) $e^{-x} > 1-x$.

41 Find the minimum value of $y=x^x$ for $x>0$. Show from d^2y/dx^2 that the curve is concave upward.

42 Find the slope of $y=x^{1/x}$ and the point where $dy/dx=0$. Check d^2y/dx^2 to show that the maximum of $x^{1/x}$ is ______.

43 If $dy/dx=y$ find the derivative of $e^{-x}y$ by the product rule. Deduce that $y(x)=Ce^x$ for some constant C.

44 Prove that $x^e = e^x$ has only one positive solution.

Evaluate the integrals in 45–54. With infinite limits, 49–50 are "improper."

45 $\int_0^1 e^{2x}\,dx$

46 $\int_0^{\pi} \sin x\, e^{\cos x}\,dx$

47 $\int_{-1}^1 2^x\,dx$

48 $\int_{-1}^1 2^{-x}\,dx$

49 $\int_0^{\infty} e^{-x}\,dx$

50 $\int_0^{\infty} xe^{-x^2}\,dx$

51 $\int_0^1 e^{1+x}\,dx$

52 $\int_0^1 e^{1+x^2}\,x\,dx$

53 $\int_0^{\pi} 2^{\sin x}\cos x\,dx$

54 $\int_0^1 (1-e^x)^{10}\,e^x\,dx$

55 Integrate the integrals that can be integrated:

$$\int \frac{e^u}{du/dx}\,dx \qquad \int \frac{du/dx}{e^u}\,dx$$

$$\int e^u\Big(\frac{du}{dx}\Big)^2 dx \qquad \int (e^u)^2\frac{du}{dx}\,dx$$

56 Find a function that solves $y'(x) = 5y(x)$ with $y(0) = 2$.

57 Find a function that solves $y'(x) = 1/y(x)$ with $y(0) = 2$.

58 With electronic help graph the function $(1+1/x)^x$. What are its asymptotes? Why?

59 This exercise shows that $F(x) = x^n/e^x \to 0$ as $x \to \infty$.

(a) Find dF/dx. Notice that $F(x)$ decreases for $x > n > 0$. The maximum of x^n/e^x, at $x = n$, is n^n/e^n.

(b) $F(2x) = (2x)^n/e^{2x} = 2^n x^n/e^x \cdot e^x \leqslant 2^n n^n/e^n \cdot e^x$. Deduce that $F(2x) \to 0$ as $x \to \infty$. Thus $F(x) \to 0$.

60 With $n = 6$, graph $F(x) = x^6/e^x$ on a calculator or computer. Estimate its maximum. Estimate x when you reach $F(x) = 1$. Estimate x when you reach $F(x) = \frac{1}{2}$.

61 Stirling's formula says that $n! \approx \sqrt{2\pi n}\, n^n/e^n$. Use it to estimate $6^6/e^6$ to the nearest whole number. Is it correct? How many decimal digits in 10!?

62 $x^6/e^x \to 0$ is also proved by I'Hôpital's rule (at $x = \infty$):

$$\lim x^6/e^x = \lim 6x^5/e^x = \underline{\text{fill this in}} = 0.$$

6.3 Growth and Decay in Science and Economics

The derivative of $y = e^{cx}$ has taken time and effort. ***The result was*** $y' = ce^{cx}$, ***which means that*** $y' = cy$. That computation brought others with it, virtually for free—the derivatives of b^x and x^x and $e^{u(x)}$. But I want to stay with $y' = cy$—which is the most important differential equation in applied mathematics.

Compare $y' = x$ ***with*** $y' = y$. The first only asks for an antiderivative of x. We quickly find $y = \frac{1}{2}x^2 + C$. The second has dy/dx equal to y itself—which we rewrite as $dy/y = dx$. The integral is in $y = x + C$. Then y itself is $e^x e^c$. Notice that the first solution is $\frac{1}{2}x^2$ *plus* a constant, and the second solution is e^x times a constant.

There is a way to graph slope x versus slope y. Figure 6.7 shows "*tangent arrows*," which give the slope at each x and y. For parabolas, the arrows grow steeper as x grows—because $y' = \text{slope} = x$. For exponentials, the arrows grow steeper as y grows—the equation is $y' = \text{slope} = y$. Now the arrows are connected by $y = Ae^x$. ***A differential equation gives a field of arrows*** (slopes). *Its solution is a curve that stays tangent to the arrows*—then the curve has the right slope.

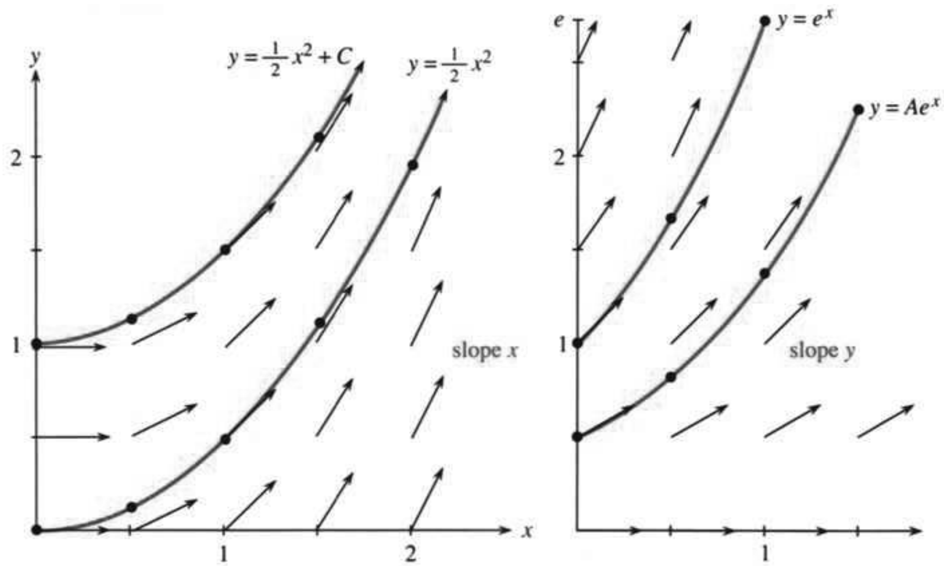


Fig. 6.7 The slopes are $y' = x$ and $y' = y$. The solution curves fit those slopes.

A field of arrows can show many solutions at once (this comes in a differential equations course). Usually a single y_0 is not sacred. To understand the equation we start from many y_0—on the left the parabolas stay parallel, on the right the heights stay proportional. For $y' = -y$ all solution curves go to zero.

From $y' = y$ it is a short step to $y' = cy$. To make c appear in the derivative, *put c into the exponent*. The derivative of $y = e^{cx}$ is ce^{cx}, which is c times y. We have reached the key equation, which comes with an *initial condition*—a starting value y_0:

$$dy/dt = cy \text{ with } y = y_0 \text{ at } t = 0. \tag{1}$$

A small change: x has switched to t. In most applications *time* is the natural variable, rather than space. The factor c becomes the "growth rate" or "decay rate"—and e^{cx} converts to e^{ct}.

The last step is to match the initial condition. The problem requires $y = y_0$ at $t = 0$. Our e^{ct} starts from $e^{c0} = 1$. *The constant of integration is needed now—the solutions are* $y = Ae^{ct}$. By choosing $A = y_0$, we match the initial condition and solve equation (1). ***The formula to remember is*** $y_0 e^{ct}$.

6I The ***exponential law*** $y = y_0 e^{ct}$ solves $y' = cy$ starting from y_0.

The rate of growth or decay is c. May I call your attention to a basic fact? *The formula* $y_0 e^{ct}$ *contains three quantities* y_0, c, t. If two of them are given, plus one additional piece of information, the third is determined. Many applications have one of these three forms: ***find*** t, ***find*** c, ***find*** y_0.

1. Find the doubling time T if $c = 1/10$. At that time $y_0 e^{cT}$ equals $2y_0$:

$$e^{cT} = 2 \text{ yields } cT = \ln 2 \text{ so that } T = \frac{\ln 2}{c} \approx \frac{.7}{.1}. \tag{2}$$

The question asks for an exponent T. The answer involves logarithms. If a cell grows at a continuous rate of $c = 10\%$ per day, it takes about $.7/.1 = 7$ days to double in size. (Note that $.7$ is close to $\ln 2$.) If a savings account earns 10% continuous interest, it doubles in 7 years.

In this problem we knew c. In the next problem we know T.

2. Find the decay constant c for carbon-14 if $y = \frac{1}{2}y_0$ in $T = 5568$ years.

$$e^{cT} = \tfrac{1}{2} \text{ yields } cT = \ln \tfrac{1}{2} \text{ so that } c \approx \left(\ln \tfrac{1}{2}\right)/5568. \tag{3}$$

After the half-life $T = 5568$, the factor e^{cT} equals $\frac{1}{2}$. Now c is negative ($\ln \frac{1}{2} = -\ln 2$).

Question **1** was about growth. Question **2** was about decay. Both answers found e^{cT} as the ratio $y(T)/y(0)$. Then cT is its logarithm. Note how c sticks to T. T has the units of time, c has the units of "$1/$time."

Main point: The doubling time is $(\ln 2)/c$, because $cT = \ln 2$. The time to multiply by e is $1/c$. The time to multiply by 10 is $(\ln 10)/c$. The time to divide by e is $-1/c$, when a negative c brings decay.

3. Find the initial value y_0 if $c = 2$ and $y(1) = 5$:

$$y(t) = y_0 e^{ct} \text{ yields } y_0 = y(t)e^{-ct} = 5e^{-2}.$$

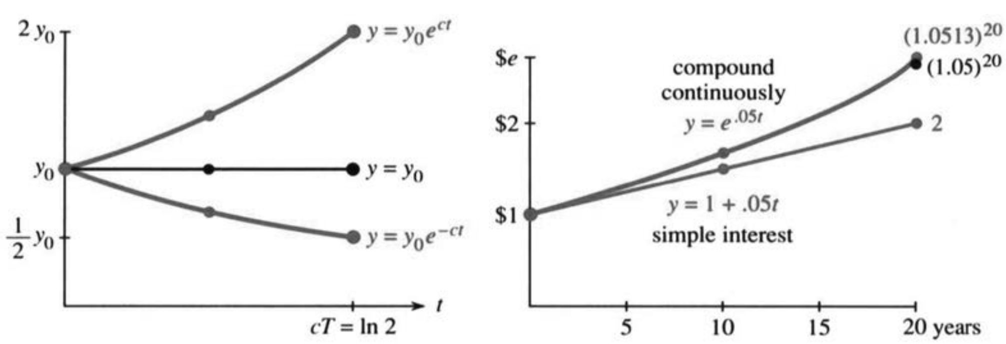


Fig. 6.8 Growth ($c > 0$) and decay ($c < 0$). Doubling time $T = (\ln 2)/c$. Future value at 5%.

All we do is run the process backward. Start from 5 and go back to y_0. With time reversed, e^{ct} becomes e^{-ct}. The product of e^2 and e^{-2} is 1—growth forward and decay backward.

Equally important is $T+t$. ***Go forward to time*** T ***and go on to*** $T+t$:

$$y(T+t) \text{ is } y_0 e^{c(T+t)} \text{ which is } (y_0 e^{cT})e^{ct}. \tag{4}$$

Ever step t, at the start or later, multiplies by the same e^{ct}. This uses the fundamental property of exponentials, that $e^{T+t} = e^T\, e^t$.

EXAMPLE 1 ***Population growth from birth rate*** b ***and death rate*** d (both constant):

$$dy/dt = by - dy = cy \quad \text{(the net rate is } c = b - d).$$

The population in this model is $y_0 e^{ct} = y_0 e^{bt} e^{-dt}$. It grows when $b > d$ (which makes $c > 0$). One estimate of the growth rate is $c = 0.02/\text{year}$:

$$\textbf{\textit{The earth's population doubles in about }} T = \frac{\ln 2}{c} \approx \frac{.7}{.02} = 35\, \textbf{\textit{years}}.$$

First comment: We predict the future based on c. We count the past population to find c. Changes in c are a serious problem for this model.

Second comment: $y_0 e^{ct}$ is not a whole number. You may prefer to think of bacteria instead of people. (***This section begins a major application of mathematics to economics and the life sciences***.) Malthus based his theory of human population on this equation $y' = cy$—and with large numbers a fraction of a person doesn't matter so much. To use calculus we go from discrete to continuous. The theory must fail when t is very large, since populations cannot grow exponentially forever. Section 6.5 introduces the *logistic* equation $y' = cy - by^2$, with a competition term $-by^2$ to slow the growth.

Third comment: The dimensions of b, c, d are "$1/\text{time}$." The dictionary gives birth rate = number of births per person in a unit of time. It is a *relative* rate—people divided by people and time. The product ct is dimensionless and e^{ct} makes sense (also dimensionless). Some texts replace c by λ (lambda). Then $1/\lambda$ is the growth time or decay time or drug elimination time or diffusion time.

EXAMPLE 2 *Radioactive dating* A gram of charcoal from the cave paintings in France gives 0.97 disintegrations per minute. A gram of living wood gives 6.68 disintegrations per minute. Find the age of those Lascaux paintings.

The charcoal stopped adding radiocarbon when it was burned (at $t = 0$). The amount has decayed to $y_0 e^{ct}$. In living wood this amount is still y_0, because cosmic rays maintain the balance. *Their ratio is* $e^{ct} = 0.97/6.68$. Knowing the decay rate c from Question **2** above, we know the present time t:

$$ct = \ln\left(\frac{0.97}{6.68}\right) \quad \text{yields} \quad t = \frac{5568}{-.7} \ln\left(\frac{0.97}{6.68}\right) = 14{,}400 \text{ years}.$$

Here is a related problem—*the age of uranium*. Right now there is 140 times as much U-238 as U-235. Nearly equal amounts were created, with half-lives of $(4.5)10^9$ and $(0.7)10^9$ years. ***Question***: How long since uranium was created? *Answer*: Find t by substituting $c = (\ln \frac{1}{2})/(4.5)10^9$ and $C = (\ln \frac{1}{2})/(0.7)10^9$:

$$e^{ct}/e^{Ct} = 140 \Rightarrow ct - Ct = \ln 140 \Rightarrow t = \frac{\ln 140}{c - C} = 6(10^9) \text{ years}.$$

EXAMPLE 3 ***Calculus in Economics*: *price inflation and the value of money***

We begin with two inflation rates—a *continuous rate* and an *annual rate*. For the price change Δy over a year, use the annual rate:

$$\Delta y = (\textit{annual rate}) \text{ times } (y) \text{ times } (\Delta t). \tag{5}$$

Calculus applies the continuous rate to each instant dt. ***The price change is*** dy:

$$dy = (\textit{continuous rate}) \text{ times } (y) \text{ times } (dt). \tag{6}$$

Dividing by dt, this is a differential equation for the price:

$$dy/dt = (\textbf{\textit{continuous rate}}) \text{ times } (y) = .05y.$$

The solution is $y_0 e^{.05t}$. Set $t = 1$. Then $e^{.05} \approx 1.0513$ and the annual rate is 5.13%.

When you ask a bank what interest they pay, they give both rates: 8% and 8.33%. The higher one they call the "effective rate." It comes from compounding (and depends how often they do it). If the compounding is continuous, every dt brings an increase of dy—and $e^{.08}$ is near 1.0833.

Section 6.6 returns to compound interest. The interval drops from a month to a day to a second. That leads to $(1 + 1/n)^n$, and in the limit to e. Here we compute the effect of 5% continuous interest:

Future value A dollar now has the same value as $e^{.05T}$ dollars in T years.

Present value A dollar in T years has the same value as $e^{-.05T}$ dollars now.

Doubling time Prices double ($e^{.05T} = 2$) in $T = \ln 2/.05 \approx 14$ years.

With no compounding, the doubling time is 20 years. Simple interest adds on 20 times $5\% = 100\%$. With continuous compounding the time is reduced by the factor $\ln 2 \approx .7$, regardless of the interest rate.

EXAMPLE 4 In 1626 the Indians sold Manhattan for \$24. Our calculations indicate that they knew what they were doing. Assuming 8% compound interest, the original \$24 is multiplied by $e^{.08t}$. After $t = 365$ years the multiplier is $e^{29.2}$ and the \$24 has grown to 115 trillion dollars. With that much money they could buy back the land and pay off the national debt.

This seems farfetched. Possibly there is a big flaw in the model. It is absolutely true that Ben Franklin left money to Boston and Philadelphia, to be invested for 200 years. In 1990 it yielded millions (not trillions, that takes longer). Our next step is a new model.

Question How can you estimate $e^{29.2}$ with a \$24 calculator (log but not ln)?
Answer Multiply 29.2 by $\log_{10} e = .434$ to get 12.7. This is the exponent to base 10. After that base change, we have $10^{12.7}$ or more than a trillion.

GROWTH OR DECAY WITH A SOURCE TERM

The equation $y' = y$ will be given a new term. Up to now, all growth or decay has started from y_0. No deposit or withdrawal was made later. The investment grew by itself—a pure exponential. ***The new term*** s ***allows you to add or subtract from the***

account. It is a “source”—or a “sink” if s is negative. The source $s=5$ adds $5dt$, proportional to dt but not to y:

$$\textit{Constant source:} \quad dy/dt = y+5 \text{ starting from } y=y_0.$$

Notice y on both sides! My first guess $y=e^{t+5}$ failed completely. Its derivative is e^{t+5} again, which is not $y+5$. The class suggested $y=e^t+5t$. But its derivative e^t+5 is still not $y+5$. We tried other ways to produce 5 in dy/dt. This idea is doomed to failure. *Finally we thought of* $y=Ae^t-5$. That has $y'=Ae^t=y+5$ as required.

Important: A ***is not*** y_0. Set $t=0$ to find $y_0=A-5$. The source contributes $5e^t-5$:

$$\textbf{\textit{The solution is }} (y_0+5)e^t-5. \textbf{\textit{ That is the same as }} y_0e^t+5(e^t-1).$$

$s=5$ multiplies the growth term e^t-1 that starts at zero. y_0e^t grows as before.

EXAMPLE 5 $dy/dt=-y+5$ has $y=(y_0-5)e^{-t}+5$. This is $y_0e^{-t}+5(1-e^{-t})$.

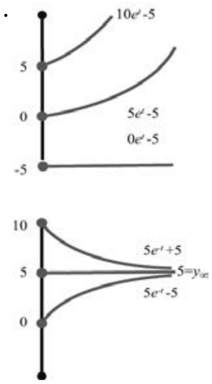


Fig. 6.9

That final term from the source is still positive. The other term y_0e^{-t} decays to zero. ***The limit as*** $t\to\infty$ *is* $y_\infty=5$. A negative c leads to a steady state y_∞.

Based on these examples with $c=1$ and $c=-1$, we can find y for any c and s.

$$\textbf{EQUATION WITH SOURCE} \qquad \frac{dy}{dt}=cy+s \text{ starts from } y=y_0 \text{ at } t=0 \tag{7}$$

The source could be a deposit of $s=\$1000/\text{year}$, after an initial investment of $y_0=\$8000$. Or we can withdraw funds at $s=-\$200/\text{year}$. The units are “dollars per year” to match dy/dt. The equation feeds in \$1000 or removes \$200 *continuously*—not all at once.

Note again that $y=e^{(c+s)t}$ is not a solution. Its derivative is $(c+s)y$. The combination $y=e^{ct}+s$ is also not a solution (but closer). ***The analysis of*** $y'=cy+s$ ***will be our main achievement for differential equations*** (in this section). The equation is not restricted to finance—far from it—but that produces excellent examples.

I propose to find y in four ways. You may feel that one way is enough.† The first way is the fastest—only three lines—but please give the others a chance. There is no point in preparing for real problems if we don’t solve them.

Solution by Method 1 (fast way) Substitute the combination $y=Ae^{ct}+B$. ***The solution has this form—exponential plus constant***. From two facts we find A and B:

$$\text{the equation } y'=cy+s \text{ gives } cAe^{ct}=c(Ae^{ct}+B)+s$$
$$\text{the initial value at } t=0 \text{ gives } A+B=y_0.$$

The first line has cAe^{ct} on both sides. Subtraction leaves $cB+s=0$, or $B=-s/c$. Then the second line becomes $A=y_0-B=y_0+(s/c)$:

$$\textbf{KEY FORMULA} \qquad y=\left(y_0+\frac{s}{c}\right)e^{ct}-\frac{s}{c} \quad \textbf{or} \quad y=y_0e^{ct}+\frac{s}{c}\left(e^{ct}-1\right). \tag{8}$$

With $s=0$ this is the old solution y_0e^{ct} (no source). The example with $c=1$ and $s=5$ produced $(y_0+5)e^t-5$. Separating the source term gives $y_0e^t+5(e^t-1)$.

†My class says one way is *more* than enough. They just want the answer. Sometimes I cave in and write down the formula: y is y_0e^{ct} plus $s(e^{ct}-1)/c$ from the source term.

Solution by Method 2 (slow way) The input y_0 produces the output y_0e^{ct}. After t years any deposit is multiplied by e^{ct}. *That also applies to deposits made after the account is opened*. If the deposit enters at time T, the growing time is only $t-T$. Therefore the multiplying factor is only $e^{c(t-T)}$. This growth factor applies to the small deposit (amount $s\,dT$) made between time T and $T+dT$.

Now add up all outputs at time t. The output from y_0 is y_0e^{ct}. The small deposit $s\,dT$ near time T grows to $e^{c(t-T)}s\,dT$. The total is an integral:

$$y(t) = y_0e^{ct} + \int_{T=0}^{t} e^{c(t-T)}s\,dT. \tag{9}$$

This principle of Duhamel would still apply when the source s varies with time. Here s is constant, and the integral divides by c:

$$s\int_{T=0}^{t} e^{c(t-T)}dT = \left.\frac{se^{c(t-T)}}{-c}\right]_0^t = -\frac{s}{c} + \frac{s}{c}e^{ct}. \tag{10}$$

That agrees with the source term from Method 1, at the end of equation (8). There we looked for "exponential plus constant," here we added up outputs.

Method 1 was easier. It succeeded because we knew the form $Ae^{ct}+B$—with "undetermined coefficients." Method 2 is more complete. The form for y is part of the output, not the input. The source s is a continuous supply of new deposits, all growing separately. Section 6.5 starts from scratch, by directly integrating $y' = cy + s$.

Remark Method 2 is often described in terms of an *integrating factor*. First write the equation as $y' - cy = s$. Then multiply by a magic factor that makes integration possible:

$(y'-cy)e^{-ct} = se^{-ct}$	multiply by the factor e^{-ct}
$\left.ye^{-ct}\right]_0^t = \left.-\frac{s}{c}\,e^{-ct}\right]_0^t$	integrate both sides
$ye^{-ct} - y_0 = -\frac{s}{c}\,(e^{-ct}-1)$	substitute 0 and t
$y = e^{ct}y_0 + \frac{s}{c}\,(e^{ct}-1)$	isolate y to reach formula (8)

The integrating factor produced a perfect derivative in line 1. I prefer Duhamel's idea, that all inputs y_0 and s grow the same way. Either method gives formula (8) for y.

THE MATHEMATICS OF FINANCE (AT A CONTINUOUS RATE)

The question from finance is this: *What inputs give what outputs*? The inputs can come at the start by y_0, or continuously by s. The output can be paid at the end or continuously. There are six basic questions, two of which are already answered.

The future value is y_0e^{ct} from a deposit of y_0. To produce y in the future, deposit the present value ye^{-ct}. *Questions* **3–6** *involve the source term s.* We fix the continuous rate at 5% per year ($c = .05$), and start the account from $y_0 = 0$. The answers come fast from equation (8).

Question 3 With deposits of $s = \$1000/\text{year}$, how large is y after 20 years?

$$y = \frac{s}{c}\,(e^{ct}-1) = \frac{1000}{.05}\,(e^{(.05)(20)}-1) = 20{,}000(e-1) \approx \$34{,}400.$$

One big deposit yields $20{,}000e \approx \$54{,}000$. The same $20{,}000$ via s yields $\$34{,}400$.

Notice a small by-product (for mathematicians). When the interest rate is $c = 0$, our formula $s(e^{ct} - 1)/c$ turns into $0/0$. We are absolutely sure that depositing $\$1000/\text{year}$ with no interest produces $\$20,000$ after 20 years. But this is not obvious from $0/0$. By l'Hôpital's rule we take c-derivatives in the fraction:

$$\lim_{c \to 0} \frac{s(e^{ct} - 1)}{c} = \lim_{c \to 0} \frac{ste^{ct}}{1} = st. \text{ This is } (1000)(20) = 20,000. \tag{11}$$

Question 4 What continuous deposit of s per year yields $\$20,000$ after 20 years?

$$20,000 = \frac{s}{.05}\,(e^{(.05)(20)} - 1) \text{ requires } s = \frac{1000}{e - 1} \approx 582.$$

Deposits of $\$582$ over 20 years total $\$11,640$. A single deposit of $y_0 = 20,000/e = \$7,360$ produces the same $\$20,000$ at the end. Better to be rich at $t = 0$.

Questions **1** and **2** had $s = 0$ (no source). Questions **3** and **4** had $y_0 = 0$ (no initial deposit). Now we come to $y = 0$. In **5**, everything is paid out by an ***annuity***. In **6**, everything is paid up on a ***loan***.

Question 5 What deposit y_0 provides $\$1000/\text{year}$ for 20 years? End with $y = 0$.

$$y = y_0 e^{ct} + \frac{s}{c}\,(e^{ct} - 1) = 0 \text{ requires } y_0 = \frac{-s}{c}\,(1 - e^{-ct}).$$

Substituting $s = -1000$, $c = .05$, $t = 20$ gives $y_0 \approx 12,640$. If you win $\$20,000$ in a lottery, and it is paid over 20 years, the lottery only has to put in $\$12,640$. Even less if the interest rate is above 5%.

Question 6 What payments s will clear a loan of $y_0 = \$20,000$ in 20 years?

Unfortunately, s exceeds \$1000 per year. The bank gives up more than the \$20,000 to buy your car (and pay tuition). *It also gives up the interest on that money*. You pay that back too, but you don't have to stay even at every moment. Instead you repay at a *constant rate* for 20 years. Your payments mostly cover interest at the start and principal at the end. After $t = 20$ years you are even and your debt is $y = 0$.

This is like Question **5** (also $y = 0$), but now we know y_0 and we want s:

$$y = y_0 e^{ct} + \frac{s}{c}\,(e^{ct} - 1) = 0 \text{ requires } s = -cy_0 e^{ct}/(e^{ct} - 1).$$

The loan is $y_0 = \$20,000$, the rate is $c = .05/\text{year}$, the time is $t = 20$ years. Substituting in the formula for s, your payments are \$1582 per year.

Puzzle How is $s = \$1582$ for loan payments related to $s = \$582$ for deposits?

$$0 \to \$582 \text{ per year} \to \$20,000 \quad \text{and} \quad \$20,000 \to -\$1582 \text{ per year} \to 0.$$

That difference of exactly 1000 cannot be an accident. 1582 and 582 came from

$$1000\,\frac{e}{e - 1} \text{ and } 1000\frac{1}{e - 1} \text{ with difference } 1000\frac{e - 1}{e - 1} = 1000.$$

Why? Here is the real reason. Instead of repaying 1582 we can pay only 1000 (to keep even with the interest on $20,000$). The other 582 goes into a separate account. After 20 years the continuous 582 has built up to $20,000$ (including interest as in Question **4**). From that account we pay back the loan.

Section 6.6 deals with daily compounding—which differs from continuous compounding by only a few cents. Yearly compounding differs by a few dollars.

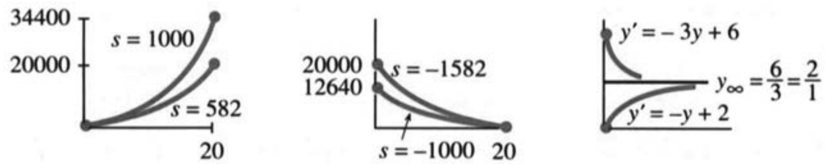


Fig. 6.10 Questions **3**−**4** deposit s. Questions **5**−**6** repay loan or annuity. Steady state $-s/c$.

TRANSIENTS VS. STEADY STATE

Suppose there is decay instead of growth. The constant c is negative and y_0e^{ct} dies out. That is the "*transient*" term, which disappears as $t \to \infty$. What is left is the "*steady state*." We denote that limit by y_∞.

Without a source, y_∞ is zero (total decay). When s is present, $y_\infty = -s/c$:

6J The solution $y = \left(y_0 + \frac{s}{c}\right)e^{ct} - \frac{s}{c}$ approaches $y_\infty = -\frac{s}{c}$ when $e^{ct} \to 0$.

At this steady state, the source s exactly balances the decay cy. In other words $cy + s = 0$. From the left side of the differential equation, this means $dy/dt = 0$. *There is no change*. That is why y_∞ is steady.

Notice that y_∞ depends on the source and on c—*but not on* y_0.

EXAMPLE 6 Suppose Bermuda has a birth rate $b = .02$ and death rate $d = .03$. The net decay rate is $c = -.01$. There is also immigration from outside, of $s = 1200/\text{year}$. The initial population might be $y_0 = 5$ thousand or $y_0 = 5$ million, but that number has no effect on y_∞. ***The steady state is independent of*** y_0.

In this case $y_\infty = -s/c = 1200/.01 = 120,000$. The population grows to $120,000$ if y_0 is smaller. It decays to $120,000$ if y_0 is larger.

EXAMPLE 7 ***Newton's Law of Cooling***: $dy/dt = c(y - y_\infty)$. (12)

This is back to physics. The temperature of a body is y. The temperature around it is y_∞. Then y starts at y_0 and approaches y_∞, following Newton's rule: *The rate is proportional to* $y - y_\infty$. The bigger the difference, the faster heat flows.

The equation has $-cy_\infty$ where before we had s. That fits with $y_\infty = -s/c$. For the solution, replace s by $-cy_\infty$ in formula (8). Or use this new method:

Solution by Method 3 ***The new idea is to look at the difference*** $y - y_\infty$. Its derivative is dy/dt, since y_∞ is constant. But dy/dt is $c(y - y_\infty)$—this is our equation. The difference starts from $y_0 - y_\infty$, and grows or decays as a pure exponential:

$$\frac{d}{dt}(y - y_\infty) = c(y - y_\infty) \quad \textit{has the solution} \quad (y - y_\infty) = (y_0 - y_\infty)e^{ct}. \quad (13)$$

This solves the law of cooling. We repeat Method 3 using the letters s and c:

$$\frac{d}{dt}\left(y + \frac{s}{c}\right) = c\left(y + \frac{s}{c}\right) \quad \textit{has the solution} \quad \left(y + \frac{s}{c}\right) = \left(y_0 + \frac{s}{c}\right)e^{ct}. \quad (14)$$

Moving s/c to the right side recovers formula (8). There is a *constant term* and an *exponential term*. In a differential equations course, those are the "***particular solution***" and the "***homogeneous solution***." In a calculus course, it's time to stop.

EXAMPLE 8 In a $70°$ room, Newton's corpse is found with a temperature of $90°$. A day later the body registers $80°$. When did he stop integrating (at $98.6°$)?

Solution Here $y_\infty = 70$ and $y_0 = 90$. Newton's equation (13) is $y = 20e^{ct} + 70$. Then $y = 80$ at $t = 1$ gives $20e^c = 10$. The rate of cooling is $c = \ln \frac{1}{2}$. Death occurred when $20e^{ct} + 70 = 98.6$ or $e^{ct} = 1.43$. The time was $t = \ln 1.43 / \ln \frac{1}{2} =$ half a day earlier.

6.3 EXERCISES

Read-through exercises

If $y' = cy$ then $y(t) =$ __a__. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t) =$ __b__. This solution reaches 8 at $t =$ __c__. If the doubling time is T then $c =$ __d__. If $y' = 3y$ and $y(1) = 9$ then y_0 was __e__. When c is negative, the solution approaches __f__ as $t \to \infty$.

The constant solution to $dy/dt = y + 6$ is $y =$ __g__. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then $A =$ __h__. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B =$ __i__. The output from the source s is __j__. An input at time T grows by the factor __k__ at time t.

At $c = 10\%$, the interest in time dt is $dy =$ __l__. This equation yields $y(t) =$ __m__. With a source term instead of y_0, a continuous deposit of $s = 4000/\text{year}$ yields $y =$ __n__ after 10 years. The deposit required to produce $10,000$ in 10 years is $s =$ __o__ (exactly or approximately). An income of $4000/\text{year}$ forever (!) comes from $y_0 =$ __p__. The deposit to give $4000/\text{year}$ for 20 years is $y_0 =$ __q__. The payment rate s to clear a loan of $10,000$ in 10 years is __r__.

The solution to $y' = -3y + s$ approaches $y_\infty =$ __s__.

Solve 1–4 starting from $y_0 = 1$ and from $y_0 = -1$. Draw both solutions on the same graph.

1 $\dfrac{dy}{dt} = 2t$ **2** $\dfrac{dy}{dt} = -t$ **3** $\dfrac{dy}{dt} = 2y$ **4** $\dfrac{dy}{dt} = -y$

Solve 5–8 starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

5 $\dfrac{dy}{dt} = 4y$ **6** $\dfrac{dy}{dt} = 4t$ **7** $\dfrac{dy}{dt} = e^{4t}$ **8** $\dfrac{dy}{dt} = e^{-4t}$

9 Draw a field of "tangent arrows" for $y' = -y$, with the solution curves $y = e^{-x}$ and $y = -e^{-x}$.

10 Draw a direction field of arrows for $y' = y - 1$, with solution curves $y = e^x + 1$ and $y = 1$.

Problems 11–27 involve $y_0 e^{ct}$. They ask for c or t or y_0.

11 If a culture of bacteria doubles in two hours, how many hours to multiply by 10? First find c.

12 If bacteria increase by factor of ten in ten hours, how many hours to increase by 100? What is c?

13 How old is a skull that contains $\frac{1}{5}$ as much radiocarbon as a modern skull?

14 If a relic contains 90% as much radiocarbon as new material, could it come from the time of Christ?

15 The population of Cairo grew from 5 million to 10 million in 20 years. From $y' = cy$ find c. When was $y = 8$ million?

16 The populations of New York and Los Angeles are growing at 1% and 1.4% a year. Starting from 8 million (NY) and 6 million (LA), when will they be equal?

17 Suppose the value of \$1 in Japanese yen decreases at 2% per year. Starting from \$1 = Y240, when will 1 dollar equal 1 yen?

18 The effect of advertising decays exponentially. If 40% remember a new product after three days, find c. How long will 20% remember it?

19 If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$ (exponential growth), what was y_0 at $t = 0$?

20 If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$ (exponential decay) when will $y = 1$? What was y_0?

21 Atmospheric pressure decreases with height according to $dp/dh = cp$. The pressures at $h = 0$ (sea level) and $h = 20$ km are 1013 and 50 millibars. Find c. Explain why $p = \sqrt{1013 \cdot 50}$ halfway up at $h = 10$.

22 For exponential decay show that $y(t)$ is the square root of $y(0)$ times $y(2t)$. How could you find $y(3t)$ from $y(t)$ and $y(2t)$?

23 Most drugs in the bloodstream decay by $y' = cy$ (***first-order kinetics***). (a) The half-life of morphine is 3 hours. Find its decay constant c (with units). (b) The half-life of nicotine is 2 hours. After a six-hour flight what fraction remains?

24 How often should a drug be taken if its dose is 3 mg, it is cleared at $c = .01$/hour, and 1 mg is required in the bloodstream at all times? (The doctor decides this level based on body size.)

25 The antiseizure drug dilantin has constant clearance rate $y' = -a$ until $y = y_1$. Then $y' = -ay/y_1$. Solve for $y(t)$ in two pieces from y_0. When does y reach y_1?

26 The actual elimination of nicotine is multiexponential: $y = Ae^{ct} + Be^{Ct}$. The first-order equation $(d/dt - c)y = 0$ changes to the second-order equation $(d/dt - c)(d/dt - C)y = 0$. Write out this equation starting with y'', and show that it is satisfied by the given y.

27 ***True or false*. If false, say what's true.**

(a) The time for $y = e^{ct}$ to double is $(\ln 2)/(\ln c)$.
(b) If $y' = cy$ and $z' = cz$ then $(y+z)' = 2c(y+z)$.
(c) If $y' = cy$ and $z' = cz$ then $(y/z)' = 0$.
(d) If $y' = cy$ and $z' = Cz$ then $(yz)' = (c+C)yz$.

28 A rocket has velocity v. Burnt fuel of mass Δm leaves at velocity $v - 7$. Total momentum is constant:

$$mv = (m - \Delta m)(v + \Delta v) + \Delta m(v - 7).$$

What differential equation connects m to v? Solve for $v(m)$ not $v(t)$, starting from $v_0 = 20$ and $m_0 = 4$.

Problems 29–36 are about solutions of $y' = cy + s$.

29 Solve $y' = 3y + 1$ with $y_0 = 0$ by assuming $y = Ae^{3t} + B$ and determining A and B.

30 Solve $y' = 8 - y$ starting from y_0 and $y = Ae^{-t} + B$.

Solve 31–34 with $y_0 = 0$ and graph the solution.

31 $\dfrac{dy}{dt} = y + 1$ **32** $\dfrac{dy}{dt} = y - 1$

33 $\dfrac{dy}{dt} = -y + 1$ **34** $\dfrac{dy}{dt} = -y - 1$

35 (a) What value $y =$ constant solves $dy/dt = -2y + 12$?
(b) Find the solution with an arbitrary constant A.
(c) What solutions start from $y_0 = 0$ and $y_0 = 10$?
(d) What is the steady state y_∞?

36 Choose $\pm$ signs in $dy/dt = \pm 3y \pm 6$ to achieve the following results starting from $y_0 = 1$. Draw graphs.

(a) y increases to ∞ (b) y increases to 2
(c) y decreases to -2 (d) y decreases to $-\infty$

37 What value $y =$ constant solves $dy/dt = 4 - y$? Show that $y(t) = Ae^{-t} + 4$ is also a solution. Find $y(1)$ and y_∞ if $y_0 = 3$.

38 Solve $y' = y + e^t$ from $y_0 = 0$ by Method 2, where the deposit e^T at time T is multiplied by e^{t-T}. The total output at time t is $y(t) = \int_0^t e^T e^{t-T}\, dT =$ ______. Substitute back to check $y' = y + e^t$.

39 Rewrite $y' = y + e^t$ as $y' - y = e^t$. Multiplying by e^{-t}, the left side is the derivative of ______. Integrate both sides from $y_0 = 0$ to find $y(t)$.

40 Solve $y' = -y + 1$ from $y_0 = 0$ by rewriting as $y' + y = 1$, multiplying by e^t, and integrating both sides.

41 Solve $y' = y + t$ from $y_0 = 0$ by assuming $y = Ae^t + Bt + C$.

Problems 42–57 are about the mathematics of finance.

42 Dollar bills decrease in value at $c = -.04$ per year because of inflation. If you hold \$1000, what is the decrease in dt years? At what rate s should you print money to keep even?

43 If a bank offers annual interest of $7\frac{1}{2}$% or continuous interest of $7\frac{1}{4}$%, which is better?

44 What continuous interest rate is equivalent to an annual rate of 9%? Extra credit: Telephone a bank for both rates and check their calculation.

45 At 100% interest ($c = 1$) how much is a continuous deposit of s per year worth after one year? What initial deposit y_0 would have produced the same output?

46 To have \$50,000 for college tuition in 20 years, what gift y_0 should a grandparent make now? Assume $c = 10\%$. What continuous deposit should a parent make during 20 years? If the parent saves $s = \$1000$ per year, when does he or she reach \$50,000 arid retire?

47 Income per person grows 3%, the population grows 2%, the total income grows ______. Answer if these are (a) annual rates (b) continuous rates.

48 When $dy/dt = cy + 4$, how much is the deposit of $4\,dT$ at time T worth at the later time t? What is the value at $t = 2$ of deposits $4\,dT$ from $T = 0$ to $T = 1$?

49 Depositing $s = \$1000$ per year leads to \$34,400 after 20 years (Question 3). To reach the same result, when should you deposit \$20,000 all at once?

50 For how long can you withdraw $s = \$500$/year after depositing $y_0 = \$5000$ at 8%, before you run dry?

51 What continuous payment s clears a \$1000 loan in 60 days, if a loan shark charges 1% per day continuously?

52 You are the loan shark. What is \$1 worth after a year of continuous compounding at 1% per day?

53 You can afford payments of $s = \$100$ per month for 48 months. If the dealer charges $c = 6\%$, how much can you borrow?

54 Your income is $I_0 e^{2ct}$ per year. Your expenses are $E_0 e^{ct}$ per year. (a) At what future time are they equal? (b) If you borrow the difference until then, how much money have you borrowed?

55 If a student loan in your freshman year is repaid plus 20% four years later, what was the effective interest rate ?

56 Is a variable rate mortgage with $c = .09 + .001t$ for 20 years better or worse than a fixed rate of 10% ?

57 At 10% instead of 8%, the \$24 paid for Manhattan is worth ______ after 365 years.

Problems 58–65 approach a steady state y_∞ as $t \to \infty$.

58 If $dy/dt = -y + 7$ what is y_∞ ? What is the derivative of $y - y_\infty$? Then $y - y_\infty$ equals $y_0 - y_\infty$ times ______ .

59 Graph $y(t)$ when $y' = 3y - 12$ and y_0 is

(a) below 4 (b) equal to 4 (c) above 4

60 The solutions to $dy/dt = c(y - 12)$ converge to $y_\infty =$ ______ provided c is ______ .

61 Suppose the time unit in $dy/dt = cy$ changes from minutes to hours. How does the equation change ? How does $dy/dt = -y + 5$ change ? How does y_∞ change ?

62 ***True or false***, when y_1 and y_2 both satisfy $y' = cy + s$.

(a) The sum $y = y_1 + y_2$ also satisfies this equation.
(b) The average $y = \frac{1}{2}(y_1 + y_2)$ satisfies the same equation.
(c) The derivative $y = y_1'$ satisfies the same equation.

63 If Newton's coffee cools from 80° to 60° in 12 minutes (room temperature 20°), find c. When was the coffee at 100° ?

64 If $y_0 = 100$ and $y(1) = 90$ and $y(2) = 84$, what is y_∞ ?

65 If $y_0 = 100$ and $y(1) = 90$ and $y(2) = 81$, what is y_∞ ?

66 To cool down coffee, should you add milk now or later ? The coffee is at 70°C, the milk is at 10°, the room is at 20°.

(a) Adding 1 part milk to 5 parts coffee makes it 60°. With $y_\infty = 20°$, the white coffee cools to $y(t) =$ ______ .

(b) The black coffee cools to $y_c(t) =$ ______ . The milk warms to $y_m(t) =$ ______ . Mixing at time t gives $(5y_c + y_m)/6 =$ ______ .

6.4 Logarithms

We have given first place to e^x and a lower place to $\ln x$. In applications that is absolutely correct. But logarithms have one important theoretical advantage (plus many applications of their own). The advantage is that the derivative of $\ln x$ is $1/x$, whereas the derivative of e^x is e^x. We can't define e^x as its own integral, without circular reasoning. But we can and do define $\ln x$ (the ***natural logarithm***) as the integral of the "-1 power" which is $1/x$:

$$\ln x = \int_1^x \frac{1}{x}\,dx \qquad \text{or} \qquad \ln y = \int_1^y \frac{1}{u}\,du. \tag{1}$$

Note the dummy variables, first x then u. Note also the live variables, first x then y. Especially note the lower limit of integration, which is 1 and not 0. ***The logarithm is the area measured from*** 1. Therefore $\ln 1 = 0$ at that starting point—as required.

Earlier chapters integrated all powers except this "-1 power." The logarithm is that missing integral. The curve in Figure 6.11 has height $y = 1/x$—it is a hyperbola. At $x = 0$ the height goes to infinity and the area becomes infinite: $\log 0 = -\infty$. The minus sign is because the integral goes backward from 1 to 0. The integral does not extend past zero to negative x. We are defining $\ln x$ only for $x > 0$.†

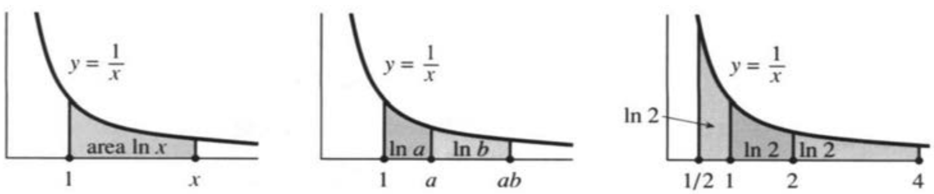


Fig. 6.11 ***Logarithm as area***. Neighbors $\ln a + \ln b = \ln ab$.
Equal areas: $-\ln\frac{1}{2} = \ln 2 = \frac{1}{2}\ln 4$.

With this new approach, $\ln x$ has a direct definition. *It is an integral* (or an area). Its two key properties must follow from this definition. That step is a beautiful application of the theory behind integrals.

***Property* 1**: $\ln ab = \ln a + \ln b$. The areas from 1 to a and from a to ab combine into a single area (1 to ab in the middle figure):

$$\text{Neighboring areas}: \int_1^a \frac{1}{x}\,dx + \int_a^{ab} \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx. \tag{2}$$

The right side is $\ln ab$, from definition (1). The first term on the left is $\ln a$. The problem is to show that the second integral (a to ab) is $\ln b$:

$$\int_a^{ab} \frac{1}{x}\,dx \overset{(?)}{=} \int_1^b \frac{1}{u}\,du \;=\; \ln b. \tag{3}$$

We need $u = 1$ when $x = a$ (the lower limit) and $u = b$ when $x = ab$ (the upper limit). The choice $u = x/a$ satisfies these requirements. Substituting $x = au$ and $dx = a\,du$ yields $dx/x = du/u$. Equation (3) gives $\ln b$, and equation (2) is $\ln a + \ln b = \ln ab$.

† The logarithm of -1 is πi (an imaginary number). That is because $e^{\pi i} = -1$. The logarithm of i is also imaginary—it is $\frac{1}{2}\pi i$. In general, logarithms are complex numbers.

***Property* 2**: $\ln b^n = n \ln b$. These are the left and right sides of

$$\int_1^{b^n} \frac{1}{x}\,dx \overset{(?)}{=} n\int_1^b \frac{1}{u}\,du. \tag{4}$$

This comes from the substitution $x = u^n$. The lower limit $x = 1$ corresponds to $u = 1$, and $x = b^n$ corresponds to $u = b$. The differential dx is $nu^{n-1}du$. Dividing by $x = u^n$ leaves $dx/x = n\ du/u$. Then equation (4) becomes $\ln b^n = n \ln b$.

Everything comes logically from the definition as an area. Also definite integrals:

EXAMPLE 1 Compute $\displaystyle\int_x^{3x} \frac{1}{t}\,dt$. Solution: $\ln 3x - \ln x = \ln \dfrac{3x}{x} = \ln 3$.

EXAMPLE 2 Compute $\displaystyle\int_{.1}^{1} \frac{1}{x}\,dx$. Solution: $\ln 1 - \ln .1 = \ln 10$. (Why ?)

EXAMPLE 3 Compute $\displaystyle\int_1^{e^2} \frac{1}{u}\,du$. Solution: $\ln e^2 = 2$. The area from 1 to e^2 is 2.

Remark While working on the theory this is a chance to straighten out old debts. The book has discussed and computed (and even differentiated) the functions e^x and b^x and x^n, without defining them properly. When the exponent is an irrational number like π, ***how do we multiply*** e ***by itself*** π ***times***? One approach (not taken) is to come closer and closer to π by rational exponents like $22/7$. Another approach (taken now) is to determine the number $e^\pi = 23.1 \ldots$ by its logarithm.† Start with e itself:

e is (by definition) the number whose logarithm is 1

e^π is (by definition) the number whose logarithm is π.

When the area in Figure 6.12 ***reaches*** 1, ***the basepoint is*** e. When the area reaches π, the basepoint is e^π. We are constructing the inverse function (which is e^x). But how do we know that the area reaches π or 1000 or -1000 at exactly one point ? (The area is 1000 far out at e^{1000}. The area is -1000 very near zero at e^{-1000}.) To define e we have to know that somewhere the area equals 1!

For a proof in two steps, go back to Figure 6.11c. The area from 1 to 2 is *more than* $\frac{1}{2}$ (because $1/x$ is more than $\frac{1}{2}$ on that interval of length one). The combined area from 1 to 4 is more than 1. ***We come to*** area $= 1$ ***before reaching*** 4. (Actually at $e = 2.718\ldots$.) Since $1/x$ is positive, the area is increasing and never comes back to 1.

To double the area we have to square the distance. The logarithm creeps upwards:

$$\ln x \to \infty \quad \text{but} \quad \frac{\ln x}{x} \to 0. \tag{5}$$

The logarithm grows slowly because e^x ***grows so fast*** (and vice versa—they are inverses). Remember that e^x goes past every power x^n. Therefore $\ln x$ is passed by every root $x^{1/n}$. Problems 60 and 61 give two proofs that $(\ln x)/x^{1/n}$ approaches zero.

We might compare $\ln x$ with $\sqrt{x}$. At $x = 10$ they are close (2.3 versus 3.2). But out at $x = e^{10}$ the comparison is 10 against e^5, and $\ln x$ loses to $\sqrt{x}$.

†Chapter 9 goes on to *imaginary exponents*, and proves the remarkable formula $e^{\pi i} = -1$.

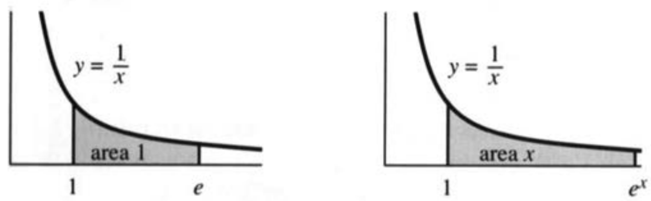


Fig. 6.12 Area is logarithm of basepoint.

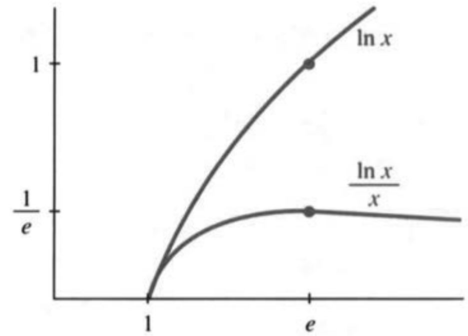


Fig. 6.13 $\ln x$ grows more slowly than x.

APPROXIMATION OF LOGARITHMS

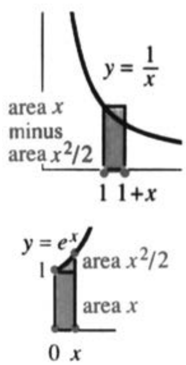


Fig. 6.14

The limiting cases $\ln 0 = -\infty$ and $\ln \infty = +\infty$ are important. More important are logarithms near the starting point $\ln 1 = 0$. Our question is: ***What is*** $\ln(1+x)$ ***for*** x ***near zero***? The exact answer is an area. The approximate answer is much simpler. If x (positive or negative) is small, then

$$\ln(1+x) \approx x \qquad \text{and} \qquad e^x \approx 1+x. \tag{6}$$

The calculator gives $\ln 1.01 = .0099503$. This is close to $x = .01$. Between 1 and $1+x$ the area under the graph of $1/x$ is nearly a rectangle. Its base is x and its height is 1. So the curved area $\ln(1+x)$ is close to the rectangular area x. Figure 6.14 shows how a small triangle is chopped off at the top.

The difference between .0099503 (actual) and .01 (linear approximation) is $-.0000497$. That is predicted almost exactly by the second derivative: $\frac{1}{2}$ times $(\Delta x)^2$ times $(\ln x)''$ is $\frac{1}{2}(.01)^2(-1) = -.00005$. ***This is the area of the small triangle***!

$$\ln(1+x) \approx \textbf{\textit{rectangular area minus triangular area}} = x - \tfrac{1}{2}x^2.$$

The remaining mistake of .0000003 is close to $\frac{1}{3}x^3$ (Problem 65).

May I switch to e^x? Its slope starts at $e^0 = 1$, so its linear approximation is $1+x$. Then $\ln(e^x) \approx \ln(1+x) \approx x$. *Two wrongs do make a right*: $\ln(e^x) = x$ exactly.

The calculator gives $e^{.01}$ as 1.0100502 (actual) instead of 1.01 (approximation). The second-order correction is again a small triangle: $\frac{1}{2}x^2 = .00005$. The complete series for $\ln(1+x)$ and e^x are in Sections 10.1 and 6.6:

$$\ln(1+x) = x - x^2/2 + x^3/3 - \ldots \qquad e^x = 1 + x + x^2/2 + x^3/6 + \ldots.$$

DERIVATIVES BASED ON LOGARITHMS

Logarithms turn up as antiderivatives very often. To build up a collection of integrals, we now differentiate $\ln u(x)$ by the chain rule.

> **6K** The derivative of $\ln x$ is $\dfrac{1}{x}$. ***The derivative of*** $\ln u(x)$ ***is*** $\dfrac{1}{u}\dfrac{du}{dx}$.

The slope of $\ln x$ was hard work in Section 6.2. With its new definition (the integral of $1/x$) the work is gone. By the Fundamental Theorem, the slope must be $1/x$.

For $\ln u(x)$ the derivative comes from the chain rule. The inside function is u, the outside function is $\ln$. (Keep $u > 0$ to define $\ln u$.) The chain rule gives

$$\frac{d}{dx}\ln cx = \frac{1}{cx}c = \frac{1}{x}(!) \qquad \frac{d}{dx}\ln x^3 = 3x^2/x^3 = \frac{3}{x}$$

$$\frac{d}{dx}\ln(x^2+1) = 2x/(x^2+1) \qquad \frac{d}{dx}\ln\cos x = \frac{-\sin x}{\cos x} = -\tan x$$

$$\frac{d}{dx}\ln e^x = e^x/e^x = 1 \qquad \frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{1}{x}.$$

Those are worth another look, especially the first. Any reasonable person would expect the slope of $\ln 3x$ to be $3/x$. ***Not so***. The 3 cancels, and $\ln 3x$ has the same slope as $\ln x$. (The real reason is that $\ln 3x = \ln 3 + \ln x$.) The antiderivative of $3/x$ is not $\ln 3x$ but $3 \ln x$, which is $\ln x^3$.

Before moving to integrals, here is a new method for derivatives: ***logarithmic differentiation*** or **LD**. It applies to *products* and *powers*. The product and power rules are always available, but sometimes there is an easier way.

Main idea: The logarithm of a product $p(x)$ is a *sum of logarithms*. Switching to $\ln p$, the sum rule just adds up the derivatives. But there is a catch at the end, as you see in the example.

EXAMPLE 4 Find dp/dx if $p(x) = x^x\sqrt{x-1}$. Here $\ln p(x) = x\ln x + \frac{1}{2}\ln(x-1)$.

Take the derivative of $\ln p$: $\quad \dfrac{1}{p}\dfrac{dp}{dx} = x\cdot\dfrac{1}{x} + \ln x + \dfrac{1}{2(x-1)}.$

Now multiply by $p(x)$: $\quad \dfrac{dp}{dx} = p\left(1 + \ln x + \dfrac{1}{2(x-1)}\right).$

The catch is that last step. Multiplying by p complicates the answer. This can't be helped—logarithmic differentiation contains no magic. The derivative of $p = fg$ is the same as from the product rule: $\ln p = \ln f + \ln g$ gives

$$\frac{p'}{p} = \frac{f'}{f} + \frac{g'}{g} \quad \text{and} \quad p' = p\left(\frac{f'}{f} + \frac{g'}{g}\right) = f'g + fg'. \tag{7}$$

For $p = xe^x \sin x$, with three factors, the sum has three terms:

$$\ln p = \ln x + x + \ln\sin x \text{ and } p' = p\left[\frac{1}{x} + 1 + \frac{\cos x}{\sin x}\right].$$

We multiply p ***times*** p'/p (the derivative of $\ln p$). ***Do the same for powers***:

EXAMPLE 5 $\quad p = x^{1/x} \Rightarrow \ln p = \dfrac{1}{x}\ln x \Rightarrow \dfrac{dp}{dx} = p\left[\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right].$

EXAMPLE 6 $\quad p = x^{\ln x} \Rightarrow \ln p = (\ln x)^2 \Rightarrow \dfrac{dp}{dx} = p\left[\dfrac{2\ln x}{x}\right].$

EXAMPLE 7 $\quad p = x^{1/\ln x} \Rightarrow \ln p = \dfrac{1}{\ln x}\ln x = 1 \Rightarrow \dfrac{dp}{dx} = 0 \quad (!)$

INTEGRALS BASED ON LOGARITHMS

Now comes an important step. Many integrals produce logarithms. The foremost example is $1/x$, whose integral is $\ln x$. In a certain way that is the only example, but its range is enormously extended by the chain rule. The derivative of $\ln u(x)$ is u'/u, so the integral goes from u'/u back to $\ln u$:

$$\int \frac{du/dx}{u(x)}\,dx = \ln u(x) \quad \text{or equivalently} \quad \int \frac{du}{u} = \ln u.$$

Try to choose $u(x)$ ***so that the integral contains*** du/dx ***divided by*** u.

EXAMPLES $$\int \frac{dx}{x+7} = \ln|x+7| \qquad \int \frac{dx}{cx+7} = \frac{1}{c}\ln|cx+7|$$

Final remark When u is negative, $\ln u$ cannot be the integral of $1/u$. The logarithm is not defined when $u<0$. But the integral can go forward by switching to $-u$:

$$\int \frac{du/dx}{u}\,dx = \int \frac{-du/dx}{-u}\,dx = \ln(-u). \tag{8}$$

Thus $\ln(-u)$ succeeds when $\ln u$ fails.† ***The forbidden case is*** $u=0$. The integrals $\ln u$ and $\ln(-u)$, on the plus and minus sides of zero, can be combined as $\ln|u|$. Every integral that gives a logarithm allows $u<0$ by changing to the absolute value $|u|$:

$$\int_{-e}^{-1} \frac{dx}{x} = \Big[\ln|x|\Big]_{-e}^{-1} = \ln 1 - \ln e \qquad \int_2^4 \frac{dx}{x-5} = \Big[\ln|x-5|\Big]_2^4 = \ln 1 - \ln 3.$$

The areas are -1 and $-\ln 3$. The graphs of $1/x$ and $1/(x-5)$ are below the x axis. We do *not* have logarithms of negative numbers, and we will not integrate $1/(x-5)$ from 2 to 6. That crosses the forbidden point $x=5$, with infinite area on both sides.

The ratio du/u leads to important integrals. When $u=\cos x$ or $u=\sin x$, we are integrating the ***tangent*** and ***cotangent***. When there is a possibility that $u<0$, write the integral as $\ln|u|$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| \qquad \int \frac{x\,dx}{x^2+7} = \frac{1}{2}\ln(x^2+7)$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| \qquad \int \frac{dx}{x\ln x} = \ln|\ln x|$$

Now we report on the ***secant*** and ***cosecant***. The integrals of $1/\cos x$ and $1/\sin x$ also surrender to an attack by logarithms—based on a crazy trick:

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx = \ln|\sec x+\tan x|. \tag{9}$$

$$\int \csc x\,dx = \int \csc x\left(\frac{\csc x-\cot x}{\csc x-\cot x}\right)dx = \ln|\csc x+\cot x|. \tag{10}$$

Here $u=\sec x+\tan x$ is in the denominator; $du/dx = \sec x\tan x+\sec^2 x$ is above it. The integral is $\ln|u|$. Similarly (10) contains du/dx over $u=\csc x-\cot x$.

†The integral of $1/x$ (odd function) is $\ln|x|$ (even function). Stay clear of $x=0$.

In closing we integrate $\ln x$ itself. The derivative of $x \ln x$ is $\ln x + 1$. To remove the extra 1, subtract x from the integral: $\int \ln x\, dx = x \ln x - x$.

In contrast, the area under $1/(\ln x)$ has no elementary formula. Nevertheless it is the key to the greatest approximation in mathematics—the ***prime number theorem***. *The area* $\int_a^b dx/\ln x$ *is approximately the number of primes between* a *and* b. Near e^{1000}, about $1/1000$ of the integers are prime.

6.4 EXERCISES

Read-through questions

The natural logarithm of x is $\int_1^x$ ___a___. This definition leads to $\ln xy =$ ___b___ and $\ln x^n =$ ___c___. Then e is the number whose logarithm (area under $1/x$ curve) is ___d___. Similarly e^x is now defined as the number whose natural logarithm is ___e___. As $x \to \infty$, $\ln x$ approaches ___f___. But the ratio $(\ln x)/\sqrt{x}$ approaches ___g___. The domain and range of $\ln x$ are ___h___.

The derivative of $\ln x$ is ___i___. The derivative of $\ln(1+x)$ is ___j___. The tangent approximation to $\ln(1+x)$ at $x=0$ is ___k___. The quadratic approximation is ___l___. The quadratic approximation to e^x is ___m___.

The derivative of $\ln u(x)$ by the chain rule is ___n___. Thus $(\ln \cos x)' =$ ___o___. An antiderivative of $\tan x$ is ___p___. The product $p = xe^{5x}$ has $\ln p =$ ___q___. The derivative of this equation is ___r___. Multiplying by p gives $p' =$ ___s___, which is **LD** or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is ___t___. The integral of $2x/(x^2+4)$ is ___u___. The integral of $1/cx$ is ___v___. The integral of $1/(ct+s)$ is ___w___. The integral of $1/\cos x$, after a trick, is ___x___. We should write $\ln|x|$ for the antiderivative of $1/x$, since this allows ___y___. Similarly $\int du/u$ should be written ___z___.

Find the derivative dy/dx in 1–10.

1 $y = \ln(2x)$

2 $y = \ln(2x+1)$

3 $y = (\ln x)^{-1}$

4 $y = (\ln x)/x$

5 $y = x \ln x - x$

6 $y = \log_{10} x$

7 $y = \ln(\sin x)$

8 $y = \ln(\ln x)$

9 $y = 7 \ln 4x$

10 $y = \ln((4x)^7)$

Find the indefinite (or definite) integral in 11–24.

11 $\displaystyle\int \frac{dt}{3t}$

12 $\displaystyle\int \frac{dx}{1+x}$

13 $\displaystyle\int_0^1 \frac{dx}{3+x}$

14 $\displaystyle\int_0^1 \frac{dt}{3+2t}$

15 $\displaystyle\int_0^2 \frac{x\,dx}{x^2+1}$

16 $\displaystyle\int_0^2 \frac{x^3\,dx}{x^2+1}$

17 $\displaystyle\int_2^e \frac{dx}{x(\ln x)}$

18 $\displaystyle\int_2^e \frac{dx}{x(\ln x)^2}$

19 $\displaystyle\int \frac{\cos x\,dx}{\sin x}$

20 $\displaystyle\int_0^{\pi/4} \tan x\,dx$

21 $\displaystyle\int \tan 3x\,dx$

22 $\displaystyle\int \cot 3x\,dx$

23 $\displaystyle\int \frac{(\ln x)^2 dx}{x}$

24 $\displaystyle\int \frac{dx}{x(\ln x)(\ln \ln x)}$

25 Graph $y = \ln(1+x)$

26 Graph $y = \ln(\sin x)$

Compute dy/dx by differentiating $\ln y$. This is LD:

27 $y = \sqrt{x^2+1}$

28 $y = \sqrt{x^2+1}\sqrt{x^2-1}$

29 $y = e^{\sin x}$

30 $y = x^{-1/x}$

31 $y = e^{(e^x)}$

32 $y = x^e$

33 $y = x^{(e^x)}$

34 $y = (\sqrt{x})(\sqrt[3]{x})(\sqrt[6]{x})$

35 $y = x^{-1/\ln x}$

36 $y = e^{-\ln x}$

Evaluate 37–42 by any method.

37 $\displaystyle\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

38 $\displaystyle\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

39 $\displaystyle\frac{d}{dx}\int_x^1 \frac{dt}{t}$

40 $\displaystyle\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$

41 $\displaystyle\frac{d}{dx}\ln(\sec x + \tan x)$

42 $\displaystyle\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

Verify the derivatives 43–46, which give useful antiderivatives:

43 $\displaystyle\frac{d}{dx}\ln(x+\sqrt{x^2+1}) = \frac{1}{\sqrt{1+x^2}}$

44 $\displaystyle\frac{d}{dx}\ln\left(\frac{x-a}{x+a}\right) = \frac{2a}{(x^2-a^2)}$

45 $\displaystyle\frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$

46 $\displaystyle\frac{d}{dx}\ln(x+\sqrt{x^2-a^2}) = \frac{1}{\sqrt{x^2-a^2}}$

Estimate 47–50 to linear accuracy, then quadratic accuracy, by $e^x \approx 1 + x + \frac{1}{2}x^2$. **Then use a calculator**.

47 $\ln(1.1)$ **48** $e^{\cdot 1}$ **49** $\ln(.99)$ **50** e^2

51 Compute $\lim_{x \to 0} \dfrac{\ln(1+x)}{x}$ **52** Compute $\lim_{x \to 0} \dfrac{e^x - 1}{x}$

53 Compute $\lim_{x \to 0} \dfrac{\log_b(1+x)}{x}$ **54** Compute $\lim_{x \to 0} \dfrac{b^x - 1}{x}$

55 Find the area of the "hyperbolic quarter-circle" enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

56 Estimate the area under $y = 1/x$ from 4 to 8 by four upper rectangles and four lower rectangles. Then average the answers (trapezoidal rule). What is the exact area?

57 Why is $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$ near $\ln n$? Is it above or below?

58 Prove that $\ln x \leqslant 2(\sqrt{x} - 1)$ for $x > 1$. Compare the integrals of $1/t$ and $1/\sqrt{t}$, from 1 to x.

59 Dividing by x in Problem 58 gives $(\ln x)/x \leqslant 2(\sqrt{x} - 1)/x$. Deduce that $(\ln x)/x \to 0$ as $x \to \infty$. Where is the maximum of $(\ln x)/x$?

60 Prove that $(\ln x)/x^{1/n}$ also approaches zero. (Start with $(\ln x^{1/n})/x^{1/n} \to 0$.) Where is its maximum?

61 For any power n, Problem 6.2.59 proved $e^x > x^n$ for large x. Then by logarithms, $x > n \ln x$. Since $(\ln x)/x$ goes below $1/n$ and stays below, it converges to ______.

62 Prove that $y \ln y$ approaches zero as $y \to 0$, by changing y to $1/x$. Find the limit of y^y (take its logarithm as $y \to 0$). What is $.1^{\cdot 1}$ on your calculator?

63 Find the limit of $\ln x / \log_{10} x$ as $x \to \infty$.

64 We know the integral $\int_1^x t^{h-1} dt = [t^h/h]_1^x = (x^h - 1)/h$. Its limit as $h \to 0$ is ______.

65 Find linear approximations near $x = 0$ for e^{-x} and 2^x.

66 The x^3 correction to $\ln(1+x)$ yields $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. Check that $\ln 1.01 \approx .0099503$ and find $\ln 1.02$.

67 An ant crawls at 1 foot/second along a rubber band whose original length is 2 feet. The band is being stretched at 1 foot/second by pulling the other end. At what time T, *if ever*, does the ant reach the other end?

One approach: The band's length at time t is $t + 2$. Let $y(t)$ be the *fraction* of that length which the ant has covered, and explain

(a) $y' = 1/(t+2)$ (b) $y = \ln(t+2) - \ln 2$ (c) $T = 2e - 2$.

68 If the rubber band is stretched at 8 feet/second, when if ever does the same ant reach the other end?

69 A weaker ant slows down to $2/(t+2)$ feet/second, so $y' = 2/(t+2)^2$. Show that the other end is never reached.

70 The slope of $p = x^x$ comes two ways from $\ln p = x \ln x$:

1 Logarithmic differentiation (**LD**): Compute $(\ln p)'$ and multiply by p.

2 Exponential differentiation (**ED**): Write x^x as $e^{x \ln x}$, take its derivative, and put back x^x.

71 If $p = 2^x$ then $\ln p =$ ______. **LD** gives $p' = (p)(\ln p)' =$ ______. **ED** gives $p = e$ and then $p' =$ ______.

72 Compute $\ln 2$ by the trapezoidal rule and/or Simpson's rule, to get five correct decimals.

73 Compute $\ln 10$ by either rule with $\Delta x = 1$, and compare with the value on your calculator.

74 Estimate $1/\ln 90{,}000$, the fraction of numbers near $90{,}000$ that are prime. (879 of the next $10{,}000$ numbers are actually prime.)

75 Find a pair of positive integers for which $x^y = y^x$. Show how to change this equation to $(\ln x)/x = (\ln y)/y$. So look for two points at the same height in Figure 6.13. Prove that you have discovered all the integer solutions.

*__76__ Show that $(\ln x)/x = (\ln y)/y$ is satisfied by

$$x = \left(\frac{t+1}{t}\right)^t \text{ and } y = \left(\frac{t+1}{1}\right)^{t+1}$$

with $t \neq 0$. Graph those points to show the curve $x^y = y^x$. It crosses the line $y = x$ at $x =$ ______, where $t \to \infty$.

6.5 Separable Equations Including the Logistic Equation

This section begins with the integrals that solve two basic differential equations:

$$\frac{dy}{dt} = cy \qquad \text{and} \qquad \frac{dy}{dt} = cy + s. \tag{1}$$

We already know the solutions. What we don't know is how to discover those solutions, when a suggestion "*try* e^{ct}" has not been made. Many important equations, including these, separate into a y-integral and a t-integral. The answer comes directly from the two separate integrations. When a differential equation is reduced that far—to integrals that we know or can look up—it is solved.

One particular equation will be emphasized. The ***logistic equation*** describes the speedup and slowdown of growth. Its solution is an **S-*curve***, which starts slowly, rises quickly, and levels off. (The 1990's are near the middle of the **S**, if the prediction is correct for the world population.) **S**-curves are solutions to ***nonlinear*** equations, and we will be solving our first nonlinear model. It is highly important in biology and all life sciences.

SEPARABLE EQUATIONS

The equations $dy/dt = cy$ and $dy/dt = cy + s$ (with constant source s) can be solved by a direct method. ***The idea is to separate y from t***:

$$\frac{dy}{y} = c\ dt \qquad \text{and} \qquad \frac{dy}{y + (s/c)} = c\ dt. \tag{2}$$

All y's are on the left side. All t's are on the right side (and c can be on either side). This separation would not be possible for $dy/dt = y + t$.

Equation (2) contains differentials. They suggest integrals. The t-integrals give ct and the y-integrals give logarithms:

$$\ln y = ct + \text{constant} \qquad \text{and} \qquad \ln\left(y + \frac{s}{c}\right) = ct + \text{constant}. \tag{3}$$

The constant is determined by the initial condition. At $t = 0$ we require $y = y_0$, and the right constant will make that happen:

$$\ln y = ct + \ln y_0 \qquad \text{and} \qquad \ln\left(y + \frac{s}{c}\right) = ct + \ln\left(y_0 + \frac{s}{c}\right). \tag{4}$$

Then the final step isolates y. The goal is a formula for y itself, not its logarithm, so take the exponential of both sides ($e^{\ln y}$ is y):

$$y = y_0 e^{ct} \qquad \text{and} \qquad y + \frac{s}{c} = \left(y_0 + \frac{s}{c}\right) e^{ct}. \tag{5}$$

It is wise to substitute y back into the differential equation, as a check.

This is our fourth method for $y' = cy + s$. Method 1 assumed from the start that $y = Ae^{ct} + B$. Method 2 multiplied all inputs by their growth factors $e^{c(t-T)}$ and added up outputs. Method 3 solved for $y - y_\infty$. Method 4 is ***separation of variables*** (and all methods give the same answer). This separation method is so useful that we repeat its main idea, and then explain it by using it.

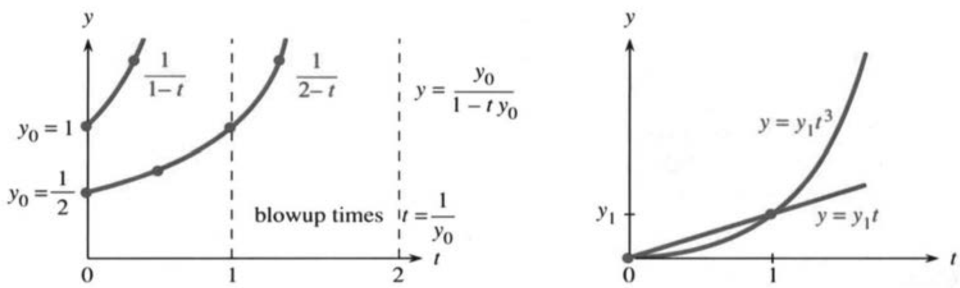


Fig. 6.15 The solutions to separable equations $\frac{dy}{dt} = y^2$ and $\frac{dy}{dt} = n\frac{y}{t}$ or $\frac{dy}{y} = n\frac{dt}{t}$.

To solve $dy/dt = u(y)v(t)$, ***separate*** $dy/u(y)$ ***from*** $v(t)dt$ ***and integrate both sides:***

$$\int dy/u(y) = \int v(t)dt + C. \tag{6}$$

Then substitute the initial condition to determine C, ***and solve for*** $y(t)$.

EXAMPLE 1 $dy/dt = y^2$ separates into $dy/y^2 = dt$. Integrate to reach $-1/y = t + C$. Substitute $t = 0$ and $y = y_0$ to find $C = -1/y_0$. Now solve for y:

$$-\frac{1}{y} = t - \frac{1}{y_0} \qquad \text{and} \qquad y = \frac{y_0}{1 - ty_0}.$$

This solution blows up (Figure 6.15a) when t reaches $1/y_0$. If the bank pays interest on your deposit *squared* ($y' = y^2$), you soon have all the money in the world.

EXAMPLE 2 $dy/dt = ty$ separates into $dy/y = t\ dt$. Then by integration $\ln y = \frac{1}{2}t^2 + C$. Substitute $t = 0$ and $y = y_0$ to find $C = \ln y_0$. The exponential of $\frac{1}{2}t^2 + \ln y_0$ gives $y = y_0 e^{t^2/2}$. When the interest rate is $c = t$, the exponent is $t^2/2$.

EXAMPLE 3 $dy/dt = y + t$ is *not separable*. Method 1 survives by assuming $y = Ae^t + B + Dt$—with an extra coefficient D in Problem 23. Method 2 also succeeds—but not the separation method.

EXAMPLE 4 Separate $dy/dt = ny/t$ into $dy/y = n\ dt/t$. By integration $\ln y = n \ln t + C$. Substituting $t = 0$ produces $\ln 0$ and disaster. This equation cannot start from time zero (it divides by t). However y can start from y_1 at $t = 1$, which gives $C = \ln y_1$. ***The solution is a power function*** $y = y_1 t^n$.

This was the first differential equation in the book (Section 2.2). The ratio of dy/y to dt/t is the "*elasticity*" in economics. These relative changes have units like dollars/dollars—they are dimensionless, and $y = t^n$ has constant elasticity n.

On log–log paper the graph of $\ln y = n \ln t + C$ is a *straight line with slope* n.

THE LOGISTIC EQUATION

The simplest model of population growth is $dy/dt = cy$. The growth rate c is the birth rate minus the death rate. If c is constant the growth goes on forever—beyond the point where the model is reasonable. A population can't grow all the way to infinity! Eventually there is competition for food and space, and $y = e^{ct}$ must slow down.

The true rate c ***depends on the population size*** y. It is a function $c(y)$ not a constant. The choice of the model is at least half the problem:

Problem in biology or ecology: Discover $c(y)$.

Problem in mathematics: Solve $dy/dt = c(y)y$.

Every model looks linear over a small range of y's—but not forever. When the rate drops off, two models are of the greatest importance. The *Michaelis-Menten* equation has $c(y) = c/(y+K)$. The *logistic* equation has $c(y) = c - by$. It comes first.

The nonlinear effect is from "interaction." For two populations of size y and z, the number of interactions is proportional to y times z. ***The Law of Mass Action produces a quadratic term*** byz. It is the basic model for interactions and competition. Here we have one population competing within itself, so z is the same as y. This competition slows down the growth, because $-by^2$ goes into the equation.

The basic model of *growth versus competition* is known as the ***logistic equation***:

$$dy/dt = cy - by^2. \tag{7}$$

Normally b is very small compared to c. The growth begins as usual (close to e^{ct}). The competition term by^2 is much smaller than cy, *until* y *itself gets large*. Then by^2 (with its minus sign) slows the growth down. The solution follows an **S**-curve that we can compute exactly.

What are the numbers b and c for human population? Ecologists estimate the natural growth rate as $c = .029/\text{year}$. That is not the actual rate, because of b. About 1930, the world population was 3 billion. The cy term predicts a yearly increase of $(.029)(3\text{ billion}) = 87\text{ million}$. The actual growth was more like $dy/dt = 60\text{ million/year}$. That difference of 27 million/year was by^2:

$$27\text{ million/year} = b(3\text{ billion})^2 \text{ leads to } b = 3\cdot 10^{-12}/\text{year}.$$

Certainly b is a small number (three trillionths) but its effect is not small. It reduces 87 to 60. What is fascinating is to calculate the ***steady state***, when the new term by^2 equals the old term cy. When these terms cancel each other, $dy/dt = cy - by^2$ is zero. The loss from competition balances the gain from new growth: $cy = by^2$ and $y = c/b$. The growth stops at this equilibrium point—the top of the **S**-curve:

$$y_\infty = \frac{c}{b} = \frac{.029}{3}10^{12} \approx 10\text{ billion people.}$$

According to Verhulst's logistic equation, *the world population is converging to* 10 *billion*. That is from the model. From present indications we are growing much faster. We will very probably go beyond 10 billion. The United Nations report in Section 3.3 predicts 11 billion to 14 billion.

Notice a special point halfway to $y_\infty = c/b$. (In the model this point is at 5 billion.) It is the *inflection point* where the **S**-curve begins to bend down. The second derivative d^2y/dt^2 is zero. The slope dy/dt is a maximum. It is easier to find this point from the differential equation (which gives dy/dt) than from y. Take one more derivative:

$$y'' = (cy - by^2)' = cy' - 2byy' = (c - 2by)y'. \tag{8}$$

The factor $c - 2by$ is zero at the inflection point $y = c/2b$, halfway up the **S**-curve.

THE S-CURVE

The logistic equation is solved by separating variables y and t:

$$dy/dt = cy - by^2 \text{ becomes } \int dy/(cy - by^2) = \int dt. \tag{9}$$

The first question is whether we recognize this y-integral. *No*. The second question is whether it is listed in the cover of the book. *No*. The nearest is $\int dx/(a^2 - x^2)$, which can be reached with considerable manipulation (Problem 21). The third question is whether a general method is available. *Yes*. "Partial fractions" is perfectly suited to $1/(cy - by^2)$, and Section 7.4 gives the following integral of equation (9):

$$\ln \frac{y}{c - by} = ct + C \qquad \text{and then} \qquad \ln \frac{y_0}{c - by_0} = C. \tag{10}$$

That constant C makes the solution correct at $t = 0$. The logistic equation is integrated, but the solution can be improved. Take exponentials of both sides to remove the logarithms:

$$\frac{y}{c - by} = e^{ct} \frac{y_0}{c - by_0}. \tag{11}$$

This contains the same growth factor e^{ct} as in linear equations. But the logistic equation is not linear—it is not y that increases so fast. According to (11), it is $y/(c - by)$ that grows to infinity. This happens when $c - by$ approaches zero.

The growth stops at $y = c/b$. That is the final population of the world (10 billion ?).

We still need a formula for y. *The perfect* **S**-*curve is the graph of* $y = 1/(1 + e^{-t})$. It equals 1 when $t = \infty$, it equals $\frac{1}{2}$ when $t = 0$, it equals 0 when $t = -\infty$. It satisfies $y' = y - y^2$, with $c = b = 1$. The general formula cannot be so beautiful, because it allows any c, b, and y_0. To find the **S**-curve, multiply equation (11) by $c - by$ and solve for y:

$$y = \frac{c}{b + e^{-ct}(c - by_0)/y_0} \qquad \text{or} \qquad y = \frac{c}{b + de^{-ct}}. \tag{12}$$

When t approaches infinity, e^{-ct} approaches zero. The complicated part of the formula disappears. Then y approaches its steady state c/b, the asymptote in Figure 6.16. The **S**-shape comes from the inflection point halfway up.

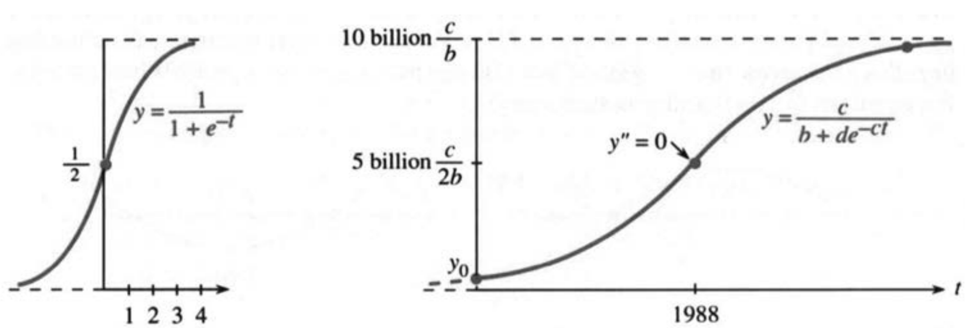


Fig. 6.16 The standard **S**-curve $y = 1/(1 + e^{-t})$. The population **S**-curve (with prediction).

Surprising observation: $z = 1/y$ *satisfies a linear equation*. By calculus $z' = -y'/y^2$. So

$$z' = \frac{-cy + by^2}{y^2} = -\frac{c}{y} + b = -cz + b. \tag{13}$$

This equation $z' = -cz + b$ is solved by an exponential e^{-ct} plus a constant:

$$z = Ae^{-ct} + \frac{a}{b} = \left(\frac{1}{y_0} - \frac{b}{c}\right)e^{-ct} + \frac{b}{c}. \tag{14}$$

Turned upside down, $y = 1/z$ is the **S**-curve (12). As z approaches b/c, the **S**-curve approaches c/b. Notice that z starts at $1/y_0$.

Year	US Population		Model
1790	3.9	=	3.9
1800	5.3		5.3
1810	7.2		7.2
1820	9.6		9.8
1830	12.9		13.1
1840	17.1		17.5
1850	23.2	=	23.2
1860	31.4		30.4
1870	38.6		39.4
1880	50.2		50.2
1890	62.9		62.8
1900	76.0		76.9
1910	92.0	=	92.0
1920	105.7		107.6
1930	122.8		123.1
1940	131.7	≠	136.7
1950	150.7		149.1

EXAMPLE 1 (**United States population**) The table shows the actual population and the model. Pearl and Reed used census figures for $1790, 1850$, and 1910 to compute c and b. In between, the fit is good but not fantastic. One reason is war—another is depression. Probably more important is immigration.† In fact the Pearl-Reed steady state c/b is below 200 million, which the US has already passed. Certainly their model can be and has been improved. ***The*** 1990 ***census predicted a stop before*** 300 ***million***. For constant immigration s we could still solve $y' = cy - by^2 + s$ by partial fractions—but in practice the computer has taken over. The table comes from Braun's book *Differential Equations* (Springer 1975).

Remark For good science the y^2 term should be explained and justified. It gave a nonlinear model that could be completely solved, but simplicity is not necessarily truth. The basic justification is this: In a population of size y, *the number of encounters is proportional to* y^2. If those encounters are fights, the term is $-by^2$. If those encounters *increase* the population, as some like to think, the sign is changed. There is a cooperation term $+by^2$, and the population increases very fast.

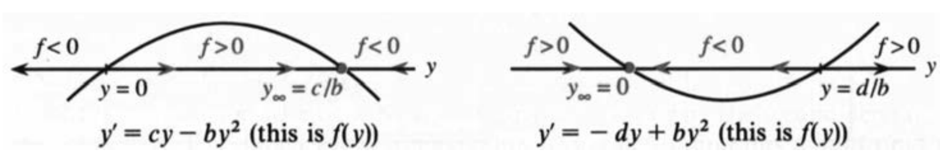
EXAMPLE 5 $y' = cy + by^2$: *y goes to infinity in a finite time.*

EXAMPLE 6 $y' = -dy + by^2$: *y dies to zero if $y_0 < d/b$.*

In Example 6 death wins. A small population dies out before the cooperation by^2 can save it. A population below d/b is an endangered species.

The logistic equation can't predict oscillations—those go beyond $dy/dt = f(y)$.

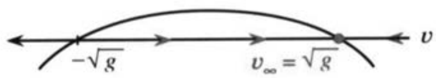
The y line Here is a way to understand every nonlinear equation $y' = f(y)$. Draw a "y *line*." Add arrows to show the sign of $f(y)$. When $y' = f(y)$ is positive, y is increasing (***it follows the arrow to the right***). When f is negative, y goes to the left. When f is zero, the equation is $y' = 0$ and y is stationary:



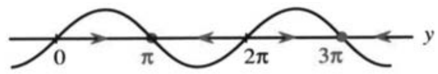
The arrows take you left or right, to the steady state or to infinity. Arrows go *toward* stable steady states. The arrows go *away,* when the stationary point is unstable. The ***y line*** shows which way y moves and where it stops.

† Immigration does not enter for the world population model (at least not yet).

The terminal velocity of a falling body is $v_\infty = \sqrt{g}$ in Problem 6.7.54. For $f(y) = \sin y$ there are several steady states:



falling body: $dv/dt = g - v^2$



$dy/dt = \sin y$

EXAMPLE 7 Kinetics of a chemical reaction $mA + nB \to pC$.

The reaction combines m molecules of A with n molecules of B to produce p molecules of C. The numbers m, n, p are $1, 1, 2$ for hydrogen chloride: $H_2 + Cl_2 = 2\,HCl$. The ***Law of Mass Action*** says that the reaction rate is proportional to the product of the concentrations $[A]$ and $[B]$. Then $[A]$ decays as $[C]$ grows:

$$d[A]/dt = -r[A][B] \qquad \text{and} \qquad d[C]/dt = +k[A][B]. \tag{15}$$

Chemistry measures r and k. Mathematics solves for $[A]$ and $[C]$. Write y for the concentration $[C]$, the number of molecules in a unit volume. Forming those y molecules drops the concentration $[A]$ from a_0 to $a_0 - (m/p)y$. Similarly $[B]$ drops from b_0 to $b_0 - (n/p)y$. The mass action law (15) contains y^2:

$$\frac{dy}{dt} = k\left(a_0 - \frac{m}{p}y\right)\left(b_0 - \frac{n}{p}y\right). \tag{16}$$

This fits our nonlinear model (Problem $33-34$). We now find this same mass action in biology. You recognize it whenever there is a product of two concentrations.

THE MM EQUATION $dy/dt = -cy/(y+K)$

Biochemical reactions are the keys to life. They take place continually in every living organism. Their mathematical description is not easy! Engineering and physics go far with linear models, while biology is quickly nonlinear. It is true that $y' = cy$ is extremely effective in first-order kinetics (Section 6.3), but nature builds in a nonlinear regulator.

It is ***enzymes*** that speed up a reaction. Without them, your life would be in slow motion. Blood would take years to clot. Steaks would take decades to digest. Calculus would take centuries to learn. The whole system is awesomely beautiful—DNA tells amino acids how to combine into useful proteins, and we get enzymes and elephants and Isaac Newton.

Briefly, the enzyme enters the reaction and comes out again. It is the ***catalyst***. Its combination with the substrate is an unstable intermediate, which breaks up into a new product and the enzyme (which is ready to start over).

Here are examples of catalysts, some good and some bad.

1. The platinum in a catalytic converter reacts with pollutants from the car engine. (But platinum also reacts with lead—ten gallons of leaded gasoline and you can forget the platinum.)
2. Spray propellants (CFC's) catalyze the change from ozone (O_3) into ordinary oxygen (O_2). This wipes out the ozone layer—our shield in the atmosphere.
3. Milk becomes yoghurt and grape juice becomes wine.

4. Blood clotting needs a whole cascade of enzymes, amplifying the reaction at every step. In hemophilia—the "Czar's disease"—the enzyme called Factor VIII is missing. A small accident is disaster; the bleeding won't stop.

5. Adolph's Meat Tenderizer is a protein from papayas. It predigests the steak. The same enzyme (chymopapain) is injected to soften herniated disks.

6. Yeast makes bread rise. Enzymes put the sour in sourdough.

Of course, it takes enzymes to make enzymes. The maternal egg contains the material for a cell, and also half of the DNA. The fertilized egg contains the full instructions.

We now look at the Michaelis–Menten (MM) equation, to describe these reactions. It is based on the ***Law of Mass Action***. An enzyme in concentration z converts a substrate in concentration y by $dy/dt = -byz$. The rate constant is b, and you see the product of "enzyme times substrate." A similar law governs the other reactions (some go backwards). The equations are nonlinear, with no exact solution. It is typical of applied mathematics (and nature) that a pattern can still be found.

What happens is that the enzyme concentration $z(t)$ quickly drops to $z_0 K/(y + K)$. The *Michaelis constant* K depends on the rates (like b) in the mass action laws. Later the enzyme reappears ($z_\infty = z_0$). But by then the first reaction is over. Its law of mass action is effectively

$$\frac{dy}{dt} = -byz = -\frac{cy}{y+K} \tag{17}$$

with $c = bz_0 K$. This is the ***Michaelis–Menten equation***—basic to biochemistry.

The rate dy/dt is all-important in biology. Look at the function $cy/(y+K)$:

when y is large, $dy/dt \approx -c$ when y is small, $dy/dt \approx -cy/K$.

The start and the finish operate at different rates, depending whether y dominates K or K dominates y. The fastest rate is c.

A biochemist solves the MM equation by separating variables:

$$\int \frac{y+K}{y}\, dy = -\int c\ dt \quad \text{gives} \quad y + K \ln y = -ct + C. \tag{18}$$

Set $t = 0$ as usual. Then $C = y_0 + K \ln y_0$. The exponentials of the two sides are

$$e^y y^K = e^{-ct} e^{y_0} y_0^K. \tag{19}$$

We don't have a simple formula for y. We are lucky to get this close. A computer can quickly graph $y(t)$—and we see the dynamics of enzymes.

Problems $27-32$ follow up the Michaelis–Menten theory. In science, concentrations and rate constants come with units. In mathematics, variables can be made dimensionless and constants become 1. We solve $dY/dT = Y/(Y+1)$ and then switch back to y, t, c, K. This idea applies to other equations too.

Essential point: ***Most applications of calculus come through differential equations***. That is the language of mathematics—with populations and chemicals and epidemics obeying the same equation. Running parallel to $dy/dt = cy$ are the difference equations that come next.

6.5 EXERCISES

Read-through questions

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called __a__ because we can separate y from t. Integration of $\int dy/y = \int c\,dt$ gives __b__. Integration of $\int dy/(y+s/c) = \int c\,dt$ gives __c__. The equation $dy/dx = -x/y$ leads to __d__. Then $y^2 + x^2 =$ __e__ and the solution stays on a circle.

The logistic equation is $dy/dt =$ __f__. The new term $-by^2$ represents __g__ when cy represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the y-integral is $1/c$ times ln __h__. Substituting y_0 at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}$(__i__). As $t \to \infty$, y approaches __j__. That is the steady state where $cy - by^2 =$ __k__. The graph of y looks like an __l__, because it has an inflection point at $y =$ __m__.

In biology and chemistry, concentrations y and z react at a rate proportional to y times __n__. This is the Law of __o__. In a model equation $dy/dt = c(y)y$, the rate c depends on __p__. The MM equation is $dy/dt =$ __q__. Separating variables yields $\int$ __r__ $dy =$ __s__ $= -ct + C$.

Separate, integrate, and solve equations 1–8.

1 $dy/dt = y + 5, \quad y_0 = 2$

2 $dy/dt = 1/y, \quad y_0 = 1$

3 $dy/dx = x/y^2, \quad y_0 = 1$

4 $dy/dx = y^2 + 1, \quad y_0 = 0$

5 $dy/dx = (y+1)/(x+1), \quad y_0 = 0$

6 $dy/dx = \tan y \cos x, \quad y_0 = 1$

7 $dy/dt = y \sin t, \quad y_0 = 1$

8 $dy/dt = e^{t-y}, \quad y_0 = e$

9 Suppose the rate of growth is proportional to $\sqrt{y}$ instead of y. Solve $dy/dt = c\sqrt{y}$ starting from y_0.

10 The equation $dy/dx = ny/x$ for constant elasticity is the same as $d(\ln y)/d(\ln x) =$ ______. The solution is $\ln y =$ ______.

11 When $c = 0$ in the logistic equation, the only term is $y' = -by^2$. What is the steady state y_∞? How long until y drops from y_0 to $\frac{1}{2}y_0$?

12 Reversing signs in Problem 11, suppose $y' = +by^2$. At what time does the population explode to $y = \infty$, starting from $y_0 = 2$ (*Adam+Eve*)?

Problems 13–26 deal with logistic equations $y' = cy - by^2$.

13 Show that $y = 1/(1+e^{-t})$ solves the equation $y' = y - y^2$. Draw the graph of y from starting values $\frac{1}{2}$ and $\frac{1}{3}$.

14 (a) What logistic equation is solved by $y = 2/(1+e^{-t})$?

(b) Find c and b in the equation solved by $y = 1/(1+e^{-3t})$.

15 Solve $z' = -z + 1$ with $z_0 = 2$. Turned upside down as in (13), what is $y = 1/z$?

16 By algebra find the **S**-curve (12) from $y = 1/z$ in (14).

17 How many years to grow from $y_0 = \frac{1}{2}c/b$ to $y = \frac{3}{4}c/b$? Use equation (10) for the time t since the inflection point in 1988. When does y reach 9 billion $= .9c/b$?

18 Show by differentiating $u = y/(c - by)$ that if $y' = cy - by^2$ then $u' = cu$. This explains the logistic solution (11)—it is $u = u_0 e^{ct}$.

19 Suppose Pittsburgh grows from $y_0 = 1$ million people in 1900 to $y = 3$ million in the year 2000. If the growth rate is $y' = 12{,}000$/year in 1900 and $y' = 30{,}000$/year in 2000, substitute in the logistic equation to find c and b. What is the steady state? Extra credit: When does $y = y_\infty/2 = c/2b$?

20 Suppose $c = 1$ but $b = -1$, giving cooperation $y' = y + y^2$. Solve for $y(t)$ if $y_0 = 1$. When does y become infinite?

21 Draw an **S**-curve through $(0,0)$ with horizontal asymptotes $y = -1$ and $y = 1$. Show that $y = (e^t - e^{-t})/(e^t + e^{-t})$ has those three properties. The graph of y^2 is shaped like ______.

22 To solve $y' = cy - by^3$ change to $u = 1/y^2$. Substitute for y' in $u' = -2y'/y^3$ to find a linear equation for u. Solve it as in (14) but with $u_0 = 1/y_0^2$. Then $y = 1/\sqrt{u}$.

23 With $y = rY$ and $t = sT$, the equation $dy/dt = cy - by^2$ changes to $dY/dT = Y - Y^2$. Find r and s.

24 In a change to $y = rY$ and $t = sT$, how are the initial values y_0 and y_0' related to Y_0 and Y_0'?

25 A rumor spreads according to $y' = y(N - y)$. If y people know, then $N - y$ don't know. The product $y(N - y)$ measures the number of meetings (to pass on the rumor).

(a) Solve $dy/dt = y(N - y)$ starting from $y_0 = 1$.

(b) At what time T have $N/2$ people heard the rumor?

(c) This model is terrible because T goes to ______ as $N \to \infty$. A better model is $y' = by(N - y)$.

26 Suppose b and c are both multiplied by 10. Does the middle of the **S**-curve get steeper or flatter?

Problems 27–34 deal with mass action and the MM equation $y' = -cy/(y + K)$.

27 Most drugs are eliminated according to $y' = -cy$ but aspirin follows the MM equation. With $c = K = y_0 = 1$, does aspirin decay faster?

28 If you take aspirin at a constant rate d (the maintenance dose), find the steady state level where $d = cy/(y + K)$. Then $y' = 0$.

29 Show that the rate $R = cy/(y+K)$ in the MM equation increases as y increases, and find the maximum as $y \to \infty$.

30 Graph the rate R as a function of y for $K = 1$ and $K = 10$. (Take $c = 1$.) As the Michaelis constant increases, the rate ______. *At what value of y is $R = \frac{1}{2}c$?*

31 With $y = KY$ and $ct = KT$, find the "nondimensional" MM equation for dY/dT. From the solution $e^Y Y = e^{-T} e^{Y_0} Y_0$ recover the y, t solution (19).

32 Graph $y(t)$ in (19) for different c and K (by computer).

33 The Law of Mass Action for $A + B \to C$ is $y' = k(a_0 - y)(b_0 - y)$. Suppose $y_0 = 0$, $a_0 = b_0 = 3$, $k = 1$. Solve for y and find the time when $y = 2$.

34 In addition to the equation for $d[C]/dt$, the mass action law gives $d[A]/dt =$ ______.

35 Solve $y' = y + t$ from $y_0 = 0$ by assuming $y = Ae^t + B + Dt$. Find A, B, D.

36 Rewrite $cy - by^2$ as $a^2 - x^2$, with $x = \sqrt{b}y - c/2\sqrt{b}$ and $a =$ ______. Substitute for a and x in the integral taken from tables, to obtain the y-integral in the text:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} \qquad \int \frac{dy}{cy - by^2} = \frac{1}{c} \ln \frac{y}{c - by}$$

37 (Important) Draw the y-lines (with arrows as in the text) for $y' = y/(1-y)$ and $y' = y - y^3$. Which steady states are approached from which initial values y_0 ?

38 Explain in your own words how the y-line works.

39 (a) Solve $y' = \tan y$ starting from $y_0 = \pi/6$ to find $\sin y = \frac{1}{2}e^t$.

(b) Explain why $t = 1$ is never reached.

(c) Draw arrows on the y-line to show that y approaches $\pi/2$—when does it get there ?

40 Write the logistic equation as $y' = cy(1 - y/K)$. As y' approaches zero, y approaches ______. Find y, y', y'' at the inflection point.

6.6 Powers Instead of Exponentials

You may remember our first look at e. It is the special base for which e^x has slope 1 at $x=0$. That led to the great equation of exponential growth: *The derivative of e^x equals e^x*. But our look at the actual number $e=2.71828\ldots$ was very short. It appeared as the limit of $(1+1/n)^n$. This seems an unnatural way to write down such an important number.

I want to show how $(1+1/n)^n$ and $(1+x/n)^n$ arise naturally. They give ***discrete growth in finite steps***—with applications to compound interest. Loans and life insurance and money market funds use the discrete form of $y'=cy+s$. (We include extra information about bank rates, hoping this may be useful some day.) The applications in science and engineering are equally important. Scientific computing, like accounting, has *difference equations* in parallel with differential equations.

Knowing that this section will be full of formulas, I would like to jump ahead and tell you the best one. It is an infinite series for e^x. What makes the series beautiful is that ***its derivative is itself***.

Start with $y=1+x$. This has $y=1$ and $y'=1$ at $x=0$. But y'' is zero, not one. Such a simple function doesn't stand a chance! No polynomial can be its own derivative, because the highest power x^n drops down to nx^{n-1}. The only way is *to have no highest power*. We are forced to consider infinitely many terms—***a power series***—to achieve "derivative equals function."

To produce the derivative $1+x$, we need $1+x+\frac{1}{2}x^2$. Then $\frac{1}{2}x^2$ is the derivative of $\frac{1}{6}x^3$, which is the derivative of $\frac{1}{24}x^4$. The best way is to write the whole series at once:

$$\textbf{\textit{Infinite series}}\quad e^x=1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\tfrac{1}{24}x^4+\cdots. \tag{1}$$

This must be the greatest power series ever discovered. Its derivative is itself:

$$de^x/dx=0+1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\cdots=e^x. \tag{2}$$

The derivative of each term is the term before it. The integral of each term is the one after it (so $\int e^x\,dx=e^x+C$). The approximation $e^x\approx 1+x$ appears in the first two terms. Other properties like $(e^x)(e^x)=e^{2x}$ are not so obvious. (Multiplying series is hard but interesting.) *It is not even clear why the sum is* $2.718\ldots$ *when* $x=1$. Somehow $1+1+\frac{1}{2}+\frac{1}{6}+\cdots$ equals e. That is where $(1+1/n)^n$ will come in.

Notice that x^n is divided by the product $1\cdot 2\cdot 3\cdot\cdots\cdot n$. This is "*$n$ factorial*." Thus *x^4 is divided by* $1\cdot 2\cdot 3\cdot 4=4!=24$, and x^5 is divided by $5!=120$. The derivative of $x^5/120$ is $x^4/24$, because 5 from the derivative cancels 5 from the factorial. In general $x^n/n!$ has derivative $x^{n-1}/(n-1)!$ Surprisingly $0!$ is 1.

Chapter 10 emphasizes that $x^n/n!$ becomes extremely small as n increases. The infinite series adds up to a finite number—which is e^x. We turn now to discrete growth, which produces the same series in the limit.

This headline was on page one of the New York Times for May 27, 1990.

213 Years After Loan, Uncle Sam is Dunned

San Antonio, May 26—More than 200 years ago, a wealthy Pennsylvania merchant named Jacob DeHaven lent $\$450{,}000$ to the Continental Congress to rescue the troops at Valley Forge. That loan was apparently never repaid.

So Mr. DeHaven's descendants are taking the United States Government to court to collect what they believe they are owed. The total: $141 billion if the interest is compounded daily at 6 percent, the going rate at the time. If compounded yearly, the bill is only $98 billion.

The thousands of family members scattered around the country say they are not being greedy. "It's not the money—it's the principle of the thing," said Carolyn Cokerham, a DeHaven on her father's side who lives in San Antonio. "You have to wonder whether there would even be a United States if this man had not made the sacrifice that he did. He gave everything he had."

The descendants say that they are willing to be flexible about the amount of settlement. But they also note that interest is accumulating at $190 a second.

"None of these people have any intention of bankrupting the Government," said Jo Beth Kloecker, a lawyer from Stafford, Texas. Fresh out of law school, Ms. Kloecker accepted the case for less than the customary 30 percent contingency.

It is unclear how many descendants there are. Ms. Kloecker estimates that based on 10 generations with four children in each generation, there could be as many as half a million.

The initial suit was dismissed on the ground that the statute of limitations is six years for a suit against the Federal Government. The family's appeal asserts that this violates Article 6 of the Constitution, which declares as valid all debts owed by the Government before the Constitution was adopted.

Mr. DeHaven died penniless in 1812. He had no children.

COMPOUND INTEREST

The idea of compound interest can be applied right away. Suppose you invest $1000 at a rate of 100% (hard to do). If this is the *annual rate*, the interest after a year is another $1000. You receive $2000 in all. But if the interest is *compounded* you receive more:

after six months: Interest of $500 is reinvested to give $1500

end of year: New interest of $750 (50% of 1500) gives $2250 total.

The bank multiplied twice by 1.5 (1000 to 1500 to 2250). Compounding *quarterly* multiplies *four times* by 1.25 (1 for principal, .25 for interest):

$$\begin{aligned}
\text{after one quarter the total is } & 1000+(.25)(1000)=1250\\
\text{after two quarters the total is } & 1250+(.25)(1250)=1562.50\\
\text{after nine months the total is } & 1562.50+(.25)(1562.50)=1953.12\\
\text{after a full year the total is } & 1953.12+(.25)(1953.12)=2441.41
\end{aligned}$$

Each step multiplies by $1+(1/n)$, to add one nth of a year's interest—still at 100%:

$$\begin{aligned}
\text{quarterly conversion: } & (1+1/4)^4\times 1000 &= 2441.41\\
\text{monthly conversion: } & (1+1/12)^{12}\times 1000 &= 2613.04\\
\text{daily conversion: } & (1+1/365)^{365}\times 1000 &= 2714.57.
\end{aligned}$$

Many banks use 360 days in a year, although computers have made that obsolete. Very few banks use minutes (525,600 per year). Nobody compounds every second

$(n = 31,536,000)$. But some banks offer ***continuous compounding***. This is the limiting case $(n \to \infty)$ that produces e:

$$\left(1+\frac{1}{n}\right)^n \times 1000 \ \textit{approaches} \ e \times 1000 = 2718.28.$$

1. *Quick method for* $(1+1/n)^n$: ***Take its logarithm. Use*** $\ln(1+x) \approx x$ ***with*** $x = \frac{1}{n}$:

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) \approx n\left(\frac{1}{n}\right) = 1. \tag{3}$$

As $1/n$ gets smaller, this approximation gets better. The limit is 1. Conclusion: $(1+1/n)^n$ approaches the number whose logarithm is 1. Sections 6.2 and 6.4 define the same number (which is e).

2. *Slow method for* $(1+1/n)^n$: ***Multiply out all the terms. Then let*** $n \to \infty$.

This is a brutal use of the binomial theorem. It involves nothing smart like logarithms, but the result is a fantastic new formula for e.

$$\text{Practice for } n=3: \quad \left(1+\frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + \frac{3\cdot 2}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3.$$

Binomial theorem for any positive integer n:

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n. \tag{4}$$

Each term in equation (4) approaches a limit as $n \to \infty$. Typical terms are

$$\frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 \to \frac{1}{1\cdot 2} \qquad \text{and} \qquad \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 \to \frac{1}{1\cdot 2\cdot 3}.$$

Next comes $1/1\cdot 2\cdot 3\cdot 4$. The sum of all those limits in (4) is our new formula for e:

$$\lim\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots = e. \tag{5}$$

In summation notation this is $\Sigma_{k=0}^{\infty} 1/k! = e$. The factorials give fast convergence:

$$1 + 1 + .5 + .16667 + .04167 + .00833 + .00139 + .00020 + .00002 = 2.71828.$$

Those nine terms give an accuracy that was not reached by $n = 365$ compoundings. A limit is still involved (to add up the whole series). ***You never see*** e ***without a limit***! It can be defined by derivatives or integrals or powers $(1+1/n)^n$ or by an infinite series. Something goes to zero or infinity, and care is required.

All terms in equation (4) are below (or equal to) the corresponding terms in (5). ***The power*** $(1+1/n)^n$ ***approaches*** e ***from below***. There is a steady increase with n. Faster compounding yields more interest. Continuous compounding at 100% yields e, as each term in (4) moves up to its limit in (5).

Remark Change $(1+1/n)^n$ to $(1+x/n)^n$. Now the binomial theorem produces e^x:

$$\left(1+\frac{x}{n}\right)^n = 1+n\left(\frac{x}{n}\right)+\frac{n(n-1)}{1\cdot 2}\left(\frac{x}{n}\right)^2+\cdots \text{ approaches } 1+x+\frac{x^2}{1\cdot 2}+\cdots. \quad (6)$$

Please recognize e^x on the right side! It is the infinite power series in equation (1). The next term is $x^3/6$ (x can be positive or negative). This is a final formula for e^x:

6L The limit of $(1+x/n)^n$ is e^x. At $x=1$ we find e.

The logarithm of that power is $n\,\ln(1+x/n) \approx n(x/n) = x$. The power approaches e^x.

To summarize: The quick method proves $(1+1/n)^n \to e$ by logarithms. The slow method (multiplying out every term) led to the infinite series. Together they show the agreement of all our definitions of e.

DIFFERENCE EQUATIONS VS. DIFFERENTIAL EQUATIONS

We have the chance to see an important part of applied mathematics. This is not a course on differential equations, and it cannot become a course on difference equations. But it is a course with a purpose—*we aim to use what we know*. Our main application of e was to solve $y'=cy$ and $y'=cy+s$. Now we solve the corresponding difference equations.

Above all, the goal is to see the connections. *The purpose of mathematics is to understand and explain patterns*. The path from "discrete to continuous" is beautifully illustrated by these equations. Not every class will pursue them to the end, but I cannot fail to show the pattern in a ***difference equation***:

$$y(t+1) = a\,y(t). \quad (7)$$

Each step multiplies by the same number a. The starting value y_0 is followed by ay_0, a^2y_0, and a^3y_0. The solution at discrete times $t=0,1,2,\ldots$ is $y(t)=a^ty_0$.

This formula a^ty_0 replaces the continuous solution $e^{ct}y_0$ of the differential equation.

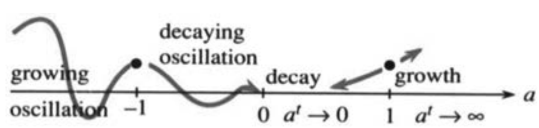


Fig. 6.17 Growth for $|a|>1$, decay for $|a|<1$. Growth factor a compares to e^c.

A source or sink (birth or death, deposit or withdrawal) is like $y'=cy+s$:

$$y(t+1) = a\,y(t)+s. \quad (8)$$

Each step multiplies by *a and adds s*. The first outputs are

$$y(1)=ay_0+s, \quad y(2)=a^2y_0+as+s, \quad y(3)=a^3y_0+a^2s+as+s.$$

We saw this pattern for differential equations—***every input s becomes a new starting point***. It is multiplied by powers of a. Since s enters later than y_0, the powers stop at $t-1$. Algebra turns the sum into a clean formula by adding the geometric series:

$$y(t) = a^t y_0 + s\left[a^{t-1} + a^{t-2} + \cdots + a + 1\right] = a^t y_0 + s(a^t - 1)/(a-1). \tag{9}$$

EXAMPLE 1 Interest at 8% from annual IRA deposits of $s = \$2000$ (here $y_0 = 0$).

The first deposit is at year $t = 1$. In a year it is multiplied by $a = 1.08$, because 8% is added. At the same time a new $s = 2000$ goes in. At $t = 3$ the first deposit has been multiplied by $(1.08)^2$, the second by 1.08, and there is another $s = 2000$. After year t,

$$y(t) = 2000(1.08^t - 1)/(1.08 - 1). \tag{10}$$

With $t = 1$ this is 2000. With $t = 2$ it is 2000 $(1.08 + 1)$—two deposits. Notice how $a - 1$ (the interest rate .08) appears in the denominator.

EXAMPLE 2 Approach to steady state when $|a| < 1$. Compare with $c < 0$.

With $a > 1$, everything has been increasing. That corresponds to $c > 0$ in the differential equation (which is growth). But things die, and money is spent, so a can be smaller than one. In that case $a^t y_0$ approaches zero—the starting balance disappears. What happens if there is also a source? *Every year half of the balance $y(t)$ is spent and a new* \$2000 *is deposited*. Now $a = \frac{1}{2}$:

$$y(t+1) = \tfrac{1}{2} y(t) + 2000 \quad \text{yields} \quad y(t) = \left(\tfrac{1}{2}\right)^t y_0 + 2000\left[\left(\left(\tfrac{1}{2}\right)^t - 1\right)/\left(\tfrac{1}{2} - 1\right)\right].$$

The limit as $t \to \infty$ is an equilibrium point. As $\left(\frac{1}{2}\right)^t$ goes to zero, $y(t)$ stabilizes to

$$y_\infty = 2000(0-1)/\left(\tfrac{1}{2} - 1\right) = 4000 = \textbf{\textit{steady state}}. \tag{11}$$

Why is 4000 steady? Because half is lost and the new 2000 makes it up again. ***The iteration is*** $y_{n+1} = \frac{1}{2} y_n + 2000$. ***Its fixed point is where*** $y_\infty = \frac{1}{2} y_\infty + 2000$.

In general the steady equation is $y_\infty = a y_\infty + s$. Solving for y_∞ gives $s/(1-a)$. Compare with the steady differential equation $y' = cy + s = 0$:

$$y_\infty = -\frac{s}{c} \text{(differential equation)} \quad \textit{vs.} \quad y_\infty = \frac{s}{1-a} \text{ (difference equation)}. \tag{12}$$

EXAMPLE 3 Demand equals supply when the price is right.

Difference equations are basic to economics. Decisions are made every year (by a farmer) or every day (by a bank) or every minute (by the stock market). There are three assumptions:

1. Supply next time depends on price this time: $S(t+1) = cP(t)$.
2. Demand next time depends on price next time: $D(t+1) = -dP(t+1) + b$.
3. Demand next time equals supply next time: $D(t+1) = S(t+1)$.

Comment on **3**: the price sets itself to make ***demand = supply***. The demand slope $-d$ is negative. The supply slope c is positive. Those lines intersect at the competitive price, where supply equals demand. To find the difference equation, substitute **1** and **2** into **3**:

Difference equation: $\quad -dP(t+1) + b = cP(t)$

Steady state price: $\quad -dP_\infty + b = cP_\infty$. Thus $P_\infty = b/(c+d)$.

If the price starts above P_∞, the difference equation brings it down. If below, the price goes up. When the price is P_∞, it stays there. This is not news—economic theory depends on approach to a steady state. But convergence only occurs if $c < d$. *If supply is less sensitive than demand, the economy is stable.*

Blow-up example: $c = 2$, $b = d = 1$. The difference equation is $-P(t+1)+1 = 2P(t)$. From $P(0) = 1$ the price oscillates as it grows: $P = -1, 3, -5, 11, \ldots$.

Stable example: $c = 1/2$, $b = d = 1$. The price moves from $P(0) = 1$ to $P(\infty) = 2/3$:

$$-P(t+1)+1 = \frac{1}{2}P(t) \quad \text{yields} \quad P = 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots, \text{ approaching } \frac{2}{3}.$$

Increasing d gives greater stability. That is the effect of price supports. For $d = 0$ (fixed demand regardless of price) the economy is out of control.

THE MATHEMATICS OF FINANCE

It would be a pleasure to make this supply-demand model more realistic—with curves, not straight lines. Stability depends on the slope—calculus enters. But we also have to be realistic about class time. I believe the most practical application is to solve ***the fundamental problems of finance***. Section 6.3 answered six questions about continuous interest. We now answer the same six questions when the annual rate is $x = .05 = 5\%$ and ***interest is compounded*** n ***times a year***.

First we compute ***effective rates***, higher than .05 because of compounding:

$$\text{compounded } \textit{quarterly} \left(1+\frac{.05}{4}\right)^4 = 1.0509 \quad \left[\textit{effective rate } .0509 = 5.09\%\right]$$

$$\text{compounded } \textit{continuously} \qquad e^{.05} = 1.0513 \quad \left[\textit{effective rate } 5.13\%\right]$$

Now come the six questions. Next to the new answer (discrete) we write the old answer (continuous). One is algebra, the other is calculus. The time period is 20 years, so simple interest on y_0 would produce $(.05)(20)(y_0)$. That equals y_0—money doubles in 20 years at 5% simple interest.

Questions **1** and **2** ask for the ***future value*** y and ***present value*** y_0 *with compound interest* n *times a year*:

1. y growing from y_0: $\quad y = \left(1+\frac{.05}{n}\right)^{20n} y_0 \qquad y = e^{(.05)(20)} y_0$

2. deposit y_0 to reach y: $\quad y_0 = \left(1+\frac{.05}{n}\right)^{-20n} y \qquad y_0 = e^{-(.05)(20)} y$

Each step multiplies by $a = (1+.05/n)$. There are $20n$ steps in 20 years. Time goes backward in Question **2**. We divide by the growth factor instead of multiplying. The future value is greater than the present value (unless the interest rate is negative!). As $n \to \infty$ the discrete y on the left approaches the continuous y on the right.

Questions **3** and **4** connect y to s (with $y_0 = 0$ at the start). As soon as each s is deposited, it starts growing. Then $y = s + as + a^2 s + \cdots$.

3. y growing from deposits s: $\quad y = s\left[\frac{(1+.05/n)^{20n}-1}{.05/n}\right] \qquad y = s\left[\frac{e^{(.05)(20)}-1}{.05}\right]$

4. deposits s to reach y: $\quad s = y\left[\frac{.05/n}{(1+.05/n)^{20n}-1}\right] \qquad s = y\left[\frac{.05}{e^{(.05)(20)}-1}\right]$

Questions **5** and **6** connect y_0 to s. This time y is zero—***there is nothing left at the end***. Everything is paid. The deposit y_0 is just enough to allow payments of s. This is an ***annuity***, where the bank earns interest on your y_0 while it pays you s (n times a year for 20 years). So your deposit in Question **5** is less than $20ns$.

Question **6** is the opposite—a ***loan***. At the start you borrow y_0 (instead of giving the bank y_0). You can earn interest on it as you pay it back. Therefore your payments have to total more than y_0. This is the calculation for car loans and mortgages.

5. ***Annuity***: Deposit y_0 to receive $20n$ payments of s:

$$y_0 = s\left[\frac{1-(1+.05/n)^{-20n}}{.05/n}\right] \qquad y_0 = s\left[\frac{1-e^{-(.05)(20)}}{.05}\right]$$

6. ***Loan***: Repay y_0 with $20n$ payments of s:

$$s = y_0\left[\frac{.05/n}{1-(1+.05/n)^{-20n}}\right] \qquad s = y_0\left[\frac{.05}{1-e^{-(.05)(20)}}\right]$$

Questions $\mathbf{2},\mathbf{4},\mathbf{6}$ are the inverses of $\mathbf{1},\mathbf{3},\mathbf{5}$. Notice the pattern: There are three numbers y, y_0, and s. *One of them is zero each time*. If all three are present, go back to equation (9).

The algebra for these lines is in the exercises. *It is not calculus because Δt is not dt*. All factors in brackets $[\quad]$ are listed in tables, and the banks keep copies. It might also be helpful to know their symbols. If a bank has interest rate i per period over N periods, then in our notation $a = 1+i = 1+.05/n$ and $t = N = 20n$:

$$\text{future value of } y_0 = \$1 \text{ (line } \mathbf{1}) : y(N) = (1+i)^N$$

$$\text{present value of } y = \$1 \text{ (line } \mathbf{2}) : y_0 = (1+i)^{-N}$$

$$\text{future value of } s = \$1 \text{ (line } \mathbf{3}) : y(N) = s_{\overline{N}|i} = \left[(1+i)^N - 1\right]/i$$

$$\text{present value of } s = \$1 \text{ (line } \mathbf{5}) : y_0 = a_{\overline{N}|i} = \left[1-(1+i)^{-N}\right]/i$$

To tell the truth, I never knew the last two formulas until writing this book. The mortgage on my home has $N = (12)(25)$ monthly payments with interest rate $i = .07/12$. In 1972 the present value was $\$42,000 =$ amount borrowed. I am now going to see if the bank is honest.†

Remark In many loans, the bank computes interest on the amount paid back instead of the amount received. This is called ***discounting***. A loan of \$1000 at 5% for one year costs \$50 interest. Normally you receive \$1000 and pay back \$1050. *With discounting you receive* \$950 (called the proceeds) *and you pay back* \$1000. The true interest rate is higher than 5%—because the \$50 interest is paid on the smaller amount \$950. In this case the "discount rate" is $50/950 = 5.26\%$.

SCIENTIFIC COMPUTING: DIFFERENTIAL EQUATIONS BY DIFFERENCE EQUATIONS

In biology and business, most events are discrete. In engineering and physics, time and space are continuous. Maybe at some quantum level it's all the same, but the

† It's not. s is too big. I knew it.

equations of physics (starting with Newton's law $F = ma$) are differential equations. The great contribution of calculus is to model the rates of change we see in nature. But to *solve that model with a computer*, it needs to be made digital and discrete.

These paragraphs work with $dy/dt = cy$. It is the test equation that all analysts use, as soon as a new computing method is proposed. Its solution is $y = e^{ct}$, starting from $y_0 = 1$. Here we test Euler's method (nearly ancient, and not well thought of). He replaced dy/dt by $\Delta y/\Delta t$:

$$\textbf{Euler's Method} \qquad \frac{y(t+\Delta t) - y(t)}{\Delta t} = cy(t). \tag{13}$$

The left side is dy/dt, in the limit $\Delta t \to 0$. We stop earlier, when $\Delta t > 0$.

The problem is to solve (13). Multiplying by Δt, the equation is

$$y(t+\Delta t) = (1 + c\Delta t)\,y(t) \qquad (\text{with } y(0) = 1).$$

Each step multiplies by $a = 1 + c\Delta t$, so n steps multiply by a^n:

$$y = a^n = (1 + c\Delta t)^n \text{ at time } n\Delta t. \tag{14}$$

This is growth or decay, *depending on* a. The correct e^{ct} is growth or decay, *depending on* c. ***The question is whether*** a^n ***and*** e^{ct} ***stay close***. Can one of them grow while the other decays? We expect the difference equation to copy $y' = cy$, but we might be wrong.

A good example is $y' = -y$. Then $c = -1$ and $y = e^{-t}$—the true solution decays. The calculator gives the following answers a^n for $n = 2, 10, 20$:

Δt	$a = 1 + c\Delta t$	a^2	a^{10}	a^{20}
3	-2	4	1024	1048576
1	0	0	0	0
$1/10$	.90	.81	.35	.12
$1/20$	.95	.90	.60	.36

The big step $\Delta t = 3$ shows total instability (top row). The numbers blow up when they should decay. The row with $\Delta t = 1$ is equally useless (all zeros). In practice the magnitude of $c\Delta t$ must come down to .10 or .05. For accurate calculations it would have to be even smaller, unless we change to a better difference equation. That is the right thing to do.

Notice the two reasonable numbers. They are .35 and .36, approaching $e^{-1} = .37$. They come from $n = 10$ (with $\Delta t = 1/10$) and $n = 20$ (with $\Delta t = 1/20$). Those have the same clock time $n\Delta t = 1$:

$$\left(1 - \frac{1}{10}\right)^{10} = .35 \qquad \left(1 - \frac{1}{20}\right)^{20} = .36 \qquad \left(1 - \frac{1}{n}\right)^{n} \to e^{-1} = .37.$$

The main diagonal of the table is executing $(1 + x/n)^n \to e^x$ in the case $x = -1$.

Final question: ***How quickly are*** .35 ***and*** .36 ***converging to*** $e^{-1} = .37$? With $\Delta t = .10$ the error is .02. With $\Delta t = .05$ the error is .01. Cutting the time step in half cuts the error in half. We are not keeping enough digits to be sure, but *the* *error seems close to* $\frac{1}{5}\Delta t$. To test that, apply the "quick method" and estimate $a^n = (1-\Delta t)^n$ from its logarithm:

$$\ln(1-\Delta t)^n = n\,\ln(1-\Delta t) \approx n\Big[-\Delta t - \tfrac{1}{2}\big(\Delta t\big)^2\Big] = -1 - \tfrac{1}{2}\Delta t. \qquad (15)$$

The clock time is $n\Delta t = 1$. Now take exponentials of the far left and right:

$$a^n = (1-\Delta t)^n \approx e^{-1}e^{-\Delta t/2} \approx e^{-1}\big(1-\tfrac{1}{2}\Delta t\big). \qquad (16)$$

The difference between a^n and e^{-1} is the last term $\frac{1}{2}\Delta t e^{-1}$. Everything comes down to one question: Is that error the same as $\frac{1}{5}\Delta t$? *The answer is yes*, because $e^{-1}/2$ is $1/5$. If we keep only one digit, the prediction is perfect!

That took an hour to work out, and I hope it takes longer than Δt to read. I wanted you to see *in use* the properties of $\ln x$ and e^x. The exact property $\ln a^n = n \ln a$ came first. In the middle of (15) was the key approximation $\ln(1+x) \approx x - \frac{1}{2}x^2$, with $x = -\Delta t$. That x^2 term uses the second derivative (Section 6.4). At the very end came $e^x \approx 1+x$.

A linear approximation shows convergence: $(1+x/n)^n \to e^x$. A quadratic shows the error: proportional to $\Delta t = 1/n$. It is like using rectangles for areas, with error proportional to Δx. This minimal accuracy was enough to define the integral, and here it is enough to define e. It is completely unacceptable for scientific computing.

The trapezoidal rule, for integrals or for $y' = cy$, has errors of order $(\Delta x)^2$ and $(\Delta t)^2$. All good software goes further than that. Euler's first-order method could not predict the weather before it happens.

Euler's Method for $\dfrac{dy}{dt} = F(y,t): \quad \dfrac{y(t+\Delta t)-y(t)}{\Delta t} = F(y(t),t).$

6.6 EXERCISES

Read-through questions

The infinite series for e^x is ___a___. Its derivative is ___b___. The denominator $n!$ is called "___c___" and it equals ___d___. At $x=1$ the series for e is ___e___.

To match the original definition of e, multiply out $(1+1/n)^n =$ ___f___ (first three terms). As $n\to\infty$ those terms approach ___g___ in agreement with e. The first three terms of $(1+x/n)^n$ are ___h___. As $n\to\infty$ they approach ___i___ in agreement with e^x. Thus $(1+x/n)^n$ approaches ___j___. A quicker method computes $\ln(1+x/n)^n \approx$ ___k___ (first term only) and takes the exponential.

Compound interest (n times in one year at annual rate x) multiplies by (___l___)^n. As $n\to\infty$, continuous compounding multiplies by ___m___. At $x=10\%$ with continuous compounding, \$1 grows to ___n___ in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) =$ ___o___ times y_0. The equation $y(t+1) = ay(t)+s$ is solved by $y = a^t y_0 + s[1+a+\cdots+a^{t-1}]$. The sum in brackets is ___p___. When $a=1.08$ and $y_0=0$, annual deposits of $s=1$ produce $y=$ ___q___ after t years. If $a=\frac{1}{2}$ and $y_0=0$, annual deposits of $s=6$ leave ___r___ after t years, approaching $y_\infty =$ ___s___. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty =$ ___t___.

When $i =$ interest rate per period, the value of $y_0 = \$1$ after N periods is $y(N) =$ ___u___. The deposit to produce $y(N)=1$ is $y_0 =$ ___v___. The value of $s=\$1$ deposited after each period grows to $y(N) =$ ___w___. The deposit to reach $y(N)=1$ is $s=$ ___x___.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies y by ___y___. Therefore y at $t=1$ is $(1+c\Delta t)^{1/t}y_0$, which converges to ___z___ as $\Delta t\to 0$. The error is proportional to ___A___, which is too ___B___ for scientific computing.

1 Write down a power series $y = 1 - x + \cdots$ whose derivative is $-y$.

2 Write down a power series $y = 1 + 2x + \cdots$ whose derivative is $2y$.

3 Find *two* series that are equal to their second derivatives.

4 By comparing $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ with a larger series (whose sum is easier) show that $e < 3$.

5 At 5% interest compute the output from \$1000 in a year with 6-month and 3-month and weekly compounding.

6 With the quick method $\ln(1+x) \approx x$, estimate $\ln(1-1/n)^n$ and $\ln(1+2/n)^n$. Then take exponentials to find the two limits.

7 With the slow method multiply out the three terms of $(1-\frac{1}{2})^2$ and the five terms of $(1-\frac{1}{4})^4$. What are the first three terms of $(1-1/n)^n$, and what are their limits as $n \to \infty$?

8 The slow method leads to $1 - 1 + 1/2! - 1/3! + \cdots$ for the limit of $(1-1/n)^n$. What is the sum of this infinite series—the exact sum and the sum after five terms?

9 Knowing that $(1+1/n)^n \to e$, explain $(1+1/n)^{2n} \to e^2$ and $(1+2/N)^N \to e^2$.

10 What are the limits of $(1+1/n^2)^n$ and $(1+1/n)^{n^2}$? OK to use a calculator to guess these limits.

11 (a) The power $(1+1/n)^n$ (decreases) (increases) with n, as we compound more often. (b) The derivative of $f(x) = x\ln(1+1/x)$, which is ______, should be $(<0)(>0)$. This is confirmed by Problem 12.

12 Show that $\ln(1+1/x) > 1/(x+1)$ by drawing the graph of $1/t$. The area from $t = 1$ to $1 + 1/x$ is ______. The rectangle inside it has area ______.

13 Take three steps of $y(t+1) = 2y(t)$ from $y_0 = 1$.

14 Take three steps of $y(t+1) = 2y(t) + 1$ from $y_0 = 0$.

Solve the difference equations 15–22.

15 $y(t+1) = 3y(t), y_0 = 4$

16 $y(t+1) = \frac{1}{2}y(t), y_0 = 1$

17 $y(t+1) = y(t) + 1, y_0 = 0$

18 $y(t+1) = y(t) - 1, y_0 = 0$

19 $y(t+1) = 3y(t) + 1, y_0 = 0$

20 $y(t+1) = 3y(t) + s, y_0 = 1$

21 $y(t+1) = ay(t) + s, y_0 = 0$

22 $y(t+1) = ay(t) + s, y_0 = 5$

In 23–26, which initial value produces $y_1 = y_0$ (steady state)?

23 $y(t+1) = 2y(t) - 6$

24 $y(t+1) = \frac{1}{2}y(t) - 6$

25 $y(t+1) = -y(t) + 6$

26 $y(t+1) = -\frac{1}{2}y(t) + 6$

27 In Problems 23 and 24, start from $y_0 = 2$ and take three steps to reach y_3. Is this approaching a steady state?

28 For which numbers a does $(1-a^t)/(1-a)$ approach a limit as $t \to \infty$ and what is the limit?

29 The price P is determined by supply = demand or $-dP(t+1) + b = cP(t)$. Which price P is not changed from one year to the next?

30 Find $P(t)$ from the supply-demand equation with $c = 1$, $d = 2$, $b = 8$, $P(0) = 0$. What is the steady state as $t \to \infty$?

Assume 10% **interest (so $a = 1 + i = 1.1$) in Problems** $31 - 38$.

31 At 10% interest compounded quarterly, what is the effective rate?

32 At 10% interest compounded daily, what is the effective rate?

33 Find the future value in 20 years of \$100 deposited now.

34 Find the present value of \$1000 promised in twenty years.

35 For a mortgage of \$100,000 over 20 years, what is the monthly payment?

36 For a car loan of \$10,000 over 6 years, what is the monthly payment?

37 With annual compounding of deposits $s = \$1000$, what is the balance in 20 years?

38 If you repay $s = \$1000$ annually on a loan of \$8000, when are you paid up? (Remember interest.)

39 Every year two thirds of the available houses are sold, and 1000 new houses are built. What is the steady state of the housing market—how many are available?

40 If a loan shark charges 5% interest a month on the \$1000 you need for blackmail, and you pay \$60 a month, how much do you still owe after one month (and after a year)?

41 Euler charges $c = 100\%$ interest on his \$1 fee for discovering e. What do you owe (including the \$1) after a year with (a) no compounding; (b) compounding every week; (c) continuous compounding?

42 Approximate $(1+1/n)^n$ as in (15) and (16) to show that you owe Euler about $e - e/2n$. Compare Problem 6.2.5.

43 My Visa statement says monthly rate = 1.42% and yearly rate = 17%. What is the true yearly rate, since Visa compounds the interest? Give a formula or a number.

44 You borrow $y_0 = \$80,000$ at 9% to buy a house.

(a) What are your monthly payments s over 30 years?

(b) How much do you pay altogether?

6.7 Hyperbolic Functions

This section combines e^x with e^{-x}. Up to now those functions have gone separate ways—one increasing, the other decreasing. But two particular combinations have earned names of their own ($\cosh x$ and $\sinh x$):

$$\textbf{\textit{hyperbolic cosine}}\ \cosh x = \frac{e^x + e^{-x}}{2} \qquad \textbf{\textit{hyperbolic sine}}\ \sinh x = \frac{e^x - e^{-x}}{2}$$

The first name rhymes with "gosh". The second is usually pronounced "cinch".

The graphs in Figure 6.18 show that $\cosh x > \sinh x$. For large x both hyperbolic functions come extremely close to $\frac{1}{2}e^x$. When x is large and *negative*, it is e^{-x} that dominates. Cosh x still goes up to $+\infty$ while $\sinh x$ goes down to $-\infty$ (because $\sinh x$ has a minus sign in front of e^{-x}).

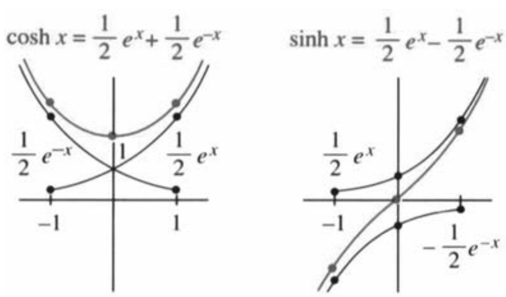


Fig. 6.18 Cosh x and $\sinh x$. The hyperbolic functions combine $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

Fig. 6.19 Gateway Arch courtesy of the St. Louis Visitors Commission.

The following facts come directly from $\frac{1}{2}\left(e^x + e^{-x}\right)$ and $\frac{1}{2}\left(e^x - e^{-x}\right)$:

$$\cosh(-x) = \cosh x \text{ and } \cosh 0 = 1 \qquad \text{(cosh is \textbf{\textit{even}} like the cosine)}$$
$$\sinh(-x) = -\sinh x \text{ and } \sinh 0 = 0 \qquad \text{(sinh is \textbf{\textit{odd}} like the sine)}$$

The graph of $\cosh x$ corresponds to a *hanging cable* (hanging under its weight). Turned upside down, it has the shape of the Gateway Arch in St. Louis. That must be the largest upside-down cosh function ever built. A cable is easier to construct than an arch, because gravity does the work. With the right axes in Problem 55, the height of the cable is a stretched-out cosh function called a ***catenary***:

$$y = a \cosh(x/a) \qquad (\text{cable tension}/\text{cable density} = a).$$

Busch Stadium in St. Louis has 96 catenary curves, to match the Arch.

The properties of the hyperbolic functions come directly from the definitions. There are too many properties to memorize—and no reason to do it! One rule is the most important. ***Every fact about sines and cosines is reflected in a corresponding fact about*** $\sinh x$ ***and*** $\cosh x$. Often the only difference is a minus sign. Here are four properties:

1. $(\cosh x)^2 - (\sinh x)^2 = 1 \quad \left[\text{instead of } (\cos x)^2 + (\sin x)^2 = 1\right]$

Check: $\left[\dfrac{e^x + e^{-x}}{2}\right]^2 - \left[\dfrac{e^x - e^{-x}}{2}\right]^2 = \dfrac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$

2. $\dfrac{d}{dx}(\cosh x) = \sinh x \quad \left[\text{instead of } \dfrac{d}{dx}(\cos x) = -\sin x\right]$

3. $\dfrac{d}{dx}(\sinh x) = \cosh x \quad \left[\text{like } \dfrac{d}{dx}\sin x = \cos x\right]$

4. $\int \sinh x\, dx = \cosh x + C$ and $\int \cosh x\, dx = \sinh x + C$

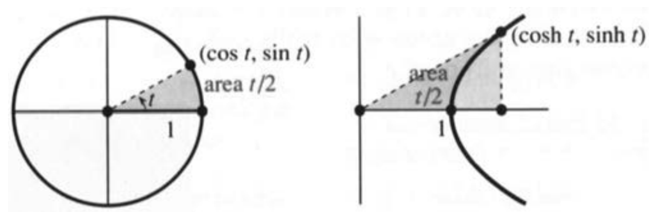


Fig. 6.20 The unit circle $\cos^2 t + \sin^2 t = 1$ and the unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Property 1 is the connection to hyperbolas. It is responsible for the "h" in cosh and sinh. Remember that $(\cos x)^2 + (\sin x)^2 = 1$ puts the point $(\cos x, \sin x)$ onto a *unit circle*. As x varies, the point goes around the circle. The ordinary sine and cosine are "circular functions." Now look at $(\cosh x, \sinh x)$. Property 1 is $(\cosh x)^2 - (\sinh x)^2 = 1$, so this point travels on the *unit hyperbola* in Figure 6.20.

You will guess the definitions of the other four hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

I think "tanh" is pronounceable, and "sech" is easy. The others are harder. Their properties come directly from $\cosh^2 x - \sinh^2 x = 1$. Divide by $\cosh^2 x$ and $\sinh^2 x$:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \operatorname{csch}^2 x$$
$$(\tanh x)' = \operatorname{sech}^2 x \quad \text{and} \quad (\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$

$$\int \tanh x\; dx = \int \frac{\sinh x}{\cosh x}\, dx = \ln(\cosh x) + C.$$

INVERSE HYPERBOLIC FUNCTIONS

You remember the angles $\sin^{-1} x$ and $\tan^{-1} x$ and $\sec^{-1} x$. In Section 4.4 we differentiated those inverse functions by the chain rule. The main application was to *integrals*. If we happen to meet $\int dx/(1+x^2)$, it is $\tan^{-1} x + C$. The situation for

$\sinh^{-1}x$ and $\tanh^{-1}x$ and $\operatorname{sech}^{-1}x$ is the same except for sign changes—which are expected for hyperbolic functions. We write down the ***three new derivatives***:

$$y = \sinh^{-1}x \text{ (meaning } x = \sinh y\text{) has } \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \tag{1}$$

$$y = \tanh^{-1}x \text{ (meaning } x = \tanh y\text{) has } \frac{dy}{dx} = \frac{1}{1-x^2} \tag{2}$$

$$y = \operatorname{sech}^{-1}x \text{ (meaning } x = \operatorname{sech} y\text{) has } \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} \tag{3}$$

Problems $44-46$ compute dy/dx from $1/(dx/dy)$. The alternative is to use logarithms. Since $\ln x$ is the inverse of e^x, we can express $\sinh^{-1}x$ and $\tanh^{-1}x$ and $\operatorname{sech}^{-1}x$ as logarithms. Here is $y = \tanh^{-1}x$:

$$y = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \text{ has slope } \frac{dy}{dx} = \frac{1}{2}\frac{1}{1+x} - \frac{1}{2}\frac{1}{1-x} = \frac{1}{1-x^2}. \tag{4}$$

The last step is an ordinary derivative of $\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$. Nothing is new except the answer. But where did the logarithms come from? In the middle of the following identity, multiply above and below by $\cosh y$:

$$\frac{1+x}{1-x} = \frac{1+\tanh y}{1-\tanh y} = \frac{\cosh y + \sinh y}{\cosh y - \sinh y} = \frac{e^y}{e^{-y}} = e^{2y}.$$

Then $2y$ is the logarithm of the left side. This is the first equation in (4), and it is the third formula in the following list:

$$\sinh^{-1}x = \ln\left[x + \sqrt{x^2+1}\right] \qquad \cosh^{-1}x = \ln\left[x + \sqrt{x^2-1}\right]$$

$$\tanh^{-1}x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \qquad \operatorname{sech}^{-1}x = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right]$$

***Remark* 1** Those are listed *only for reference*. If possible do not memorize them. The derivatives in equations (1), (2), (3) offer a choice of antiderivatives—either inverse functions or logarithms (most tables prefer logarithms). The inside cover of the book has

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] + C \quad (\text{in place of } \tanh^{-1}x + C).$$

***Remark* 2** Logarithms were not seen for $\sin^{-1}x$ and $\tan^{-1}x$ and $\sec^{-1}x$. You might wonder why. How does it happen that $\tanh^{-1}x$ is expressed by logarithms, when the parallel formula for $\tan^{-1}x$ was missing? Answer: *There must be a parallel formula*. To display it I have to reveal a secret that has been hidden throughout this section.

The secret is one of the great equations of mathematics. ***What formulas for*** $\cos x$ ***and*** $\sin x$ ***correspond to*** $\frac{1}{2}(e^x + e^{-x})$ ***and*** $\frac{1}{2}(e^x - e^{-x})$? With so many analogies (circular vs. hyperbolic) you would expect to find something. The formulas do exist, but ***they involve imaginary numbers***. Fortunately they are very simple and there is no reason to withhold the truth any longer:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \text{and} \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right). \tag{5}$$

It is the imaginary exponents that kept those identities hidden. Multiplying $\sin x$ by i and adding to $\cos x$ gives Euler's unbelievably beautiful equation

$$\cos x + i \sin x = e^{ix}. \tag{6}$$

That is parallel to the non-beautiful hyperbolic equation $\cosh x + \sinh x = e^x$.

I have to say that (6) is infinitely more important than anything hyperbolic will ever be. The sine and cosine are far more useful than the sinh and cosh. So we end our record of the main properties, with exercises to bring out their applications.

6.7 EXERCISES

Read-through questions

Cosh x = __a__ and $\sinh x$ = __b__ and $\cosh^2 x - \sinh^2 x$ = __c__. Their derivatives are __d__ and __e__ and __f__. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola __g__. A cable hangs in the shape of a catenary y = __h__.

The inverse functions $\sinh^{-1} x$ and $\tanh^{-1} x$ are equal to $\ln[x + \sqrt{x^2+1}]$ and $\frac{1}{2}\ln$ __i__. Their derivatives are __j__ and __k__. So we have two ways to write the anti __l__. The parallel to $\cosh x + \sinh x = e^x$ is Euler's formula __m__. The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves __n__ exponents. The parallel formula for $\sin x$ is __o__.

1 Find $\cosh x + \sinh x$, $\cosh x - \sinh x$, and $\cosh x \sinh x$.

2 From the definitions of $\cosh x$ and $\sinh x$, find their derivatives.

3 Show that both functions satisfy $y'' = y$.

4 By the quotient rule, verify $(\tanh x)' = \operatorname{sech}^2 x$.

5 Derive $\cosh^2 x + \sinh^2 x = \cosh 2x$, from the definitions.

6 From the derivative of Problem 5 find $\sinh 2x$.

7 The parallel to $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ is a hyperbolic formula $(\cosh x + \sinh x)^n = \cosh nx +$ ______.

8 Prove $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ by changing to exponentials. Then the x-derivative gives $\cosh(x+y) =$ ______.

Find the derivatives of the functions 9–18:

9 $\cosh(3x+1)$

10 $\sinh x^2$

11 $1/\cosh x$

12 $\sinh(\ln x)$

13 $\cosh^2 x + \sinh^2 x$

14 $\cosh^2 x - \sinh^2 x$

15 $\tanh \sqrt{x^2+1}$

16 $(1+\tanh x)/(1-\tanh x)$

17 $\sinh^6 x$

18 $\ln(\operatorname{sech} x + \tanh x)$

19 Find the minimum value of $\cosh(\ln x)$ for $x > 0$.

20 From $\tanh x = \frac{3}{5}$ find $\operatorname{sech} x$, $\cosh x$, $\sinh x$, $\coth x$, $\operatorname{csch} x$.

21 Do the same if $\tanh x = -12/13$.

22 Find the other five values if $\sinh x = 2$.

23 Find the other five values if $\cosh x = 1$.

24 Compute $\sinh(\ln 5)$ and $\tanh(2 \ln 4)$.

Find antiderivatives for the functions in 25–32:

25 $\cosh(2x+1)$

26 $x \cosh(x^2)$

27 $\cosh^2 x \sinh x$

28 $\tanh^2 x \operatorname{sech}^2 x$

29 $\dfrac{\sinh x}{1+\cosh x}$

30 $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

31 $\sinh x + \cosh x$

32 $(\sinh x + \cosh x)^n$

33 The triangle in Figure 6.20 has area $\frac{1}{2} \cosh t \sinh t$.

(a) Integrate to find the shaded area below the hyperbola

(b) For the area A in red verify that $dA/dt = \frac{1}{2}$

(c) Conclude that $A = \frac{1}{2}t + C$ and show $C = 0$.

Sketch graphs of the functions in 34–40.

34 $y = \tanh x$ (with inflection point)

35 $y = \coth x$ (in the limit as $x \to \infty$)

36 $y = \operatorname{sech} x$

37 $y = \sinh^{-1} x$

38 $y = \cosh^{-1} x$ for $x \geqslant 1$

39 $y = \operatorname{sech}^{-1} x$ for $0 < x \leqslant 1$

40 $y = \tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ for $|x| < 1$

41 (a) Multiplying $x = \sinh y = \frac{1}{2}\left(e^y - e^{-y}\right)$ by $2e^y$ gives $(e^y)^2 - 2x(e^y) - 1 = 0$. Solve as a quadratic equation for e^y.

(b) Take logarithms to find $y = \sinh^{-1} x$ and compare with the text.

42 (a) Multiplying $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ by $2e^y$ gives $(e^y)^2 - 2x(e^y) + 1 = 0$. Solve for e^y.

(b) Take logarithms to find $y = \cosh^{-1} x$ and compare with the text.

43 Turn (4) upside down to prove $y' = -1/(1-x^2)$, if $y = \coth^{-1} x$.

44 Compute $dy/dx = 1/\sqrt{x^2+1}$ by differentiating $x = \sinh y$ and using $\cosh^2 y - \sinh^2 y = 1$.

45 Compute $dy/dx = 1/(1-x^2)$ if $y = \tanh^{-1} x$ by differentiating $x = \tanh y$ and using $\operatorname{sech}^2 y + \tanh^2 y = 1$.

46 Compute $dy/dx = -1/x\sqrt{1-x^2}$ for $y = \operatorname{sech}^{-1} x$, by differentiating $x = \operatorname{sech} y$.

From formulas (1), (2), (3) or otherwise, find antiderivatives in 47–52:

47 $\int dx/(4-x^2)$

48 $\int dx/(a^2-x^2)$

49 $\int dx/\sqrt{x^2+1}$

50 $\int x\,dx/\sqrt{x^2+1}$

51 $\int dx/x\sqrt{1-x^2}$

52 $\int dx/\sqrt{1-x^2}$

53 Compute $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$ and $\displaystyle\int_0^{1} \frac{dx}{1-x^2}$.

54 A falling body with friction equal to velocity squared obeys $dv/dt = g - v^2$.

(a) Show that $v(t) = \sqrt{g} \tanh \sqrt{g}t$ satisfies the equation.

(b) Derive this v yourself, by integrating $dv/(g-v^2) = dt$.

(c) Integrate $v(t)$ to find the distance $f(t)$.

55 A cable hanging under its own weight has slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

(a) Show that $S = \sinh cx$ satisfies the equation.

(b) Integrate $dy/dx = \sinh cx$ to find the cable height $y(x)$, if $y(0) = 1/c$.

(c) Sketch the cable hanging between $x = -L$ and $x = L$ and find how far it sags down at $x = 0$.

56 The simplest nonlinear wave equation (Burgers' equation) yields a waveform $W(x)$ that satisfies $W'' = WW' - W'$. One integration gives $W' = \frac{1}{2}W^2 - W$.

(a) Separate variables and integrate:
$dx = dW/\left(\frac{1}{2}W^2 - W\right) = -dW/(2-W) - dW/W$.

(b) Check $W' = \frac{1}{2}W^2 - W$.

57 A solitary water wave has a shape satisfying the *KdV* equation $y'' = y' - 6yy'$.

(a) Integrate once to find y''. Multiply the answer by y'.

(b) Integrate again to find y' (all constants of integration are zero).

(c) Show that $y = \frac{1}{2}\operatorname{sech}^2(x/2)$ gives the shape of the "soliton."

58 Derive $\cos ix = \cosh x$ from equation (5). What is the cosine of the imaginary angle $i = \sqrt{-1}$?

59 Derive $\sin ix = i \sinh x$ from (5). What is $\sin i$?

60 The derivative of $e^{ix} = \cos x + i \sin x$ is ______.