CHAPTER 2

Derivatives

2.1 The Derivative of a Function

This chapter begins with the definition of the derivative. Two examples were in Chapter 1. When the distance is t^2, the velocity is $2t$. When $f(t) = \sin t$ we found $v(t) = \cos t$. *The velocity is now called the* ***derivative*** *of* $f(t)$. As we move to a more formal definition and new examples, we use new symbols f' and df/dt for the derivative.

2A At time t, the *derivative* $f'(t)$ or df/dt or $v(t)$ is

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}. \tag{1}$$

The ratio on the right is the average velocity over a short time Δt. The derivative, on the left side, is its limit as the step Δt (*delta* t) approaches zero.

Go slowly and look at each piece. The distance at time $t + \Delta t$ is $f(t + \Delta t)$. The distance at time t is $f(t)$. Subtraction gives the ***change in distance***, between those times. We often write Δf for this difference: $\Delta f = f(t + \Delta t) - f(t)$. ***The average velocity is the ratio*** $\Delta f/\Delta t$—change in distance divided by change in time.

The limit of the average velocity is the derivative, if this limit exists:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}. \tag{2}$$

This is the neat notation that Leibniz invented: $\Delta f/\Delta t$ *approaches* df/dt. Behind the innocent word "*limit*" is a process that this course will help you understand.

Note that Δf is not Δ times f! ***It is the change in*** f. Similarly Δt is not Δ times t. It is the time step, positive or negative and eventually small. To have a one-letter symbol we replace Δt by h.

The right sides of (1) and (2) contain average speeds. On the graph of $f(t)$, the distance *up* is divided by the distance *across*. That gives the average slope $\Delta f/\Delta t$.

The left sides of (1) and (2) are ***instantaneous*** speeds df/dt. They give the slope at the instant t. This is the derivative df/dt (when Δt and Δf shrink to zero). Look again at the calculation for $f(t) = t^2$:

$$\frac{\Delta f}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t} = \frac{t^2+2t\,\Delta t+(\Delta t)^2-t^2}{\Delta t} = 2t+\Delta t. \qquad (3)$$

Important point: Those steps are taken before Δt goes to zero. ***If we set*** $\Delta t = 0$ ***too soon***, ***we learn nothing***. The ratio $\Delta f/\Delta t$ becomes $0/0$ (which is meaningless). The numbers Δf and Δt must approach zero together, not separately. Here their ratio is $2t + \Delta t$, the average speed.

To repeat: Success came by writing out $(t+\Delta t)^2$ and subtracting t^2 and dividing by Δt. Then and only then can we approach $\Delta t = 0$. The limit is the derivative $2t$.

There are several new things in formulas (1) and (2). Some are easy but important, others are more profound. The idea of a function we will come back to, and the definition of a limit. But the notations can be discussed right away. They are used constantly and you also need to know how to read them aloud:

$f(t) =$ "f of t" $=$ the value of the function f at time t

$\Delta t =$ "delta t" $=$ the time step forward or backward from t

$f(t+\Delta t) =$ "f of t plus delta t" $=$ the value of f at time $t+\Delta t$

$\Delta f =$ "delta f" $=$ the change $f(t+\Delta t) - f(t)$

$\Delta f/\Delta t =$ "delta f over delta t" $=$ the average velocity

$f'(t) =$ "f prime of t" $=$ the value of the derivative at time t

$df/dt =$ "$d\ f\ d\ t$" $=$ the same as f' (the instantaneous velocity)

$\lim_{\Delta\to 0} =$ "limit as delta t goes to zero" $=$ the process that starts with numbers $\Delta f/\Delta t$ and produces the number df/dt.

From those last words you see what lies behind the notation df/dt. The symbol Δt indicates a nonzero (usually short) length of time. The symbol dt indicates an infinitesimal (even shorter) length of time. Some mathematicians work separately with df and dt, and df/dt is their ratio. For us df/dt is a single notation (don't cancel d and don't cancel Δ). The derivative df/dt is the limit of $\Delta f/\Delta t$. *When that notation df/dt is awkward, use f' or v.*

Remark The notation hides one thing we should mention. The time step can be *negative* just as easily as positive. We can compute the average $\Delta f/\Delta t$ over a time interval *before* the time t, instead of after. This ratio also approaches df/dt.

The notation also hides another thing: ***The derivative might not exist***. The averages $\Delta f/\Delta t$ might not approach a limit (it has to be the same limit going forward and backward from time t). In that case $f'(t)$ is not defined. At that instant there is no clear reading on the speedometer. This will happen in Example 2.

EXAMPLE 1 (Constant velocity $V = 2$) The distance f is V times t. The distance at time $t+\Delta t$ is V times $t+\Delta t$. ***The difference*** Δf ***is*** V ***times*** Δt:

$$\frac{\Delta f}{\Delta t} = \frac{V\Delta t}{\Delta t} = V \quad \text{so the limit is} \quad \frac{df}{dt} = V.$$

The derivative of Vt is V. The derivative of $2t$ is 2. The averages $\Delta f/\Delta t$ are always $V = 2$, in this exceptional case of a constant velocity.

EXAMPLE 2 Constant velocity 2 up to time $t=3$, ***then stop***.

For small times we still have $f(t)=2t$. But after the stopping time, the distance is fixed at $f(t)=6$. The graph is flat beyond time 3. Then $f(t+\Delta t)=f(t)$ and $\Delta f=0$ and ***the derivative of a constant function is zero***:

$$t>3: \quad f'(t)=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{0}{\Delta t}=0. \qquad (4)$$

In this example ***the derivative is not defined at the instant when*** $t=3$. The velocity falls suddenly from 2 to zero. The ratio $\Delta f/\Delta t$ depends, at that special moment, on whether Δt is positive or negative. The average velocity *after* time $t=3$ is zero. The average velocity *before* that time is 2. When the graph of f has a corner, the graph of v has a ***jump***. It is a ***step function***.

One new part of that example is the notation (df/dt or f' instead of v). Please look also at the third figure. It shows how the function takes t (on the left) to $f(t)$. Especially it shows Δt and Δf. At the start, $\Delta f/\Delta t$ is 2. After the stop at $t=3$, all t's go to the same $f(t)=6$. So $\Delta f=0$ and $df/dt=0$.

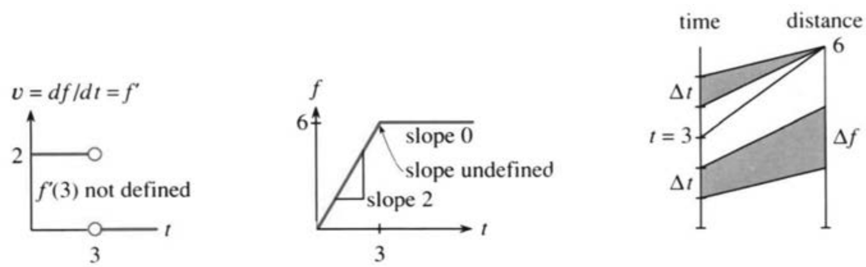


Fig. 2.1 The derivative is 2 then 0. It does not exist at $t=3$.

THE DERIVATIVE OF $1/t$

Here is a completely different slope, for the "demand function" $f(t)=1/t$. The demand is $1/t$ when the price is t. A high price t means a low demand $1/t$. Increasing the price reduces the demand. The calculus question is: ***How quickly does*** $1/t$ ***change when*** t ***changes***? The "marginal demand" is the slope of the demand curve.

The big thing is to find the derivative of $1/t$ once and for all. It is $-1/t^2$.

EXAMPLE 3 $f(t)=\dfrac{1}{t}$ has $\Delta f=\dfrac{1}{t+\Delta t}-\dfrac{1}{t}$. This equals $\dfrac{t-(t+\Delta t)}{t(t+\Delta t)}=\dfrac{-\Delta t}{t(t+\Delta t)}$.

$$\text{Divide by } \Delta t \text{ and let } \Delta t\to 0: \frac{\Delta f}{\Delta t}=\frac{-1}{t(t+\Delta t)} \text{ approaches } \frac{df}{dt}=\frac{-1}{t^2}.$$

Line 1 is algebra, line 2 is calculus. The first step in line 1 subtracts $f(t)$ from $f(t+\Delta t)$. The difference is $1/(t+\Delta t)$ minus $1/t$. The common denominator is t times $t+\Delta t$—this makes the algebra possible. We can't set $\Delta t=0$ in line 2, until we have divided by Δt.

The average is $\Delta f/\Delta t=-1/t(t+\Delta t)$. Now set $\Delta t=0$. The derivative is $-1/t^2$. Section 2.4 will discuss the first of many cases when substituting $\Delta t=0$ is not possible, and the idea of a limit has to be made clearer.

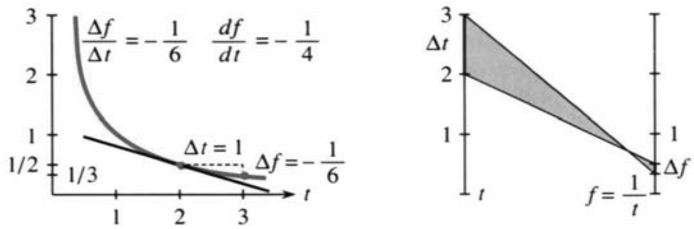


Fig. 2.2 Average slope is $-\frac{1}{6}$, true slope is $-\frac{1}{4}$. Increase in t produces decrease in f.

Check the algebra at $t = 2$ and $t + \Delta t = 3$. The demand $1/t$ drops from $1/2$ to $1/3$. The difference is $\Delta f = -1/6$, which agrees with $-1/(2)(3)$ in line 1. As the steps Δf and Δt get smaller, their ratio approaches $-1/(2)(2) = -1/4$.

This derivative is negative. The function $1/t$ is *decreasing*, and Δf is below zero. The graph is going *downward* in Figure 2.2, and its slope is negative:

An increasing $f(t)$ has positive slope. A decreasing $f(t)$ has negative slope.

The slope $-1/t^2$ is very negative for small t. A price increase severely cuts demand.

The next figure makes a small but important point. There is nothing sacred about t. Other letters can be used—especially x. A quantity can depend on ***position instead of time***. The height changes as we go west. The area of a square changes as the side changes. Those are not affected by the passage of time, and there is no reason to use t. You will often see $y = f(x)$, with x across and y up—connected by a function f.

Similarly, f is not the only possibility. Not every function is named f! That letter is useful because it stands for the word function—but we are perfectly entitled to write $y(x)$ or $y(t)$ instead of $f(x)$ or $f(t)$. The distance up is a function of the distance across. This relationship "y of x" is all-important to mathematics.

The slope is also a function. Calculus is about two functions, $y(x)$ and dy/dx.

Question If we add 1 to $y(x)$, what happens to the slope? *Answer* Nothing.

Question If we add 1 to the slope, what happens to the height? *Answer* _____ .

The symbols t and x represent ***independent variables***—they take any value they want to (in the domain). Once they are set, $f(t)$ and $y(x)$ are determined. Thus f and y represent ***dependent variables***—they *depend* on t and x. A change Δt produces a

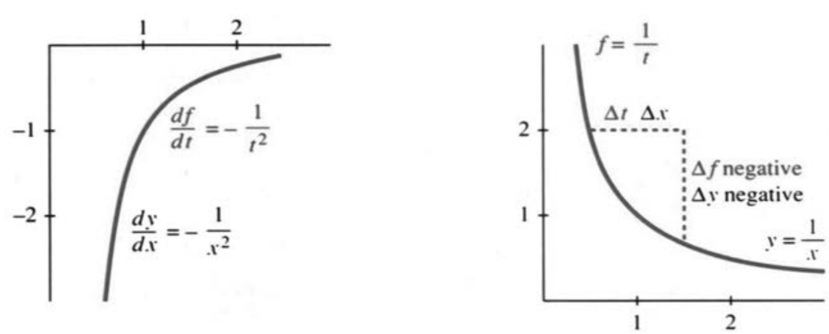


Fig. 2.3 The derivative of $1/t$ is $-1/t^2$. The slope of $1/x$ is $-1/x^2$.

change Δf. A change Δx produces Δy. The *in*dependent variable goes *in*side the parentheses in $f(t)$ and $y(x)$. It is not the letter that matters, it is the idea:

independent variable t or x
dependent variable f or g or y or z or u
derivative df/dt or df/dx or dy/dx or $\cdots$

The derivative dy/dx comes from [change in y] divided by [change in x]. The time step becomes a space step, forward or backward. The slope is the rate at which y changes with x. ***The derivative of a function is its "rate of change."***

I mention that physics books use $x(t)$ for distance. Darn it.

To emphasize the definition of a derivative, here it is again with y and x:

$$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x)-y(x)}{\Delta x} = \frac{\text{distance up}}{\text{distance across}} \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'(x).$$

The notation $y'(x)$ pins down the point x where the slope is computed. In dy/dx that extra precision is omitted. This book will try for a reasonable compromise between logical perfection and ordinary simplicity. The notation $dy/dx(x)$ is not good; $y'(x)$ is better; when x is understood it need not be written in parentheses.

You are allowed to say that the function is $y = x^2$ and the derivative is $y' = 2x$—even if the strict notation requires $y(x) = x^2$ and $y'(x) = 2x$. You can even say that the function is x^2 and its derivative is $2x$ and its ***second derivative*** is 2—provided everybody knows what you mean.

Here is an example. It is a little early and optional but terrific. You get excellent practice with letters and symbols, and out come new derivatives.

EXAMPLE 4 If $u(x)$ has slope du/dx, what is the slope of $f(x) = (u(x))^2$?

From the derivative of x^2 this will give the derivative of x^4. In that case $u = x^2$ and $f = x^4$. First point: ***The derivative of*** u^2 ***is not*** $(du/dx)^2$. We do not square the derivative $2x$. To find the "square rule" we start as we have to—with $\Delta f = f(x+\Delta x) - f(x)$:

$$\Delta f = (u(x+\Delta x))^2 - (u(x))^2 = [u(x+\Delta x)+u(x)][u(x+\Delta x)-u(x)].$$

This algebra puts Δf in a convenient form. We factored $a^2 - b^2$ into $[a+b]$ times $[a-b]$. Notice that we don't have $(\Delta u)^2$. We have Δf, the change in u^2. Now divide by Δx and take the limit:

$$\frac{\Delta f}{\Delta x} = [u(x+\Delta x)+u(x)]\left[\frac{u(x+\Delta x)-u(x)}{\Delta x}\right] \text{ approaches } 2u(x)\frac{du}{dx}. \quad (5)$$

This is the *square rule*: ***The derivative of*** $(u(x))^2$ ***is*** $2u(x)$ ***times*** du/dx. From the derivatives of x^2 and $1/x$ and $\sin x$ (all known) the examples give new derivatives.

EXAMPLE 5 $(u = x^2)$ The derivative of x^4 is $2u\ du/dx = 2(x^2)(2x) = 4x^3$.

EXAMPLE 6 $(u = 1/x)$ The derivative of $1/x^2$ is $2u\ du/dx = (2/x)(-1/x^2) = -2/x^3$.

EXAMPLE 7 $(u = \sin x,\ du/dx = \cos x)$ The derivative of $u^2 = \sin^2 x$ is $2 \sin x \cos x$.

Mathematics is really about ideas. The notation is created to express those ideas. Newton and Leibniz invented calculus independently, and Newton's friends spent a lot of time proving that he was first. He was, but it was Leibniz who thought of writing dy/dx—which caught on. It is the perfect way to suggest the limit of $\Delta y/\Delta x$. Newton was one of the great scientists of all time, and calculus was one of the great inventions of all time-but the notation must help. You now can write and speak about the derivative. What is needed is a longer list of functions and derivatives.

2.1 EXERCISES

Read-through questions

The derivative is the __a__ of $\Delta f/\Delta t$ as Δt approaches __b__. Here Δf equals __c__. The step Δt can be positive or __d__. The derivative is written v or __e__ or __f__. If $f(x)=2x+3$ and $\Delta x=4$ then $\Delta f=$ __g__. If $\Delta x=-1$ then $\Delta f=$ __h__. If $\Delta x=0$ then $\Delta f=$ __i__. The slope is not $0/0$ but $df/dx=$ __j__.

The derivative does not exist where $f(t)$ has a __k__ and $v(t)$ has a __l__. For $f(t)=1/t$ the derivative is __m__. The slope of $y=4/x$ is $dy/dx=$ __n__. A decreasing function has a __o__ derivative. The __p__ variable is t or x and the __q__ variable is f or y. The slope of y^2 (is) (is not) $(dy/dx)^2$. The slope of $(u(x))^2$ is __r__ by the square rule. The slope of $(2x+3)^2$ is __s__.

1 Which of the following numbers (*as is*) gives df/dt at time t? If in doubt test on $f(t)=t^2$.

(a) $\dfrac{f(t+\Delta t)-f(t)}{\Delta t}$ (b) $\lim\limits_{h\to 0}\dfrac{f(t+2h)-f(t)}{2h}$

(c) $\lim\limits_{\Delta t\to 0}\dfrac{f(t-\Delta t)-f(t)}{-\Delta t}$ (d) $\lim\limits_{t\to 0}\dfrac{f(t+\Delta t)-f(t)}{\Delta t}$

2 Suppose $f(x)=x^2$. Compute each ratio and set $h=0$:

(a) $\dfrac{f(x+h)-f(x)}{h}$ (b) $\dfrac{f(x+5h)-f(x)}{5h}$

(c) $\dfrac{f(x+h)-f(x-h)}{2h}$ (d) $\dfrac{f(x+1)-f(x)}{h}$

3 For $f(x)=3x$ and $g(x)=1+3x$, find $f(4+h)$ and $g(4+h)$ and $f'(4)$ and $g'(4)$. Sketch the graphs of f and g—why do they have the same slope?

4 Find three functions with the same slope as $f(x)=x^2$.

5 For $f(x)=1/x$, sketch the graphs of $f(x)+1$ and $f(x+1)$. Which one has the derivative $-1/x^2$?

6 Choose c so that the line $y=x$ is tangent to the parabola $y=x^2+c$. They have the same slope where they touch.

7 Sketch the curve $y(x)=1-x^2$ and compute its slope at $x=3$.

8 If $f(t)=1/t$, what is the average velocity between $t=\frac{1}{2}$ and $t=2$? What is the average between $t=\frac{1}{2}$ and $t=1$? What is the average (to one decimal place) between $t=\frac{1}{2}$ and $t=101/200$?

9 Find $\Delta y/\Delta x$ for $y(x)=x+x^2$. Then find dy/dx.

10 Find $\Delta y/\Delta x$ and dy/dx for $y(x)=1+2x+3x^2$.

11 When $f(t)=4/t$, simplify the difference $f(t+\Delta t)-f(t)$, divide by Δt, and set $\Delta t=0$. The result is $f'(t)$.

12 Find the derivative of $1/t^2$ from $\Delta f(t)=1/(t+\Delta t)^2-1/t^2$. Write Δf as a fraction with the denominator $t^2(t+\Delta t)^2$. Divide the numerator by Δt to find $\Delta f/\Delta t$. Set $\Delta t=0$.

13 Suppose $f(t)=7t$ to $t=1$. Afterwards $f(t)=7+9(t-1)$.

(a) Find df/dt at $t=\frac{1}{2}$ and $t=\frac{3}{2}$.

(b) Why doesn't $f(t)$ have a derivative at $t=1$?

14 Find the derivative of the derivative (the *second derivative*) of $y=3x^2$. What is the third derivative?

15 Find numbers A and B so that the straight line $y=x$ fits smoothly with the curve $Y=A+Bx+x^2$ at $x=1$. Smoothly means that $y=Y$ and $dy/dx=dY/dx$ at $x=1$.

16 Find numbers A and B so that the horizontal line $y=4$ fits smoothly with the curve $y=A+Bx+x^2$ at the point $x=2$.

17 *True* (with reason) *or false* (with example):

(a) If $f(t)<0$ then $df/dt<0$.

(b) The derivative of $(f(t))^2$ is $2df/dt$.

(c) The derivative of $2f(t)$ is $2df/dt$.

(d) The derivative is the limit of Δf divided by the limit of Δt.

18 For $f(x)=1/x$ the *centered difference* $f(x+h)-f(x-h)$ is $1/(x+h)-1/(x-h)$. Subtract by using the common denominator $(x+h)(x-h)$. Then divide by $2h$ and set $h=0$. Why divide by $2h$ to obtain the correct derivative?

19 Suppose $y=mx+b$ for negative x and $y=Mx+B$ for $x\geqslant 0$. The graphs meet if ______. The two slopes are ______. The slope at $x=0$ is ______ (what is possible?).

20 The slope of $y=1/x$ at $x=1/4$ is $y'=-1/x^2=-16$. At $h=1/12$, which of these ratios is closest to -16?

$$\frac{y(x+h)-y(x)}{h}\qquad\frac{y(x)-y(x-h)}{h}\qquad\frac{y(x+h)-y(x-h)}{2h}$$

21 Find the average slope of $y=x^2$ between $x=x_1$ and $x=x_2$. What does this average approach as x_2 approaches x_1?

22 Redraw Figure 2.1 when $f(t) = 3 - 2t$ for $t \leqslant 2$ and $f(t) = -1$ for $t \geqslant 2$. Include df/dt.

23 Redraw Figure 2.3 for the function $y(x) = 1 - (1/x)$. Include dy/dx.

24 The limit of $0/\Delta t$ as $\Delta t \to 0$ is not $0/0$. Explain.

25 Guess the limits by an informal working rule. Set $\Delta t = 0.1$ and -0.1 and imagine Δt becoming smaller:

(a) $\dfrac{1+\Delta t}{2+\Delta t}$ (b) $\dfrac{|\Delta t|}{\Delta t}$

(c) $\dfrac{\Delta t + (\Delta t)^2}{\Delta t - (\Delta t)^2}$ (d) $\dfrac{t+\Delta t}{t-\Delta t}$

***26** Suppose $f(x)/x \to 7$ as $x \to 0$. Deduce that $f(0) = 0$ and $f'(0) = 7$. Give an example other than $f(x) = 7x$.

27 What is $\lim\limits_{x\to 0} \dfrac{f(3+x) - f(3)}{x}$ if it exists? What if $x \to 1$?

Problems 28–31 use the square rule: $d(u^2)/dx = 2u(du/dx)$.

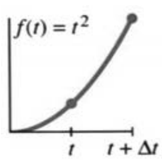
28 Take $u = x$ and find the derivative of x^2 (a new way).

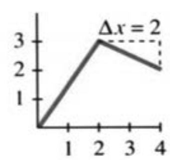
29 Take $u = x^4$ and find the derivative of x^8 (using $du/dx = 4x^3$).

30 If $u = 1$ then $u^2 = 1$. Then $d1/dx$ is 2 times $d1/dx$. How is this possible?

31 Take $u = \sqrt{x}$. The derivative of $u^2 = x$ is $1 = 2u(du/dx)$. So what is du/dx, the derivative of $\sqrt{x}$?

32 The left figure shows $f(t) = t^2$. Indicate distances $f(t+\Delta t)$ and Δt and Δf. Draw lines that have slope $\Delta f/\Delta t$ and $f'(t)$.





33 The right figure shows $f(x)$ and Δx. Find $\Delta f/\Delta x$ and $f'(2)$.

34 Draw $f(x)$ and Δx so that $\Delta f/\Delta x = 0$ but $f'(x) \neq 0$.

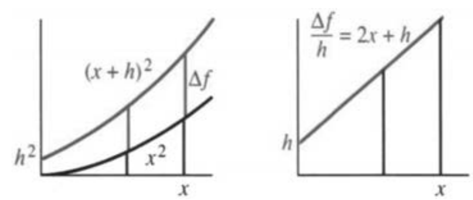
35 If $f = u^2$ then $df/dx = 2u\ du/dx$. If $g = f^2$ then $dg/dx = 2f\ df/dx$. Together those give $g = u^4$ and $dg/dx =$ ______.

36 ***True or false***, assuming $f(0) = 0$:

(a) If $f(x) \leqslant x$ for all x, then $df/dx \leqslant 1$.

(b) If $df/dx \leqslant 1$ for all x, then $f(x) \leqslant x$.

37 The graphs show Δf and $\Delta f/h$ for $f(x) = x^2$. Why is $2x+h$ the equation for $\Delta f/h$? If h is cut in half, draw in the new graphs.



38 Draw the corresponding graphs for $f(x) = \frac{1}{2}x$.

39 Draw $1/x$ and $1/(x+h)$ and $\Delta f/h$—either by hand with $h = \frac{1}{2}$ or by computer to show $h \to 0$.

40 For $y = e^x$, show on computer graphs that $dy/dx = y$.

41 *Explain the derivative in your own words.*

2.2 Powers and Polynomials

This section has two main goals. One is to find the derivatives of $f(x) = x^3$ and x^4 and x^5 (and more generally $f(x) = x^n$). The *power* or *exponent* n is at first a positive integer. Later we allow x^π and $x^{2.2}$ and every x^n.

The other goal is different. While computing these derivatives, we look ahead to their applications. In using calculus, we meet *equations with derivatives in them*—"***differential equations***." It is too early to solve those equations. But it is not too early to see the purpose of what we are doing. Our examples come from economics and biology.

With $n = 2$, the derivative of x^2 is $2x$. With $n = -1$, the slope of x^{-1} is $-1x^{-2}$. Those are two pieces in a beautiful pattern, which it will be a pleasure to discover. We begin with x^3 and its derivative $3x^2$, before jumping to x^n.

EXAMPLE 1 If $f(x) = x^3$ then $\Delta f = (x+h)^3 - x^3 = (x^3 + 3x^2h + 3xh^2 + h^3) - x^3$.

Step 1: Cancel x^3. **Step 2:** Divide by h. **Step 3:** h goes to zero.

$$\frac{\Delta f}{h} = 3x^2 + 3xh + h^2 \quad \text{approaches} \quad \frac{df}{dx} = 3x^2.$$

That is straightforward, and you see the crucial step. The power $(x+h)^3$ yields four separate terms $x^3 + 3x^2h + 3xh^2 + h^3$. (Notice 1, 3, 3, 1.) After x^3 is subtracted, we can divide by h. At the limit $(h = 0)$ we have $3x^2$.

For $f(x) = x^n$ the plan is the same. A step of size h leads to $f(x+h) = (x+h)^n$. One reason for algebra is to calculate powers like $(x+h)^n$, and if you have forgotten the binomial formula we can recapture its main point. Start with $n = 4$:

$$(x+h)(x+h)(x+h)(x+h) = x^4 + \quad ??? \quad + h^4. \tag{1}$$

Multiplying the four x's gives x^4. Multiplying the four h's gives h^4. These are the easy terms, but not the crucial ones. The subtraction $(x+h)^4 - x^4$ will remove x^4, and the limiting step $h \to 0$ will wipe out h^4 (even after division by h). ***The products that matter are those with exactly one*** h. In Example 1 with $(x+h)^3$, this key term was $3x^2h$. Division by h left $3x^2$.

With only one h, *there are* n *places it can come from*. Equation (1) has four h's in parentheses, and four ways to produce x^3h. Therefore the key term is $4x^3h$. (Division by h leaves $4x^3$.) In general there are n parentheses and n ways to produce $x^{n-1}h$, so the ***binomial formula*** contains $nx^{n-1}h$:

$$(x+h)^n = x^n + \underline{nx^{n-1}h} + \cdots + h^n. \tag{2}$$

> **2B** For $n = 1, 2, 3, 4, \ldots$, the derivative of x^n is nx^{n-1}.

Subtract x^n from (2). Divide by h. The key term is nx^{n-1}. The rest disappears as $h \to 0$:

$$\frac{\Delta f}{\Delta x} = \frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \cdots + h^n}{h} \quad \text{so} \quad \frac{df}{dx} = nx^{n-1}.$$

The terms replaced by the dots involve h^2 and h^3 and higher powers. After dividing by h, they still have at least one factor h. All those terms vanish as h approaches zero.

EXAMPLE 2 $(x+h)^4 = x^4 + \underline{4x^3h} + 6x^2h^2 + 4xh^3 + h^4$. This is $n=4$ in detail.

Subtract x^4, divide by h, let $h \to 0$. The derivative is $4x^3$. The coefficients $1,4,6,4,1$ are in Pascal's triangle below. For $(x+h)^5$ the next row is $1,5,10,$ _?_.

Remark The missing terms in the binomial formula (replaced by the dots) contain all the products $x^{n-j}h^j$. An x or an h comes from each parenthesis. The binomial coefficient "*n choose j*" is ***the number of ways to choose*** j h's ***out of*** n ***parentheses***. It involves n *factorial*, which is $n(n-1)\cdots(1)$. Thus $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$.

These are numbers that gamblers know and love:

$$\text{“}n \text{ choose } j\text{”} = \binom{n}{j} = \frac{n!}{j!(n-j)!}$$

1	***Pascal's***
1 1	***triangle***
1 2 1	
1 3 3 1	$n=3$
1 4 6 4 1	$n=4$

In the last row, the coefficient of x^3h is $4!/1!3! = 4\cdot3\cdot2\cdot1/1\cdot3\cdot2\cdot1 = 4$. For the x^2h^2 term, with $j=2$, there are $4\cdot3\cdot2\cdot1/2\cdot1\cdot2\cdot1 = 6$ ways to choose two h's. Notice that $1+4+6+4+1$ equals 16, which is 2^4. Each row of Pascal's triangle adds to a power of 2.

Choosing 6 numbers out of 49 in a lottery, the odds are $49\cdot48\cdot47\cdot46\cdot45\cdot44/6!$ to 1. That number is $N =$ "49 choose 6" $= 13,983,816$. It is the coefficient of $x^{43}h^6$ in $(x+h)^{49}$. If λ times N tickets are bought, the expected number of winners is λ. The chance of no winner is $e^{-\lambda}$. The chance of *one* winner is $\lambda e^{-\lambda}$. See Section 8.4.

Florida's lottery in September 1990 (these rules) had six winners out of 109, 163, 978 tickets.

DERIVATIVES OF POLYNOMIALS

Now we have an infinite list of functions and their derivatives:

$$x \;\; x^2 \;\; x^3 \;\; x^4 \;\; x^5 \;\; \cdots \qquad\qquad 1 \;\; 2x \;\; 3x^2 \;\; 4x^3 \;\; 5x^4 \;\; \cdots$$

The derivative of x^n ***is*** n ***times the next lower power*** x^{n-1}. ***That rule extends beyond these integers*** $1,2,3,4,5$ ***to all powers:***

$f = 1/x$	has	$f' = -1/x^2$:	Example 3 of section 2.1	$(n=-1)$
$f = 1/x^2$	has	$f' = -2/x^3$:	Example 6 of section 2.1	$(n=-2)$
$f = \sqrt{x}$	has	$f' = \frac{1}{2}x^{-1/2}$:	true but not yet checked	$(n=\frac{1}{2})$

Remember that x^{-2} means $1/x^2$ and $x^{-1/2}$ means $1/\sqrt{x}$. Negative powers lead to *decreasing* functions, approaching zero as x gets large. Their slopes have minus signs.

Question What are the derivatives of x^{10} and $x^{2.2}$ and $x^{-1/2}$?
Answer $10x^9$ and $2.2x^{1.2}$ and $-\frac{1}{2}x^{-3/2}$. Maybe $(x+h)^{2.2}$ is a little unusual. Pascal's triangle can't deal with this fractional power, but the formula stays firm: ***After*** $x^{2.2}$ ***comes*** $2.2x^{1.2}h$. The complete binomial formula is in Section 10.5.

That list is a good start, but plenty of functions are left. What comes next is really simple. A tremendous number of new functions are "linear combinations" like

$$f(x) = 6x^3 \quad \text{or} \quad 6x^3 + \frac{1}{2}x^2 \quad \text{or} \quad 6x^3 - \frac{1}{2}x^2.$$

What are their derivatives? The answers are known for x^3 and x^2, and we want to multiply by 6 or divide by 2 or add or subtract. *Do the same to the derivatives*:

$$f'(x) = 18x^2 \quad \text{or} \quad 18x^2 + x \quad \text{or} \quad 18x^2 - x.$$

2C The derivative of c times $f(x)$ is c times $f'(x)$.

2D The derivative of $f(x) + g(x)$ is $f'(x) + g'(x)$.

The number c can be any constant. We can add (or subtract) any functions. The rules allow any combination of f and g: ***The derivative of*** $9f(x) - 7g(x)$ ***is*** $9f'(x) - 7g'(x)$.

The reasoning is direct. When $f(x)$ is multiplied by c, so is $f(x+h)$. The difference Δf is also multiplied by c. All averages $\Delta f/h$ contain c, so their limit is cf'. The only incomplete step is the last one (the limit). ***We still have to say what*** "*limit*" ***means***.

Rule 2D is similar. Adding $f + g$ means adding $\Delta f + \Delta g$. Now divide by h. In the limit as $h \to 0$ we reach $f' + g'$—because a limit of sums is a sum of limits. Any example is easy and so is the proof—it is the definition of limit that needs care (Section 2.6).

You can now find the derivative of every polynomial. A "polynomial" is a combination of $1, x, x^2, \ldots, x^n$—for example $9 + 2x - x^5$. That particular polynomial has slope $2 - 5x^4$. Note that the derivative of 9 is zero! A constant just raises or lowers the graph, without changing its slope. It alters the mileage before starting the car.

The disappearance of constants is one of the nice things in differential calculus. The reappearance of those constants is one of the headaches in integral calculus. When you find v from f, the starting mileage doesn't matter. The constant in f has no effect on v. (Δf *is measured by a trip meter*; Δt *comes from a stopwatch*.) To find distance from velocity, you need to know the mileage at the start.

A LOOK AT DIFFERENTIAL EQUATIONS (FIND y FROM dy/dx)

We know that $y = x^3$ has the derivative $dy/dx = 3x^2$. Starting with the function, we found its slope. Now reverse that process. ***Start with the slope and find the function***. This is what science does all the time—and it seems only reasonable to say so.

Begin with $dy/dx = 3x^2$. The slope is given, the function y is not given.

Question Can you go backward to reach $y = x^3$?
Answer Almost but not quite. You are only entitled to say that $y = x^3 + C$. The constant C is the starting value of y (when $x = 0$). Then the ***differential equation*** $dy/dx = 3x^2$ is solved.

Every time you find a derivative, you can go backward to solve a differential equation. The function $y = x^2 + x$ has the slope $dy/dx = 2x + 1$. In reverse, the slope $2x + 1$ produces $x^2 + x$—and all the other functions $x^2 + x + C$, shifted up and down. After going from distance f to velocity v, we return to $f + C$. But there is a lot more to differential equations. Here are two crucial points:

1. We reach dy/dx by way of $\Delta y/\Delta x$, but we have no system to go backward. With $dy/dx = (\sin x)/x$ we are lost. What function has this derivative?

2. Many equations have the same solution $y = x^3$. Economics has $dy/dx = 3y/x$. Geometry has $dy/dx = 3y^{2/3}$. These equations involve y as well as dy/dx. Function and slope are mixed together! This is typical of differential equations.

To summarize: Chapters 2-4 compute and use derivatives. Chapter 5 goes in reverse. Integral calculus discovers the function from its slope. Given dy/dx we find $y(x)$. Then Chapter 6 solves the differential equation $dy/dt = y$, function mixed with slope. Calculus moves from *derivatives* to *integrals* to *differential equations*.

This discussion of the purpose of calculus should mention a specific example. Differential equations are applied to an epidemic (like AIDS). In most epidemics the number of cases grows exponentially. The peak is quickly reached by e^t, and the epidemic dies down. Amazingly, exponential growth is not happening with AIDS—the best fit to the data through 1988 is a ***cubic polynomial*** (*Los Alamos Science*, 1989):

$$\text{The number of cases fits a cubic within } 2\%: \; y = 174.6(t-1981.2)^3 + 340.$$

This is dramatically different from other epidemics. Instead of $dy/dt = y$ we have $dy/dt = 3y/t$. Before this book is printed, we may know what has been preventing e^t (fortunately). Eventually the curve will turn away from a cubic—I hope that mathematical models will lead to knowledge that saves lives.

Added in proof: In 1989 the curve for the U.S. dropped from t^3 to t^2.

MARGINAL COST AND ELASTICITY IN ECONOMICS

First point about economics: The ***marginal*** cost and ***marginal*** income are crucially important. The average cost of making automobiles may be $\$10,000$. But it is the $\$8,000$ cost of the *next car* that decides whether Ford makes it. "*The average describes the past, the marginal predicts the future*." For bank deposits or work hours or wheat, which come in smaller units, the amounts are continuous variables. Then the word "marginal" says one thing: ***Take the derivative***.†

The average pay over all the hours we ever worked may be low. We wouldn't work another hour for that! This average is rising, but the pay for each additional hour rises faster—possibly it jumps. When $\$10$/hour increases to $\$15$/hour after a 40-hour week, a 50-hour week pays $\$550$. The average income is $\$11$/hour. The marginal income is $\$15$/hour—the overtime rate.

Concentrate next on cost. Let $y(x)$ be the cost of producing x tons of steel. The cost of $x + \Delta x$ tons is $y(x + \Delta x)$. The extra cost is the difference Δy. Divide by Δx, the number of extra tons. The ratio $\Delta y/\Delta x$ is ***the average cost per extra ton***. When Δx is an ounce instead of a ton, we are near the marginal cost dy/dx.

Example: When the cost is x^2, the average cost is $x^2/x = x$. The marginal cost is $2x$. Figure 2.4 has increasing slope—an example of "diminishing returns to scale."

This raises another point about economics. The units are arbitrary. In yen per kilogram the numbers look different. The way to correct for arbitrary units is to work with *percentage change* or ***relative change***. An increase of Δx tons is a relative increase of $\Delta x/x$. A cost increase Δy is a relative increase of $\Delta y/y$. Those are *dimensionless*, the same in tons/tons or dollars/dollars or yen/yen.

A third example is *the demand y at price x*. Now dy/dx is negative. But again the units are arbitrary. The demand is in liters or gallons, the price is in dollars or pesos.

†These paragraphs show how calculus applies to economics. You do *not* have to be an economist to understand them. Certainly the author is not, probably the instructor is not, possibly the student is not. We can all use dy/dx.

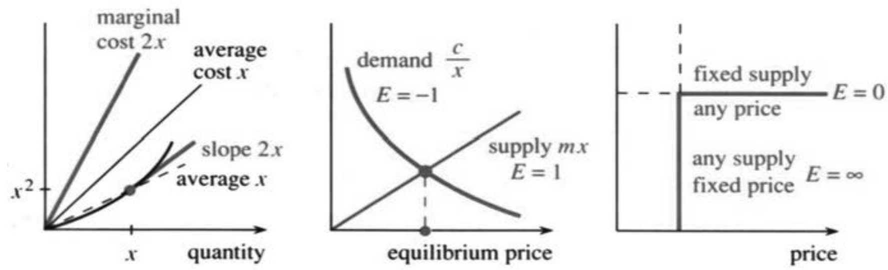


Fig. 2.4 Marginal exceeds average. Constant elasticity $E = \pm 1$. Perfectly elastic to perfectly inelastic (Γ curve).

Relative changes are better. When the price goes up by 10%, the demand may drop by 5%. If that ratio stays the same for small increases, *the elasticity of demand is* $\frac{1}{2}$.

Actually this number should be $-\frac{1}{2}$. The price rose, the demand dropped. In our definition, the elasticity *will* be $-\frac{1}{2}$. In conversation between economists the minus sign is left out (I hope not forgotten).

DEFINITION The elasticity of the demand function $y(x)$ is

$$E(x) = \lim_{\Delta x \to 0} \frac{\Delta y / y}{\Delta x / x} = \frac{dy/dx}{y/x}. \tag{3}$$

***Elasticity is* "*marginal*" *divided by* "*average*."** $E(x)$ is also relative change in y divided by relative change in x. Sometimes $E(x)$ is the same at all prices—this important case is discussed below.

EXAMPLE 4 Suppose the demand is $y = c/x$ when the price is x. The derivative $dy/dx = -c/x^2$ comes from calculus. The division $y/x = c/x^2$ is only algebra. *The ratio is* $E = -1$:

For the demand $y = c/x$***, the elasticity is*** $(-c/x^2)/(c/x^2) = -1$.

All demand curves are compared with this one. The demand is ***inelastic*** when $|E| < 1$. It is ***elastic*** when $|E| > 1$. The demand $20/\sqrt{x}$ is inelastic ($E = -\frac{1}{2}$), while x^{-3} is elastic ($E = -3$). ***The power*** $y = cx^n$, ***whose derivative we know***, ***is the function with constant elasticity*** n:

$$\text{if} \quad y = cx^n \quad \text{then} \quad dy/dx = cnx^{n-1} \quad \text{and} \quad E = cnx^{n-1}/(cx^n/x) = n.$$

It is because $y = cx^n$ sets the standard that we could come so early to economics.

In the special case when $y = c/x$, consumers spend the same at all prices. Price x times quantity y remains constant at $xy = c$.

EXAMPLE 5 The supply curve has $E > 0$—supply increases with price. Now the baseline case is $y = cx$. The slope is c and the average is $y/x = c$. *The elasticity is* $E = c/c = 1$.

Compare $E = 1$ with $E = 0$ and $E = \infty$. A constant supply is "perfectly inelastic." The power n is zero and the slope is zero: $y = c$. No more is available when the harvest is over. Whatever the price, the farmer cannot suddenly grow more wheat. Lack of elasticity makes farm economics difficult.

The other extreme $E = \infty$ is "perfectly elastic." The supply is unlimited at a fixed price x. Once this seemed true of water and timber. In reality the steep curve

$x =$ constant is leveling off to a flat curve $y =$ constant. Fixed price is changing to fixed supply, $E = \infty$ is becoming $E = 0$, and the supply of water follows a "gamma curve" shaped like Γ.

EXAMPLE 6 Demand is an increasing function of *income*—more income, more demand. The *income elasticity* is $E(I) = (dy/dI)/(y/I)$. A luxury has $E > 1$ (elastic). Doubling your income more than doubles the demand for caviar. A necessity has $E < 1$ (inelastic). The demand for bread does not double. Please recognize how the central ideas of calculus provide a language for the central ideas of economics.

Important note on supply = demand This is the basic equation of microeconomics. Where the supply curve meets the demand curve, the economy finds the equilibrium price. *Supply = demand assumes perfect competition*. With many suppliers, no one can raise the price. If someone tries, the customers go elsewhere.

The opposite case is a ***monopoly***—no competition. Instead of many small producers of wheat, there is one producer of electricity. An airport is a monopolist (and maybe the National Football League). If the price is raised, some demand remains.

Price fixing occurs when several producers act like a monopoly-which antitrust laws try to prevent. The price is not set by supply = demand. The calculus problem is different—***to maximize profit***. Section 3.2 locates the maximum where the marginal profit (the slope!) is zero.

Question on income elasticity From an income of \$10,000 you save \$500. The income elasticity of savings is $E = 2$. Out of the next dollar what fraction do you save?

Answer The savings is $y = cx^2$ because $E = 2$. The number c must give $500 = c(10{,}000)^2$, so c is $5 \cdot 10^{-6}$. Then the slope dy/dx is $2cx = 10 \cdot 10 \cdot 10^{-6} \cdot 10^4 = \frac{1}{10}$. This is the marginal savings, ten cents on the dollar. ***Average savings is*** 5%, ***marginal savings is*** 10%, ***and*** $E = 2$.

2.2 EXERCISES

Read-through questions

The derivative of $f = x^4$ is $f' =$ __a__. That comes from expanding $(x+h)^4$ into the five terms __b__. Subtracting x^4 and dividing by h leaves the four terms __c__. This is $\Delta f/h$, and its limit is __d__.

The derivative of $f = x^n$ is $f' =$ __e__. Now $(x+h)^n$ comes from the __f__ theorem. The terms to look for are $x^{n-1}h$, containing only one __g__. There are __h__ of those terms, so $(x+h)^n = x^n +$ __i__ $+ \cdots$. After subtracting __j__ and dividing by h, the limit of $\Delta f/h$ is __k__. The coefficient of $x^{n-j}h^j$, not needed here, is "n choose j" = __l__, where $n!$ means __m__.

The derivative of x^{-2} is __n__. The derivative of $x^{1/2}$ is __o__. The derivative of $3x + (1/x)$ is __p__, which uses the following rules: The derivative of $3f(x)$ is __q__ and the derivative of $f(x) + g(x)$ is __r__. Integral calculus recovers __s__ from dy/dx. If $dy/dx = x^4$ then $y(x) =$ __t__.

1 Starting with $f = x^6$, write down f' and then f''. (This is "f double prime," the derivative of f'.) After ______ derivatives of x^6 you reach a constant. What constant?

2 Find a function that has x^6 as its derivative.

Find the derivatives of the functions in 3–10. Even if n is negative or a fraction, the derivative of x^n is nx^{n-1}.

3 $x^2 + 7x + 5$

4 $1 + (7/x) + (5/x^2)$

5 $1 + x + x^2 + x^3 + x^4$

6 $(x^2+1)^2$

7 $x^n + x^{-n}$

8 $x^n/n!$

9 $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

10 $\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2}$

11 Name two functions with $df/dx = 1/x^2$.

12 ***Find the mistake***: x^2 is $x+x+\cdots+x$ (with x terms). Its derivative is $1+1+\cdots+1$ (also x terms). So the derivative of x^2 seems to be x.

13 What are the derivatives of $3x^{1/3}$ and $-3x^{-1/3}$ and $(3x^{1/3})^{-1}$?

14 The slope of $x+(1/x)$ is zero when $x=$ ______. What does the graph do at that point ?

15 Draw a graph of $y=x^3-x$. Where is the slope zero ?

16 If df/dx is negative, is $f(x)$ always negative ? Is $f(x)$ negative for large x ? If you think otherwise, give examples.

17 A rock thrown upward with velocity 16 ft/sec reaches height $f=16t-16t^2$ at time t.

(a) Find its average speed $\Delta f/\Delta t$ from $t=0$ to $t=\frac{1}{2}$.

(b) Find its average speed $\Delta f/\Delta t$ from $t=\frac{1}{2}$ to $t=1$.

(c) What is df/dt at $t=\frac{1}{2}$?

18 When f is in feet and t is in seconds, what are the units of f' and its derivative f'' ? In $f=16t-16t^2$, the first 16 is ft/sec but the second 16 is ______.

19 Graph $y=x^3+x^2-x$ from $x=-2$ to $x=2$ and estimate where it is decreasing. Check the transition points by solving $dy/dx=0$.

20 At a point where $dy/dx=0$, what is special about the graph of $y(x)$? Test case: $y=x^2$.

21 Find the slope of $y=\sqrt{x}$ by algebra (then $h\to 0$):

$$\frac{\Delta y}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}.$$

22 Imitate Problem 21 to find the slope of $y=1/\sqrt{x}$.

23 Complete Pascal's triangle for $n=5$ and $n=6$. Why do the numbers across each row add to 2^n ?

24 Complete $(x+h)^5=x^5+$ ______. What are the binomial coefficients $\binom{5}{1}$ and $\binom{5}{2}$ and $\binom{5}{3}$?

25 Compute $(x+h)^3-(x-h)^3$, divide by $2h$, and set $h=0$. *Why divide by $2h$ to find this slope* ?

26 Solve the differential equation $y''=x$ to find $y(x)$.

27 For $f(x)=x^2+x^3$, write out $f(x+\Delta x)$ and $\Delta f/\Delta x$. What is the limit at $\Delta x=0$ and what rule about sums is confirmed ?

28 The derivative of $(u(x))^2$ is ______ from Section 2.1. Test this rule on $u=x^n$.

29 What are the derivatives of x^7+1 and $(x+1)^7$? Shift the graph of x^7.

30 If df/dx is $v(x)$, what functions have these derivatives ?

(a) $4v(x)$ (b) $v(x)+1$

(c) $v(x+1)$ (d) $v(x)+v'(x)$.

31 What function $f(x)$ has fourth derivative equal to 1 ?

32 What function $f(x)$ has nth derivative equal to 1 ?

33 Suppose $df/dx=1+x+x^2+x^3$. Find $f(x)$.

34 Suppose $df/dx=x^{-2}-x^{-3}$. Find $f(x)$.

35 $f(x)$ *can be its own derivative*. In the infinite polynomial $f=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+$ ______, what numbers multiply x^4 and x^5 if df/dx equals f ?

36 Write down a differential equation $dy/dx=$ ______ that is solved by $y=x^2$. Make the right side involve y (not just $2x$).

37 *True or false*: (a) The derivative of x^π is πx^π.

(b) The derivative of ax^n/bx^n is a/b.

(c) If $df/dx=x^4$ and $dg/dx=x^4$ then $f(x)=g(x)$.

(d) $(f(x)-f(a))/(x-a)$ approaches $f'(a)$ as $x\to a$.

(e) The slope of $y=(x-1)^3$ is $y'=3(x-1)^2$.

Problems 38–44 are about calculus in economics.

38 When the cost is $y=y_0+cx$, find $E(x)=(dy/dx)/(y/x)$. It approaches ______ for large x.

39 From an income of $x=\$10,000$ you spend $y=\$1,200$ on your car. If $E=\frac{1}{2}$, what fraction of your next dollar will be spent on the car ? Compare dy/dx (marginal) with y/x (average).

40 Name a product whose price elasticity is

(a) high (b) low (c) negative (?)

41 The demand $y=c/x$ has $dy/dx=-y/x$. Show that $\Delta y/\Delta x$ is *not* $-y/x$. (Use numbers or algebra.) Finite steps miss the special feature of infinitesimal steps.

42 The demand $y=x^n$ has $E=$ ______. The revenue xy (price times demand) has elasticity $E=$ ______.

43 $y=2x+3$ grows with marginal cost 2 from the fixed cost 3. Draw the graph of $E(x)$.

44 From an income I we save $S(I)$. The *marginal* propensity to save is ______. Elasticity is not needed because S and I have the same ______. Applied to the whole economy this is (microeconomics) (macroeconomics).

45 2^t is doubled when t increases by ______. t^3 is doubled when t increases to ______ t. The doubling time for AIDS is proportional to t.

46 Biology also leads to $dy/y=n\,dx/x$, for the relative growth of the head (dy/y) and the body (dx/x). Is $n>1$ or $n<1$ for a child ?

47 What functions have $df/dx=x^9$ and $df/dx=x^n$? Why does $n=-1$ give trouble ?

48 The slope of $y=x^3$ comes from this identity:

$$\frac{(x+h)^3-x^3}{h}=(x+h)^2+(x+h)x+x^2.$$

(a) Check the algebra. Find dy/dx as $h\to 0$.

(b) Write a similar identity for $y = x^4$.

49 (Computer graphing) Find all the points where $y = x^4 + 2x^3 - 7x^2 + 3 = 0$ and where $dy/dx = 0$.

50 The graphs of $y_1(x) = x^4 + x^3$ and $y_2(x) = 7x - 5$ touch at the point where $y_3(x) =$ ______ $= 0$. Plot $y_3(x)$ to see what is special. What does the graph of $y(x)$ do at a point where $y = y' = 0$?

51 In the Massachusetts lottery you choose 6 numbers out of 36. What is your chance to win?

52 In what circumstances would it pay to buy a lottery ticket for every possible combination, so one of the tickets would win?

2.3 The Slope and the Tangent Line

Chapter 1 started with straight line graphs. The velocity was constant (at least piecewise). The distance function was linear. Now we are facing polynomials like $x^3 - 2$ or $x^4 - x^2 + 3$, with other functions to come soon. Their graphs are definitely curved. Most functions are not close to linear—except if you focus all your attention near a single point. That is what we will do.

Over a very short range a curve looks straight. Look through a microscope, or zoom in with a computer, and there is no doubt. The graph of distance versus time becomes nearly linear. Its slope is the velocity at that moment. We want to find the line that the graph stays closest to—the "***tangent line***"—before it curves away.

The tangent line is easy to describe. We are at a particular point on the graph of $y = f(x)$. At that point x equals a and y equals $f(a)$ and the slope equals $f'(a)$. ***The tangent line goes through that point*** $x = a, y = f(a)$ ***with that slope*** $m = f'(a)$. Figure 2.5 shows the line more clearly than any equation, but we have to turn the geometry into algebra. We need the equation of the line.

EXAMPLE 1 Suppose $y = x^4 - x^2 + 3$. At the point $x = a = 1$, the height is $y = f(a) = 3$. The slope is $dy/dx = 4x^3 - 2x$. At $x = 1$ the slope is $4 - 2 = 2$. That is $f'(a)$:

The numbers $x = 1$, $y = 3$, $dy/dx = 2$ ***determine the tangent line***.

The equation of the tangent line is $y - 3 = 2(x - 1)$, *and this section explains why.*

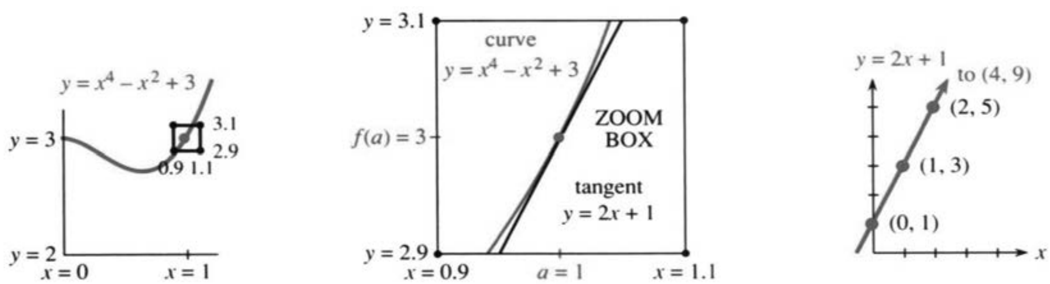


Fig. 2.5 The tangent line has the same slope 2 as the curve (especially after zoom).

THE EQUATION OF A LINE

A straight line is determined by two conditions. We know the line if we know two of its points. (We still have to write down the equation.) Also, if we know ***one point and the slope***, the line is set. That is the situation for the tangent line, which has a known slope at a known point:

1. The equation of a line has the form $y = mx + b$
2. The number m is the slope of the line, because $dy/dx = m$
3. The number b adjusts the line to go through the required point.

I will take those one at a time—first $y = mx + b$, then m, then b.

1. The graph of $y = mx + b$ is not curved. How do we know? For the specific example $y = 2x + 1$, take two points whose coordinates x, y satisfy the equation:

$$x = 0, y = 1 \quad \text{and} \quad x = 4, y = 9 \quad \text{both satisfy} \quad y = 2x + 1.$$

Those points $(0, 1)$ and $(4, 9)$ lie on the graph. *The point halfway between has* $x = 2$ *and* $y = 5$. That point also satisfies $y = 2x + 1$. ***The halfway point is on the graph***. If we subdivide again, the midpoint between $(0, 1)$ and $(2, 5)$ is $(1, 3)$. This also has $y = 2x + 1$. The graph contains all halfway points and must be straight.

2. What is the correct slope m for the tangent line? In our example it is $m = f'(a) = 2$.

The curve and its tangent line have the same slope at the crucial point: $dy/dx = 2$.

Allow me to say in another way why the line $y = mx + b$ has slope m. At $x = 0$ its height is $y = b$. At $x = 1$ its height is $y = m + b$. The graph has gone *one unit across* (0 to 1) *and* m *units up* (b to $m + b$). The whole idea is

$$\text{slope} = \frac{\text{distance up}}{\text{distance across}} = \frac{m}{1}. \tag{1}$$

Each unit across means m units up, to $2m + b$ or $3m + b$. A straight line keeps a constant slope, whereas the slope of $y = x^4 - x^2 + 3$ equals 2 only at $x = 1$.

3. Finally we decide on b. The tangent line $y = 2x + b$ must go through $x = 1$, $y = 3$. Therefore $b = 1$. With letters instead of numbers, $y = mx + b$ leads to $f(a) = ma + b$. So we know b:

> **2E** The equation of the tangent line has $b = f(a) - ma$:
>
> $$y = mx + f(a) - ma \qquad \text{or} \qquad y - f(a) = m(x - a). \tag{2}$$

That last form is the best. You see immediately what happens at $x = a$. The factor $x - a$ is zero. Therefore $y = f(a)$ as required. This is the ***point-slope form*** of the equation, and we use it constantly:

$$y - 3 = 2(x - 1) \quad \text{or} \quad \frac{y - 3}{x - 1} = \frac{\text{distance up}}{\text{distance across}} = \text{slope } 2.$$

EXAMPLE 2 The curve $y = x^3 - 2$ goes through $y = 6$ when $x = 2$. At that point $dy/dx = 3x^2 = 12$. The point-slope equation of the tangent line uses 2 and 6 and 12:

$$y - 6 = 12(x - 2) \quad \text{which is also} \quad y = 12x - 18.$$

There is another important line. It is *perpendicular* to the tangent line and *perpendicular* to the curve. This is the ***normal line*** in Figure 2.6. Its new feature is its slope. When the tangent line has slope m, the normal line has slope $-1/m$. (Rule: Slopes of perpendicular lines multiply to give -1.) Example 2 has $m = 12$, so the normal line has slope $-1/12$:

$$\textit{tangent line}\text{: } y - 6 = 12(x - 2) \qquad \textit{normal line}\text{: } y - 6 = -\tfrac{1}{12}(x - 2).$$

Light rays travel in the normal direction. So do brush fires—they move perpendicular to the fire line. Use the point-slope form! The tangent is $y = 12x - 18$, the normal is not $y = -\frac{1}{12}x - 18$.

EXAMPLE 3 You are on a roller-coaster whose track follows $y = x^2 + 4$. You see a friend at $(0,0)$ and want to get there quickly. Where do you step off ?

Solution Your path will be the tangent line (at high speed). The problem is ***to choose*** $x = a$ ***so the tangent line passes through*** $x = 0$, $y = 0$. When you step off at $x = a$,

the height is $y = a^2 + 4$ and the slope is $2a$
the equation of the tangent line is $y - (a^2 + 4) = 2a(x - a)$
this line goes through $(0,0)$ if $-(a^2 + 4) = -2a^2$ or $a = \pm 2$.

The same problem is solved by spacecraft controllers and baseball pitchers. Releasing a ball at the right time to hit a target 60 feet away is an amazing display of calculus. Quarterbacks with a moving target should read Chapter 4 on related rates.

Here is a better example than a roller-coaster. Stopping at a red light wastes gas. It is smarter to slow down early, and then accelerate. When a car is waiting in front of you, the timing needs calculus:

EXAMPLE 4 How much must you slow down when a red light is 72 meters away ? In 4 seconds it will be green. The waiting car will accelerate at 3 meters/sec^2. You cannot pass the car.

Strategy Slow down immediately to the speed V at which you will just catch that car. (If you wait and brake later, your speed will have to go below V.) At the catchup time T, the cars have the same speed and same distance. ***Two conditions***, so the distance functions in Figure 2.6d are tangent.

Solution At time T, the other car's speed is $3(T - 4)$. That shows the delay of 4 seconds. Speeds are equal when $3(T - 4) = V$ or $T = \frac{1}{3}V + 4$. Now require equal distances. Your distance is V times T. The other car's distance is $72 + \frac{1}{2}at^2$:

$$72 + \tfrac{1}{2} \cdot 3(T - 4)^2 = VT \quad \text{becomes} \quad 72 + \tfrac{1}{2} \cdot \tfrac{1}{3}V^2 = V\left(\tfrac{1}{3}V + 4\right).$$

The solution is $V = 12$ meters/second. This is 43 km/hr or 27 miles per hour.

Without the other car, you only slow down to $V = 72/4 = 18$ meters/second. As the light turns green, you go through at 65 km/hr or 40 miles per hour. Try it.

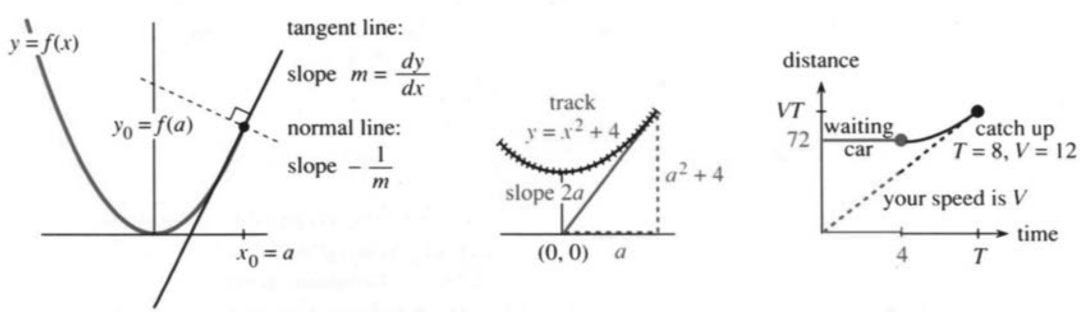


Fig. 2.6 Tangent line $y - y_0 = m(x - x_0)$. Normal line $y - y_0 = -\dfrac{1}{m}(x - x_0)$. Leaving a roller-coaster and catching up to a car.

THE SECANT LINE CONNECTING TWO POINTS ON A CURVE

Instead of the tangent line through one point, consider the ***secant line through two points***. For the tangent line the points came together. Now spread them apart. The point-slope form of a linear equation is replaced by the ***two-point form***.

The equation of the curve is still $y = f(x)$. The first point remains at $x = a$, $y = f(a)$. The other point is at $x = c$, $y = f(c)$. The secant line goes between them, and we want its equation. This time we don't start with the slope—but m is easy to find.

EXAMPLE 5 The curve $y = x^3 - 2$ goes through $x = 2$, $y = 6$. It also goes through $x = 3$, $y = 25$. The slope between those points is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{25-6}{3-2} = 19.$$

The point-slope form (at the first point) is $y - 6 = 19(x - 2)$. This line automatically goes through the second point $(3, 25)$. Check: $25 - 6$ equals $19(3 - 2)$. The secant has the right slope 19 to reach the second point. It is the *average slope* $\Delta y/\Delta x$.

A look ahead The second point is going to approach the first point. The secant slope $\Delta y/\Delta x$ will approach the tangent slope dy/dx. We discover the derivative (in the limit). That is the main point now—but not forever.

Soon you will be fast at derivatives. The exact dy/dx will be much easier than $\Delta y/\Delta x$. The situation is turned around as soon as you know that x^9 has slope $9x^8$. Near $x = 1$, the distance *up* is about 9 times the distance *across*. To find $\Delta y = 1.001^9 - 1^9$, just multiply $\Delta x = .001$ by 9. The quick approximation is .009, the calculator gives $\Delta y = .009036$. It is easier to follow the tangent line than the curve.

Come back to the secant line, and change numbers to letters. What line connects $x = a, y = f(a)$ to $x = c, y = f(c)$? A mathematician puts formulas ahead of numbers, and reasoning ahead of formulas, and ideas ahead of reasoning:

(1) The slope is $m = \dfrac{\text{distance up}}{\text{distance across}} = \dfrac{f(c) - f(a)}{c - a}$

(2) The height is $y = f(a)$ at $x = a$

(3) The height is $y = f(c)$ at $x = c$ (automatic with correct slope).

2F The ***two-point form*** uses the slope between the points:

$$\textbf{\textit{secant line}}: \quad y - f(a) = \left(\frac{f(c) - f(a)}{c - a}\right)(x - a). \qquad (3)$$

At $x = a$ the right side is zero. So $y = f(a)$ on the left side. At $x = c$ the right side has two factors $c - a$. They cancel to leave $y = f(c)$. With equation (2) for the tangent line and equation (3) for the secant line, we are ready for the moment of truth.

THE SECANT LINE APPROACHES THE TANGENT LINE

What comes now is pretty basic. It matches what we did with velocities:

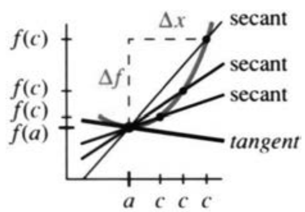
$$\text{average velocity} = \frac{\Delta \text{ distance}}{\Delta \text{ time}} = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

The limit is df/dt. We now do exactly the same thing with slopes. ***The secant line turns into the tangent line as*** c ***approaches*** a:

$$\text{slope of secant line:} \quad \frac{\Delta f}{\Delta x} = \frac{f(c) - f(a)}{c - a}$$

$$\text{slope of tangent line:} \quad \frac{df}{dx} = \text{ limit of } \frac{\Delta f}{\Delta x}.$$

There stands the fundamental idea of differential calculus! You have to imagine more secant lines than I can draw in Figure 2.7, as c comes close to a. Everybody recognizes $c - a$ as Δx. Do you recognize $f(c) - f(a)$ as $f(x + \Delta x) - f(x)$? It is Δf, the change in height. All lines go through $x = a$, $y = f(a)$. ***Their limit is the tangent line***.



secant $\quad y - f(a) = \dfrac{f(c) - f(a)}{c - a}(x - a)$

tangent $\quad y - f(a) = f'(a)(x - a)$

Fig. 2.7 Secants approach tangent as their slopes $\Delta f/\Delta x$ approach df/dx.

Intuitively, the limit is pretty clear. The two points come together, and the tangent line touches the curve at *one* point. (It could touch again at faraway points.) Mathematically this limit can be tricky—it takes us from algebra to calculus. Algebra stays away from $0/0$, but calculus gets as close as it can.

The new limit for df/dx looks different, but it is the same as before:

$$f'(a) = \lim_{c \to a} \frac{f(c) - f(a)}{c - a}. \tag{4}$$

EXAMPLE 6 Find the secant lines and tangent line for $y = f(x) = \sin x$ at $x = 0$.

The starting point is $x = 0$, $y = \sin 0$. This is the origin $(0, 0)$. The ratio of distance up to distance across is $(\sin c)/c$:

$$\textit{secant equation} \quad y = \frac{\sin c}{c} x \qquad \textit{tangent equation} \quad y = 1x.$$

As c approaches zero, the secant line becomes the tangent line. The limit of $(\sin c)/c$ is not $0/0$, which is meaningless, but 1, which is dy/dx.

EXAMPLE 7 The gold you own will be worth $\sqrt{t}$ million dollars in t years. When does the rate of increase drop to 10% of the current value, so you should sell the gold and buy a bond? At $t = 25$, how far does that put you ahead of $\sqrt{t} = 5$?

Solution The rate of increase is the derivative of $\sqrt{t}$, which is $1/2\sqrt{t}$. That is 10% of the current value $\sqrt{t}$ when $1/2\sqrt{t} = \sqrt{t}/10$. Therefore $2t = 10$ or $t = 5$. At that time you sell the gold, leave the curve, and go onto the tangent line:

$$y - \sqrt{5} = \frac{\sqrt{5}}{10}(t-5) \quad \text{becomes} \quad y - \sqrt{5} = 2\sqrt{5} \quad \text{at} \quad t = 25.$$

With straight interest on the bond, not compounded, you have reached $y = 3\sqrt{5} = 6.7$ million dollars. The gold is worth a measly five million.

2.3 EXERCISES

Read-through questions

A straight line is determined by __a__ points, or one point and the __b__. The slope of the tangent line equals the slope of the __c__. The point-slope form of the tangent equation is $y - f(a) =$ __d__.

The tangent line to $y = x^3 + x$ at $x = 1$ has slope __e__. Its equation is __f__. It crosses the y axis at __g__ and the x axis at __h__. The normal line at this point $(1,2)$ has slope __i__. Its equation is $y - 2 =$ __j__. The secant line from $(1,2)$ to $(2,$ __k__$)$ has slope __l__. Its equation is $y - 2 =$ __m__.

The point $(c, f(c))$ is on the line $y - f(a) = m(x-a)$ provided $m =$ __n__. As c approaches a, the slope m approaches __o__. The secant line approaches the __p__ line.

1 (a) Find the slope of $y = 12/x$. Find the slope of $y = 12/x$.
(b) Find the equation of the tangent line at $(2,6)$.
(c) Find the equation of the normal line at $(2,6)$.
(d) Find the equation of the secant line to $(4,3)$.

2 For $y = x^2 + x$ find equations for
(a) the tangent line and normal line at $(1,2)$;
(b) the secant line to $x = 1+h$, $y = (1+h)^2 + (1+h)$.

3 A line goes through $(1,-1)$ and $(4,8)$. Write its equation in point-slope form. Then write it as $y = mx + b$.

4 The tangent line to $y = x^3 + 6x$ at the origin is $y =$ ______. Does it cross the curve again?

5 The tangent line to $y = x^3 - 3x^2 + x$ at the origin is $y =$ ______. It is also the secant line to the point ______.

6 Find the tangent line to $x = y^2$ at $x = 4, y = 2$.

7 For $y = x^2$ the secant line from (a, a^2) to (c, c^2) has the equation ______. Do the division by $c - a$ to find the tangent line as c approaches a.

8 Construct a function that has the same slope at $x = 1$ and $x = 2$. Then find two points where $y = x^4 - 2x^2$ has the same tangent line (draw the graph).

9 Find a curve that is tangent to $y = 2x - 3$ at $x = 5$. Find the normal line to that curve at $(5,7)$.

10 For $y = 1/x$ the secant line from $(a, 1/a)$ to $(c, 1/c)$ has the equation ______. Simplify its slope and find the limit as c approaches a.

11 What are the equations of the tangent line and normal line to $y = \sin x$ at $x = \pi/2$?

12 If c and a both approach an in-between value $x = b$, then the secant slope $(f(c) - f(a))/(c-a)$ approaches ______.

13 At $x = a$ on the graph of $y = 1/x$, compute
(a) the equation of the tangent line
(b) the points where that line crosses the axes.

The triangle between the tangent line and the axes always has area ______.

14 Suppose $g(x) = f(x) + 7$. The tangent lines to f and g at $x = 4$ are ______. *True or false*: The distance between those lines is 7.

15 Choose c so that $y = 4x$ is tangent to $y = x^2 + c$. Match heights as well as slopes.

16 Choose c so that $y = 5x - 7$ is tangent to $y = x^2 + cx$.

17 For $y = x^3 + 4x^2 - 3x + 1$, find all points where the tangent is horizontal.

18 $y = 4x$ can't be tangent to $y = cx^2$. Try to match heights and slopes, or draw the curves.

19 Determine c so that the straight line joining $(0,3)$ and $(5,-2)$ is tangent to the curve $y = c/(x+1)$.

20 Choose b, c, d so that the two parabolas $y = x^2 + bx + c$ and $y = dx - x^2$ are tangent to each other at $x = 1, y = 0$.

21 The graph of $f(x) = x^3$ goes through $(1,1)$.
(a) Another point is $x = c = 1 + h$, $y = f(c) =$ ______.
(b) The change in f is $\Delta f =$ ______.
(c) The slope of the secant is $m =$ ______.
(d) As h goes to zero, m approaches ______.

22 Construct a function $y = f(x)$ whose tangent line at $x = 1$ is the same as the secant that meets the curve again at $x = 3$.

23 Draw two curves bending away from each other. Mark the points P and Q where the curves are closest. At those points, the tangent lines are ______ and the normal lines are ______.

*__24__ If the parabolas $y=x^2+1$ and $y=x-x^2$ come closest at (a,a^2+1) and $(c,c-c^2)$, set up two equations for a and c.

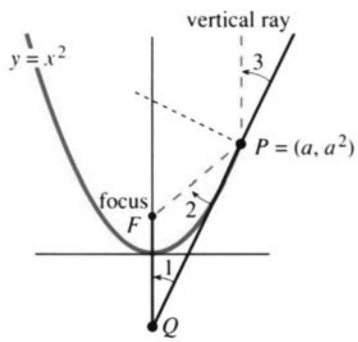
25 A light ray comes down the line $x=a$. It hits the parabolic reflector $y=x^2$ at $P=(a,a^2)$.

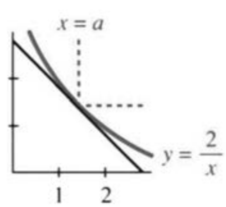
(a) Find the tangent line at P. Locate the point Q where that line crosses the y axis.

(b) Check that P and Q are the same distance from the focus at $F=(0,\frac{1}{4})$.

(c) Show from (b) that the figure has equal angles.

(d) What law of physics makes every ray reflect off the parabola to the focus at F?





26 In a bad reflector $y=2/x$, a ray down one special line $x=a$ is reflected horizontally. What is a?

27 For the parabola $4py=x^2$, where is the slope equal to 1? At that point a vertical ray will reflect horizontally. So the focus is at (0, ______).

28 Why are these statements wrong? Make them right.

(a) If $y=2x$ is the tangent line at $(1,2)$ then $y=-\frac{1}{2}x$ is the normal line.

(b) As c approaches a, the secant slope $(f(c)-f(a))/(c-a)$ approaches $(f(a)-f(a))/(a-a)$.

(c) The line through $(2,3)$ with slope 4 is $y-2=4(x-3)$.

29 A ball goes around a circle: $x=\cos t$, $y=\sin t$. At $t=3\pi/4$ the ball flies off on the tangent line. Find the equation of that line and the point where the ball hits the ground $(y=0)$.

30 If the tangent line to $y=f(x)$ at $x=a$ is the same as the tangent line to $y=g(x)$ at $x=b$, find two equations that must be satisfied by a and b.

31 Draw a circle of radius 1 resting in the parabola $y=x^2$. At the touching point (a,a^2), the equation of the normal line is ______. That line has $x=0$ when $y=$ ______. The distance to (a,a^2) equals the radius 1 when $a=$ ______. This locates the touching point.

32 Follow Problem 31 for the flatter parabola $y=\frac{1}{2}x^2$ and explain where the circle rests.

33 You are applying for a \$1000 scholarship and your time is worth \$10 a hour. If the chance of success is $1-(1/x)$ from x hours of writing, when should you stop?

34 Suppose $|f(c)-f(a)|\leqslant|c-a|$ for every pair of points a and c. Prove that $|df/dx|\leqslant 1$.

35 From which point $x=a$ does the tangent line to $y=1/x^2$ hit the x axis at $x=3$?

36 If $u(x)/v(x)=7$ find $u'(x)/v'(x)$. Also find $(u(x)/v(x))'$.

37 Find $f(c)=1.001^{10}$ in two ways—by calculator and by $f(c)-f(a)\approx f'(a)(c-a)$. Choose $a=1$ and $f(x)=x^{10}$.

38 At a distance Δx from $x=1$, how far is the curve $y=1/x$ above its tangent line?

39 At a distance Δx from $x=2$, how far is the curve $y=x^3$ above its tangent line?

40 Based on Problem 38 or 39, the distance between curve and tangent line grows like what power $(\Delta x)^p$?

41 The tangent line to $f(x)=x^2-1$ at $x_0=2$ crosses the x axis at $x_1=$ ______. The tangent line at x_1, crosses the x axis at $x_2=$ ______. Draw the curve and the two lines, which are the beginning of *Newton's method* to solve $f(x)=0$.

42 (Puzzle) The equation $y=mx+b$ requires *two* numbers, the point-slope form $y-f(a)=f'(a)(x-a)$ requires *three*, and the two-point form requires *four*: $a, f(a), c, f(c)$. How can this be?

43 Find the time T at the tangent point in Example 4, when you catch the car in front.

44 If the waiting car only accelerates at 2 meters/sec^2, what speed V must you slow down to?

45 A thief 40 meters away runs toward you at 8 meters per second. What is the smallest acceleration so that $v=at$ keeps you in front?

46 With 8 meters to go in a relay race, you slow down badly $(f=-8+6t-\frac{1}{2}t^2)$. How fast should the next runner start (choose v in $f=vt$) so you can just pass the baton?

2.4 The Derivative of the Sine and Cosine

This section does two things. One is to compute the derivatives of $\sin x$ and $\cos x$. The other is to explain why these functions are so important. They describe ***oscillation***, which will be expressed in words and equations. You will see a "***differential equation***." It involves the derivative of an unknown function $y(x)$.

The differential equation will say that the *second* derivative—***the derivative of the derivative***—is equal and opposite to y. In symbols this is $y'' = -y$. Distance in one direction leads to acceleration in the other direction. That makes y and y' and y'' all oscillate. The solutions to $y'' = -y$ are $\sin x$ and $\cos x$ and all their combinations.

We begin with the slope. The derivative of $y = \sin x$ is $y' = \cos x$. There is no reason for that to be a mystery, but I still find it beautiful. Chapter 1 followed a ball around a circle; the shadow went up and down. Its height was $\sin t$ and its velocity was $\cos t$. We now find that derivative by *the standard method of limits*, when $y(x) = \sin x$:

$$\frac{dy}{dx} = \text{limit of } \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}. \tag{1}$$

The sine is harder to work with than x^2 or x^3. Where we had $(x+h)^2$ or $(x+h)^3$, we now have $\sin(x+h)$. This calls for one of the basic "addition formulas" from trigonometry, reviewed in Section 1.5 :

$$\sin(x+h) = \sin x \cos h + \cos x \sin h \tag{2}$$

$$\cos(x+h) = \cos x \cos h - \sin x \sin h. \tag{3}$$

Equation (2) puts $\Delta y = \sin(x+h) - \sin x$ in a new form:

$$\frac{\Delta y}{\Delta x} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right). \tag{4}$$

The ratio splits into two simpler pieces on the right. Algebra and trigonometry got us this far, and now comes the calculus problem. ***What happens as*** $h \to 0$? It is no longer easy to divide by h. (I will not even mention the unspeakable crime of writing $(\sin h)/h = \sin$.) There are two critically important limits—the first is zero and the second is one:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \qquad \text{and} \qquad \lim_{h \to 0} \frac{\sin h}{h} = 1. \tag{5}$$

The careful reader will object that limits have not been defined! You may further object to computing these limits separately, before combining them into equation (4). Nevertheless—following the principle of *ideas now, rigor later*—I would like to proceed. It is entirely true that the limit of (4) comes from the two limits in (5):

$$\frac{dy}{dx} = (\sin x)(\text{first limit}) + (\cos x)(\text{second limit}) = 0 + \cos x. \tag{6}$$

The secant slope $\Delta y / \Delta x$ has approached the tangent slope dy/dx.

2G The derivative of $y = \sin x$ is $dy/dx = \cos x$.

We cannot pass over the crucial step—the two limits in (5). They contain the real ideas. ***Both ratios become*** $0/0$ ***if we just substitute*** $h=0$. Remember that the cosine of a zero angle is 1, and the sine of a zero angle is 0. Figure 2.8a shows a small angle h (as near to zero as we could reasonably draw). The edge of length $\sin h$ is close to zero, and the edge of length $\cos h$ is near 1. Figure 2.8b shows how the *ratio* of $\sin h$ to h (both headed for zero) gives the slope of the sine curve at the start.

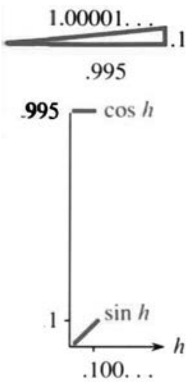


Fig. 2.8

When two functions approach zero, their ratio might do anything. We might have

$$\frac{h^2}{h} \to 0 \quad \text{or} \quad \frac{h}{h} \to 1 \quad \text{or} \quad \frac{\sqrt{h}}{h} \to \infty.$$

No clue comes from $0/0$. When matters is *whether the top or bottom goes to zero more quickly*. Roughly speaking, we want to show that $(\cos h - 1)/h$ is like h^2/h and $(\sin h)/h$ is like h/h.

Time out The graph of $\sin x$ is in Figure 2.9 (in black). The graph of $\sin(x+\Delta x)$ sits just beside it (in red). The height difference is Δf when the shift distance is Δx.

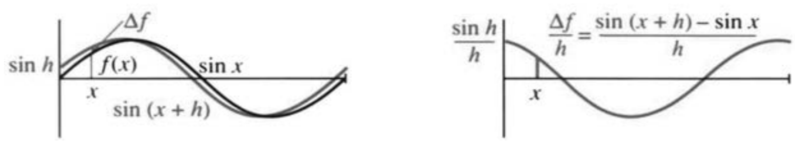


Fig. 2.9 $\sin(x+h)$ with $h = 10° = \pi/18$ radians. $\Delta f/\Delta x$ is close to $\cos x$.

Now divide by that small number Δx (or h). The second figure shows $\Delta f/\Delta x$. It is close to $\cos x$. (Look how it starts—it is not quite $\cos x$.) Mathematics will prove that the limit is $\cos x$ exactly, when $\Delta x \to 0$. Curiously, the reasoning concentrates on only one point $(x=0)$. The slope at that point is $\cos 0 = 1$.

We now prove this: $\sin \Delta x$ divided by Δx goes to 1. The sine curve starts with slope 1. By the addition formula for $\sin(x+h)$, this answer at one point will lead to the slope $\cos x$ at all points.

Question Why does the graph of $f(x+\Delta x)$ shift left from $f(x)$ when $\Delta x > 0$?
Answer When $x=0$, the shifted graph is already showing $f(\Delta x)$. In Figure 2.9a, the red graph is shifted *left* from the black graph. The red graph shows $\sin h$ when the black graph shows $\sin 0$.

THE LIMIT OF $(\sin h)/h$ IS 1

There are several ways to find this limit. The direct approach is to let a computer draw a graph. Figure 2.10a is very convincing. ***The function*** $(\sin h)/h$ ***approaches*** 1 ***at the key point*** $h=0$. So does $(\tan h)/h$. In practice, the only danger is that you might get a message like “undefined function” and no graph. (The machine may refuse to divide by zero at $h=0$. Probably you can get around that.) Because of the importance of this limit, I want to give a mathematical proof that it equals 1.

Figure 2.10b indicates, but still only graphically, that $\sin h$ *stays below* h. (The first graph shows that too; $(\sin h)/h$ is below 1.) We also see that $\tan h$ *stays above* h. Remember that the tangent is the ratio of sine to cosine. Dividing by the cosine is enough to push the tangent above h. The crucial inequalities (to be proved when h is small and positive) are

$$\sin h < h \qquad \text{and} \qquad \tan h > h. \tag{7}$$

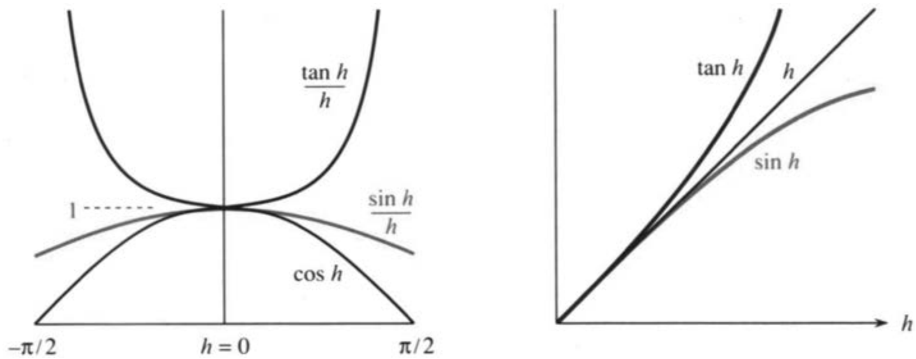


Fig. 2.10 $(\sin h)/h$ squeezed between $\cos x$ and 1; $(\tan h)/h$ decreases to 1.

Since $\tan h = (\sin h)/(\cos h)$, those are the same as

$$\frac{\sin h}{h} < 1 \qquad \text{and} \qquad \frac{\sin h}{h} > \cos h. \tag{8}$$

What happens as h goes to zero? ***The ratio*** $(\sin h)/h$ ***is squeezed between*** $\cos h$ ***and*** 1. But $\cos h$ is approaching 1! The squeeze as $h \to 0$ leaves only one possibility for $(\sin h)/h$, which is caught in between: *The ratio* $(\sin h)/h$ *approaches* 1.

Figure 2.10 shows that "squeeze play." ***If two functions approach the same limit, so does any function caught in between***. This is proved at the end of Section 2.6.

For negative values of h, which are absolutely allowed, the result is the same. To the left of zero, h reverses sign and $\sin h$ reverses sign. The ratio $(\sin h)/h$ is unchanged. (The sine is an odd function: $\sin(-h) = -\sin h$.) The ratio is an *even* function, symmetric around zero and approaching 1 from both sides.

The proof depends on $\sin h < h < \tan h$, which is displayed by the graph but not explained. We go back to right triangles.

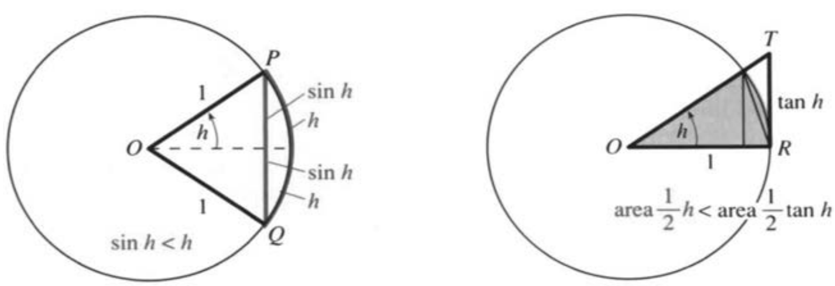


Fig. 2.11 Line shorter than arc: $2 \sin h < 2h$. Areas give $h < \tan h$.

Figure 2.11a shows why $\sin h < h$. The straight line PQ has length $2 \sin h$. The circular arc must be longer, because the shortest distance between two points is a straight line.† The arc PQ has length $2h$. (Important: *When the radius is* 1, *the arc length equals the angle*. The full circumference is 2π and the full angle is also 2π.) ***The straight distance*** $2 \sin h$ ***is less than the circular distance*** $2h$, ***so*** $\sin h < h$.

Figure 2.11b shows why $h < \tan h$. This time we look at ***areas***. The triangular area is $\frac{1}{2}$(base)(height)$= \frac{1}{2}(1)(\tan h)$. Inside that triangle is the shaded sector of the circle.

† If we try to prove that, we will be here all night. Accept it as true.

Its area is $h/2\pi$ times the area of the whole circle (because the angle is that fraction of the whole angle). The circle has area $\pi r^2 = \pi$, so multiplication by $h/2\pi$ gives $\frac{1}{2}h$ for the area of the sector. Comparing with the triangle around it, $\frac{1}{2}\tan h > \frac{1}{2}h$.

The inequalities $\sin h < h < \tan h$ are now proved. The squeeze in equation (8) produces $(\sin h)/h \to 1$. Q.E.D. Problem 13 shows how to prove $\sin h < h$ from areas.

Note All angles x and h are being measured in radians. *In degrees,* $\cos x$ *is not the derivative of* $\sin x$. A degree is much less than a radian, and dy/dx is reduced by the factor $2\pi/360$.

THE LIMIT OF $(\cos h - 1)/h$ IS 0

This second limit is different. We will show that $1 - \cos h$ shrinks to zero *more quickly* than h. Cosines are connected to sines by $(\sin h)^2 + (\cos h)^2 = 1$. We start from the known fact $\sin h < h$ and work it into a form involving cosines:

$$(1 - \cos h)(1 + \cos h) = 1 - (\cos h)^2 = (\sin h)^2 < h^2. \tag{9}$$

Note that everything is positive. Divide through by h and also by $1 + \cos h$:

$$0 < \frac{1 - \cos h}{h} < \frac{h}{1 + \cos h}. \tag{10}$$

Our ratio is caught in the middle. *The right side goes to zero because* $h \to 0$. This is another "*squeeze*"—there is no escape. Our ratio goes to zero.

For $\cos h - 1$ or for negative h, the signs change but minus zero is still zero. This confirms equation (6). The slope of $\sin x$ is $\cos x$.

Remark Equation (10) also shows that $1 - \cos h$ is approximately $\frac{1}{2}h^2$. The 2 comes from $1 + \cos h$. This is a basic purpose of calculus—to find simple approximations like $\frac{1}{2}h^2$. A "tangent parabola" $1 - \frac{1}{2}h^2$ is close to the top of the cosine curve.

THE DERIVATIVE OF THE COSINE

This will be easy. The quick way to differentiate $\cos x$ is to shift the sine curve by $\pi/2$. That yields the cosine curve (solid line in Figure 2.12b). The derivative also shifts by $\pi/2$ (dotted line). ***The derivative of*** $\cos x$ ***is*** $-\sin x$.

Notice how the dotted line (the slope) goes below zero when the solid line turns downward. The slope equals zero when the solid line is level. ***Increasing functions have positive slopes. Decreasing functions have negative slopes***. That is important, and we return to it.

There is more information in dy/dx than "function rising" or "function falling." The slope tells *how quickly* the function goes up or down. It gives the *rate of change*. The slope of $y = \cos x$ can be computed in the normal way, as the limit of $\Delta y/\Delta x$:

$$\frac{\Delta y}{\Delta x} = \frac{\cos(x+h) - \cos x}{h} = \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)$$

$$\frac{dy}{dx} = (\cos x)(0) - (\sin x)(1) = -\sin x. \tag{11}$$

The first line came from formula (3) for $\cos(x+h)$. The second line took limits, reaching 0 and 1 as before. This confirms the graphical proof that the slope of $\cos x$ is $-\sin x$.

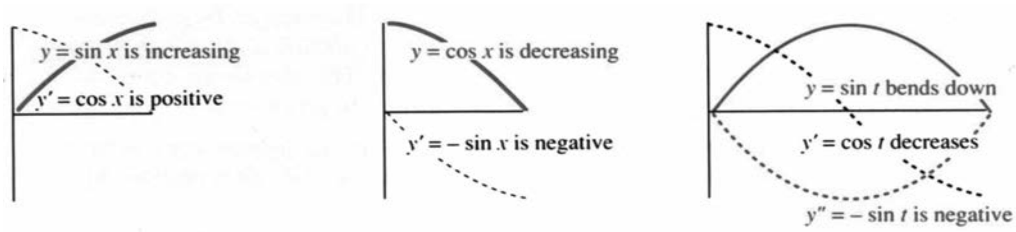


Fig. 2.12 $y(x)$ increases where y' is positive. $y(x)$ bends up where y'' is positive.

THE SECOND DERIVATIVES OF THE SINE AND COSINE

We now introduce ***the derivative of the derivative***. That is the ***second derivative*** of the original function. It tells how fast the slope is changing, not how fast y itself is changing. The second derivative is the "rate of change of the velocity." A straight line has constant slope (constant velocity), so its second derivative is zero:

$$f(t) = 5t \quad \text{has} \quad df/dt = 5 \quad \text{and} \quad d^2f/dt^2 = 0.$$

The parabola $y = x^2$ has slope $2x$ (linear) which has slope 2 (constant). Similarly

$$f(t) = \tfrac{1}{2}at^2 \quad \text{has} \quad df/dt = at \quad \text{and} \quad d^2f/dt^2 = a.$$

There stands the notation d^2f/dt^2 (or d^2y/dx^2) for the second derivative. A short form is f'' or y''. (This is pronounced f *double prime* or y *double prime*). Example: The second derivative of $y = x^3$ is $y'' = 6x$.

In the distance-velocity problem, f'' is *acceleration*. It tells how fast v is changing, while v tells how fast f is changing. Where df/dt was distance/time, the second derivative is distance/(time)^2. The acceleration due to gravity is about 32 ft/sec^2 or 9.8 m/sec^2, which means that v increases by 32 ft/sec in one second. It does not mean that the distance increases by 32 feet!

The graph of $y = \sin t$ increases at the start. Its derivative $\cos t$ is positive. However the second derivative is $-\sin t$. ***The curve is bending down while going up***. The arch is "***concave down***" because $y'' = -\sin t$ is negative.

At $t = \pi$ the curve reaches zero and goes negative. The second derivative becomes positive. *Now the curve bends upward*. The lower arch is "***concave up***."

$$y'' > 0 \text{ means that } y' \text{ increases so } y \text{ bends upward (concave up)}$$
$$y'' < 0 \text{ means that } y' \text{ decreases so } y \text{ bends down (concave down).}$$

Chapter 3 studies these things properly—here we get an advance look for $\sin t$.

The remarkable fact about the sine and cosine is that $y'' = -y$. That is unusual and special: *acceleration* $=$ $-$*distance*. The greater the distance, the greater the force pulling back:

$$y = \sin t \quad \text{has} \quad dy/dt = +\cos t \quad \text{and} \quad d^2y/dt^2 = -\sin t = -y.$$
$$y = \cos t \quad \text{has} \quad dy/dt = -\sin t \quad \text{and} \quad d^2y/dt^2 = -\cos t = -y.$$

Question Does $d^2y/dt^2 < 0$ mean that the distance $y(t)$ is decreasing?
Answer No. Absolutely not! It means that dy/dt is decreasing, not necessarily y. At the start of the sine curve, y is still increasing but $y'' < 0$.

Sines and cosines give *simple harmonic motion*—up and down, forward and back, out and in, tension and compression. Stretch a spring, and the restoring force pulls it back. Push a swing up, and gravity brings it down. These motions are controlled by a ***differential equation***:

$$\frac{d^2y}{dt^2} = -y. \tag{12}$$

All solutions are combinations of the sine and cosine: $y = A\sin t + B\cos t$.

This is not a course on differential equations. But you have to see the purpose of calculus. It models events by equations. It models oscillation by equation (12). Your heart fills and empties. Balls bounce. Current alternates. The economy goes up and down:

high prices $\rightarrow$ high production $\rightarrow$ low prices $\rightarrow$ $\cdots$

We can't live without oscillations (or differential equations).

2.4 EXERCISES

Read-through questions

The derivative of $y = \sin x$ is $y' =$ __a__. The second derivative (the __b__ of the derivative) is $y'' =$ __c__. The fourth derivative is $y'''' =$ __d__. Thus $y = \sin x$ satisfies the differential equations $y'' =$ __e__ and $y'''' =$ __f__. So does $y = \cos x$, whose second derivative is __g__.

All these derivatives come from one basic limit: $(\sin h)/h$ approaches __h__. The sine of .01 radians is very close to __i__. So is the __j__ of .01. The cosine of .01 is not .99, because $1 - \cos h$ is much __k__ than h. The ratio $(1-\cos h)/h^2$ approaches __l__. Therefore $\cos h$ is close to $1 - \frac{1}{2}h^2$ and $\cos .01 \approx$ __m__. We can replace h by x.

The differential equation $y'' = -y$ leads to __n__. When y is positive, y'' is __o__. Therefore y' is __p__. Eventually y goes below zero and y'' becomes __q__. Then y' is __r__. Examples of oscillation in real life are __s__ and __t__.

1 Which of these ratios approach 1 as $h \to 0$?

(a) $\dfrac{h}{\sin h}$ (b) $\dfrac{\sin^2 h}{h^2}$ (c) $\dfrac{\sin h}{\sin 2h}$ (d) $\dfrac{\sin(-h)}{h}$

2 (Calculator) Find $(\sin h)/h$ at $h = 0.5$ and 0.1 and .01. Where does $(\sin h)/h$ go above .99?

3 Find the limits as $h \to 0$ of

(a) $\dfrac{\sin^2 h}{h}$ (b) $\dfrac{\sin 5h}{5h}$ (c) $\dfrac{\sin 5h}{h}$ (d) $\dfrac{\sin h}{5h}$

4 Where does $\tan h = 1.01h$? Where does $\tan h = h$?

5 $y = \sin x$ has period 2π, which means that $\sin x =$ ______. The limit of $(\sin(2\pi + h) - \sin 2\pi)/h$ is 1 because ______. This gives dy/dx at $x =$ ______.

6 Draw $\cos(x + \Delta x)$ next to $\cos x$. Mark the height difference Δy. Then draw $\Delta y/\Delta x$ as in Figure 2.9.

7 The key to trigonometry is $\cos^2\theta = 1 - \sin^2\theta$. Set $\sin\theta \approx \theta$ to find $\cos^2\theta \approx 1 - \theta^2$. *The square root is* $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Reason: Squaring gives $\cos^2\theta \approx$ ______ and the correction term ______ is very small near $\theta = 0$.

8 (Calculator) Compare $\cos\theta$ with $1 - \frac{1}{2}\theta^2$ for

(a) $\theta = 0.1$ (b) $\theta = 0.5$ (c) $\theta = 30°$ (d) $\theta = 30°$.

9 Trigonometry gives $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$. The approximation $\sin\frac{1}{2}\theta \approx$ ______ leads directly to $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

10 Find the limits as $h \to 0$:

(a) $\dfrac{1-\cos h}{h^2}$ (b) $\dfrac{1-\cos^2 h}{h^2}$

(c) $\dfrac{1-\cos^2 h}{\sin^2 h}$ (d) $\dfrac{1-\cos 2h}{h}$

11 Find by calculator or calculus:

(a) $\displaystyle\lim_{h\to 0} \frac{\sin 3h}{\sin 2h}$ (b) $\displaystyle\lim_{h\to 0} \frac{1-\cos 2h}{1-\cos h}$.

12 Compute the slope at $x = 0$ directly from limits:

(a) $y = \tan x$ (b) $y = \sin(-x)$

13 The unmarked points in Figure 2.11 are P and S. Find the height PS and the area of triangle OPR. Prove by areas that $\sin h < h$.

14 The slopes of $\cos x$ and $1 - \frac{1}{2}x^2$ are $-\sin x$ and ______. The slopes of $\sin x$ and ______ are $\cos x$ and $1 - \frac{1}{2}x^2$.

15 Chapter 10 gives an infinite series for $\sin x$:

$$\sin x = \frac{x}{1} - \frac{x^3}{3\cdot2\cdot1} + \frac{x^5}{5\cdot4\cdot3\cdot2\cdot1} - \cdots.$$

From the derivative find the series for $\cos x$. Then take *its* derivative to get back to $-\sin x$.

16 A *centered difference* for $f(x) = \sin x$ is

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{\sin(x+h)-\sin(x-h)}{2h} = ?$$

Use the addition formula (2). Then let $h \to 0$.

17 Repeat Problem 16 to find the slope of $\cos x$. Use formula (3) to simplify $\cos(x+h) - \cos(x-h)$.

18 Find the tangent line to $y = \sin x$ at

(a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$

19 Where does $y = \sin x + \cos x$ have zero slope?

20 Find the derivative of $\sin(x+1)$ in two ways:

(a) Expand to $\sin x \cos 1 + \cos x \sin 1$. Compute dy/dx.

(b) Divide $\Delta y = \sin(x+1+\Delta x) - \sin(x+1)$ by Δx. Write X instead of $x+1$. Let Δx go to zero.

21 Show that $(\tan h)/h$ is squeezed between 1 and $1/\cos h$. As $h \to 0$ the limit is ______.

22 For $y = \sin 2x$, the ratio $\Delta y/h$ is

$$\frac{\sin 2(x+h) - \sin 2x}{h} = \frac{\sin 2x\,(\cos 2h - 1) + \cos 2x \sin 2h}{h}.$$

Explain why the limit dy/dx is $2\cos 2x$.

23 Draw the graph of $y = \sin \frac{1}{2}x$. State its slope at $x = 0, \pi/2, \pi$, and $2/\pi$. Does $\frac{1}{2}\sin x$ have the same slopes?

24 Draw the graph of $y = \sin x + \sqrt{3}\cos x$. Its maximum value is $y =$ ______ at $x =$ ______. The slope at that point is ______.

25 By combining $\sin x$ and $\cos x$, find a combination that starts at $x = 0$ from $y = 2$ with slope 1. This combination also solves $y'' =$ ______.

26 ***True or false***, with reason:

(a) The derivative of $\sin^2 x$ is $\cos^2 x$

(b) The derivative of $\cos(-x)$ is $\sin x$

(c) A positive function has a negative second derivative.

(d) If y' is increasing then y'' is positive.

27 Find solutions to $dy/dx = \sin 3x$ and $dy/dx = \cos 3x$.

28 If $y = \sin 5x$ then $y' = 5\cos 5x$ and $y'' = -25\sin 5x$. So this function satisfies the differential equation $y'' =$ ______.

29 If h is measured in degrees, find $\lim_{h\to 0}(\sin h)/h$. You could set your calculator in degree mode.

30 Write down a ratio that approaches dy/dx at $x = \pi$. For $y = \sin x$ and $\Delta x = .01$ compute that ratio.

31 By the square rule, the derivative of $(u(x))^2$ is $2u\,du/dx$. Take the derivative of each term in $\sin^2 x + \cos^2 x = 1$.

32 Give an example of oscillation that does not come from physics. Is it simple harmonic motion (one frequency only)?

33 Explain the second derivative in your own words.

2.5 The Product and Quotient and Power Rules

What are the derivatives of $x+\sin x$ and $x\sin x$ and $1/\sin x$ and $x/\sin x$ and $\sin^n x$? Those are made up from the familiar pieces x and $\sin x$, but we need new rules. Fortunately they are rules that apply to every function, so they can be established once and for all. If we know the separate derivatives of two functions u and v, then the derivatives of $u+v$ and uv and $1/v$ and u/v and u^n are immediately available.

This is a straightforward section, with those five rules to learn. It is also an important section, containing most of the working tools of differential calculus. But I am afraid that five rules and thirteen examples (which we need—the eyes glaze over with formulas alone) make a long list. At least the easiest rule comes first. ***When we add functions, we add their derivatives***.

Sum Rule

The derivative of the sum $u(x)+v(x)$ is $\dfrac{d}{dx}(u+v)=\dfrac{du}{dx}+\dfrac{dv}{dx}$. (1)

EXAMPLE 1 The derivative of $x+\sin x$ is $1+\cos x$. That is tremendously simple, but it is fundamental. The interpretation for distances may be more confusing (and more interesting) than the rule itself:

> Suppose a train moves with velocity 1. The distance at time t is t. On the train a professor paces back and forth (in simple harmonic motion). His distance from his seat is $\sin t$. Then the total distance from his starting point is $t+\sin t$, and his velocity (train speed plus walking speed) is $1+\cos t$.

If you add distances, you add velocities. Actually that example is ridiculous, because the professor's maximum speed equals the train speed $(=1)$. He is running like mad, not pacing. Occasionally he is standing still with respect to the ground.

The sum rule is a special case of a bigger rule called "***linearity***." It applies when we add or subtract functions and multiply them by constants—as in $3x-4\sin x$. By linearity the derivative is $3-4\cos x$. The rule works for all functions $u(x)$ and $v(x)$. A *linear combination* is $y(x)=au(x)+bv(x)$, where a and b are any real numbers. Then $\Delta y/\Delta x$ is

$$\frac{au(x+\Delta x)+bv(x+\Delta x)-au(x)-bv(x)}{\Delta x}=a\frac{u(x+\Delta x)-u(x)}{\Delta x}+b\frac{v(x+\Delta x)-v(x)}{\Delta x}.$$

The limit on the left is dy/dx. The limit on the right is $a\,du/dx+b\,dv/dx$. We are allowed to take limits separately and add. The result is what we hope for:

Rule of Linearity

The derivative of $au(x)+bv(x)$ is $\dfrac{d}{dx}(au+bv)=a\dfrac{du}{dx}+b\dfrac{dv}{dx}$. (2)

The ***product rule*** comes next. It can't be so simple—products are not linear. The sum rule is what you would have done anyway, but products give something new. ***The derivative of*** u ***times*** v ***is not*** du/dx ***times*** dv/dx. Example: The derivative of x^5 is $5x^4$. Don't multiply the derivatives of x^3 and x^2. ($3x^2$ times $2x$ is not $5x^4$.) *For a product of two functions, the derivative has two terms.*

Product Rule (the key to this section)

The derivative of $u(x)v(x)$ is $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$. (3)

EXAMPLE 2 $u = x^3$ times $v = x^2$ is $uv = x^5$. The product rule leads to $5x^4$:

$$x^3\frac{dv}{dx}+x^2\frac{du}{dx}=x^3(2x)+x^2(3x^2)=2x^4+3x^4=5x^4.$$

EXAMPLE 3 In the slope of $x\sin x$, I don't write $dx/dx = 1$ but it's there:

$$\frac{d}{dx}(x\sin x)=x\cos x+\sin x.$$

EXAMPLE 4 If $u = \sin x$ and $v = \sin x$ then $uv = \sin^2 x$. We get two equal terms:

$$\sin x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\sin x)=2\sin x\cos x.$$

This confirms the "square rule" $2u\ du/dx$, when u is the same as v. Similarly the slope of $\cos^2 x$ is $-2\cos x\sin x$ (minus sign from the slope of the cosine).

Question Those answers for $\sin^2 x$ and $\cos^2 x$ have opposite signs, so the derivative of $\sin^2 x+\cos^2 x$ is zero (sum rule). How do you see that more quickly?

EXAMPLE 5 The derivative of uvw is $uvw' + uv'w + u'vw$—one derivative at a time. The derivative of xxx is $xx + xx + xx$.

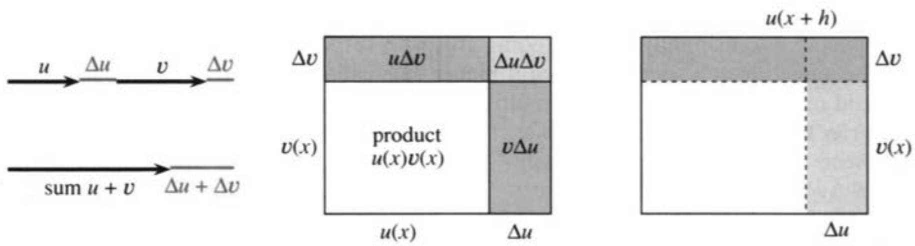


Fig. 2.13 Change in length $= \Delta u + \Delta v$. Change in area $= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$.

After those examples we prove the product rule. Figure 2.13 explains it best. The area of the big rectangle is uv. ***The important changes in area are the two strips*** $u\,\Delta v$ ***and*** $v\,\Delta u$. The corner area $\Delta u\,\Delta v$ is much smaller. When we divide by Δx, the strips give $u\,\Delta v/\Delta x$ and $v\,\Delta u/\Delta x$. The corner gives $\Delta u\,\Delta v/\Delta x$, which approaches zero.

Notice how the sum rule is in one dimension and the product rule is in two dimensions. The rule for uvw would be in three dimensions.

The extra area comes from the whole top strip plus the side strip. By algebra,

$$u(x+h)v(x+h)-u(x)v(x)=u(x+h)[v(x+h)-v(x)]+v(x)[u(x+h)-u(x)]. \quad (4)$$

This increase is $u(x+h)\Delta v + v(x)\Delta u$—top plus side. *Now divide by* h (or Δx) *and let* $h \to 0$. The left side of equation (4) becomes the derivative of $u(x)v(x)$. The right side becomes $u(x)$ times dv/dx—*we can multiply the two limits*—plus $v(x)$ times du/dx. That proves the product rule—definitely useful.

We could go immediately to the quotient rule for $u(x)/v(x)$. But start with $u = 1$. The derivative of $1/x$ is $-1/x^2$ (known). What is the derivative of $1/v(x)$?

Reciprocal Rule

$$\text{The derivative of } \frac{1}{v(x)} \text{ is } \frac{-dv/dx}{v^2}. \quad (5)$$

The proof starts with $(v)(1/v) = 1$. The derivative of 1 is 0. Apply the product rule:

$$v\frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v}\frac{dv}{dx} = 0 \quad \text{so that} \quad \frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-dv/dx}{v^2}. \tag{6}$$

It is worth checking the units—in the reciprocal rule and others. A test of dimensions is automatic in science and engineering, and a good idea in mathematics. The test ignores constants and plus or minus signs, but it prevents bad errors. If v is in dollars and x is in hours, dv/dx is in *dollars per hour*. Then dimensions agree:

$$\frac{d}{dx}\left(\frac{1}{v}\right) \approx \frac{(1/\text{dollars})}{\text{hour}} \quad \text{and also} \quad \frac{-dv/dx}{v^2} \approx \frac{\text{dollars/hour}}{(\text{dollars})^2}.$$

From this test, the derivative of $1/v$ cannot be $1/(dv/dx)$. A similar test shows that Einstein's formula $e = mc^2$ is dimensionally possible. The theory of relativity might be correct! Both sides have the dimension of (mass)(distance)^2/(time)^2, when mass is converted to energy.†

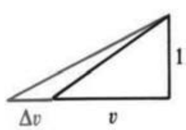
EXAMPLE 6 The derivatives of x^{-1}, x^{-2}, x^{-n} are $-1x^{-2}, -2x^{-3}, -nx^{-n-1}$.

Those come from the reciprocal rule with $v = x$ and x^2 and any x^n:

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1}.$$

The beautiful thing is that this answer $-nx^{-n-1}$ fits into the same pattern as x^n. ***Multiply by the exponent and reduce it by one***.

For negative and positive exponents the derivative of x^n ***is*** nx^{n-1}. (7)



Reciprocal $\quad \dfrac{1}{v+\Delta v} - \dfrac{1}{v} = \dfrac{-\Delta v}{v(v+\Delta v)}$

Quotient $\quad \dfrac{u+\Delta u}{v+\Delta v} - \dfrac{u}{v} = \dfrac{v\Delta u - u\Delta v}{v(v+\Delta v)}$

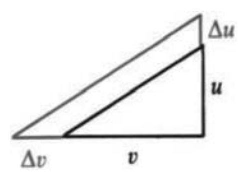


Fig. 2.14 Reciprocal rule from $(-\Delta v)/v^2$. Quotient rule from $(v\Delta u - u\Delta v)/v^2$.

EXAMPLE 7 The derivatives of $\dfrac{1}{\cos x}$ and $\dfrac{1}{\sin x}$ are $\dfrac{+\sin x}{\cos^2 x}$ and $\dfrac{-\cos x}{\sin^2 x}$.

Those come directly from the reciprocal rule. In trigonometry, $1/\cos x$ is the ***secant*** of the angle x, and $1/\sin x$ is the ***cosecant*** of x. Now we have their derivatives:

$$\frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x. \tag{8}$$

$$\frac{d}{dx}(\csc x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x. \tag{9}$$

† But only Einstein knew that the constant is 1.

Those formulas are often seen in calculus. If you have a good memory they are worth storing. Like most mathematicians, I have to check them every time before using them (maybe once a year). It is really the rules that are basic, not the formulas.

The next rule applies to the quotient $u(x)/v(x)$. That is u times $1/v$. Combining the product rule and reciprocal rule gives something new and important:

Quotient Rule

$$\text{The derivative of } \frac{u(x)}{v(x)} \text{ is } \frac{1}{v}\frac{du}{dx} - u\frac{dv/dx}{v^2} = \frac{v\,du/dx - u\,dv/dx}{v^2}.$$

You *must* memorize that last formula. The v^2 is familiar. The rest is new, but not very new. If $v=1$ the result is du/dx (of course). For $u=1$ we have the reciprocal rule. Figure 2.14b shows the difference $(u+\Delta u)/(v+\Delta v)-(u/v)$. The denominator $v(v+\Delta v)$ is responsible for v^2.

EXAMPLE 8 (only practice) If $u/v = x^5/x^3$ (which is x^2) the quotient rule gives $2x$:

$$\frac{d}{dx}\left(\frac{x^5}{x^3}\right) = \frac{x^3(5x^4)-x^5(3x^2)}{x^6} = \frac{5x^7-3x^7}{x^6} = 2x.$$

EXAMPLE 9 (important) For $u = \sin x$ and $v = \cos x$, the quotient is $\sin x/\cos x = \tan x$. ***The derivative of*** $\tan x$ ***is*** $\sec^2 x$. Use the quotient rule and $\cos^2 x + \sin^2 x = 1$:

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \qquad (11)$$

Again to memorize: $(\tan x)' = \sec^2 x$. At $x=0$, this slope is 1. The graphs of $\sin x$ and x and $\tan x$ all start with this slope (then they separate). At $x=\pi/2$ the sine curve is flat ($\cos x = 0$) and the tangent curve is vertical ($\sec^2 x = \infty$).

The slope generally blows up faster than the function. We divide by $\cos x$, once for the tangent and twice for its slope. The slope of $1/x$ is $-1/x^2$. The slope is more sensitive than the function, because of the square in the denominator.

EXAMPLE 10

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}.$$

That one I hesitate to touch at $x=0$. Formally it becomes $0/0$. In reality it is more like $0^3/0^2$, and the true derivative is zero. Figure 2.10 showed graphically that $(\sin x)/x$ is flat at the center point. The function is *even* (symmetric across the y axis) so its derivative can only be zero.

This section is full of rules, and I hope you will allow one more. It goes beyond x^n to $(u(x))^n$. A power of x changes to a power of $u(x)$—as in $(\sin x)^6$ or $(\tan x)^7$ or $(x^2+1)^8$. The derivative contains nu^{n-1} (copying nx^{n-1}), but ***there is an extra factor*** du/dx. Watch that factor in $6(\sin x)^5\cos x$ and $7(\tan x)^6\sec^2 x$ and $8(x^2+1)^7(2x)$:

Power Rule

$$\text{The derivative of } \big[u(x)\big]^n \text{ is } n\big[u(x)\big]^{n-1}\frac{du}{dx}. \qquad (12)$$

For $n=1$ this reduces to $du/dx = du/dx$. For $n=2$ we get the square rule $2u\,du/dx$. Next comes u^3. The best approach is to use ***mathematical induction***, which goes from each n to the next power $n+1$ by the product rule:

$$\frac{d}{dx}(u^{n+1}) = \frac{d}{dx}(u^n u) = u^n \frac{du}{dx} + u\left(nu^{n-1}\frac{du}{dx}\right) = (n+1)u^n \frac{du}{dx}.$$

That is exactly equation (12) for the power $n+1$. We get all positive powers this way, going up from $n=1$—then the negative powers come from the reciprocal rule.

Figure 2.15 shows the power rule for $n = 1, 2, 3$. The cube makes the point best. The three thin slabs are u by u by Δu. ***The change in volume is essentially*** $3u^2\Delta u$. From multiplying out $(u+\Delta u)^3$, the exact change in volume is $3u^2\Delta u + 3u(\Delta u)^2 + (\Delta u)^3$—which also accounts for three narrow boxes and a midget cube in the corner. This is the binomial formula in a picture.

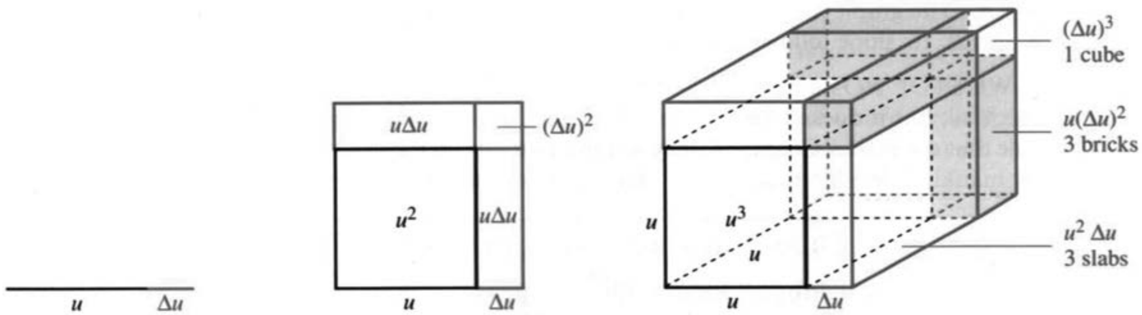


Fig. 2.15 Length change $= \Delta u$; area change $\approx 2u\Delta u$; volume change $\approx 3u^2\Delta u$.

EXAMPLE 11 $\dfrac{d}{dx}(\sin x)^n = n(\sin x)^{n-1}\cos x$. The extra factor $\cos x$ is du/dx.

Our last step finally escapes from a very undesirable restriction—that n must be a whole number. We want to allow fractional powers $n = p/q$, and keep the same formula. *The derivative of x^n is still nx^{n-1}.*

To deal with square roots I can write $(\sqrt{x})^2 = x$. Its derivative is $2\sqrt{x}(\sqrt{x})' = 1$. Therefore $(\sqrt{x})'$ is $1/2\sqrt{x}$, which fits the formula when $n = \frac{1}{2}$. Now try $n = p/q$:

Fractional powers Write $u = x^{p/q}$ as $u^q = x^p$. Take derivatives, assuming they exist:

$$qu^{q-1}\frac{du}{dx} = px^{p-1} \qquad \text{(power rule on both sides)}$$

$$\frac{du}{dx} = \frac{px^{-1}}{qu^{-1}} \qquad \text{(cancel } x^p \text{ with } u^q\text{)}$$

$$\frac{du}{dx} = nx^{n-1} \qquad \text{(replace } p/q \text{ by } n \text{ and } u \text{ by } x^n\text{)}$$

EXAMPLE 12 The slope of $x^{1/3}$ is $\frac{1}{3}x^{-2/3}$. The slope is infinite at $x=0$ and zero at $x=\infty$. But the curve in Figure 2.16 keeps climbing. It doesn't stay below an "asymptote."

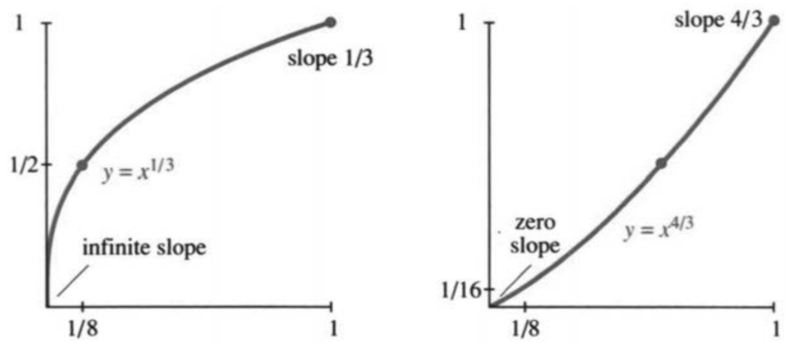


Fig. 2.16 Infinite slope of x^n versus zero slope: the difference between $0 < n < 1$ and $n > 1$.

EXAMPLE 13 The slope of $x^{4/3}$ is $\frac{4}{3}x^{1/3}$. The slope is zero at $x = 0$ and infinite at $x = \infty$. The graph climbs faster than a line and slower than a parabola ($\frac{4}{3}$ is between 1 and 2). Its slope follows the cube root curve (times $\frac{4}{3}$).

WE STOP NOW! I am sorry there were so many rules. A computer can memorize them all, but it doesn't know what they mean and you do. Together with the chain rule that dominates Chapter 4, they achieve virtually all the derivatives ever computed by mankind. We list them in one place for convenience.

Rule of Linearity	$(au + bv)' = au' + bv'$
Product Rule	$(uv)' = uv' + vu'$
Reciprocal Rule	$(1/v)' = -v'/v^2$
Quotient Rule	$(u/v)' = (vu' - uv')/v^2$
Power Rule	$(u^n)' = nu^{n-1}u'$

The power rule applies when n is *negative*, or a *fraction*, or ***any real number***. The derivative of x^π is $\pi x^{\pi-1}$, according to Chapter 6. The derivative of $(\sin x)^\pi$ is _____. And the derivatives of all six trigonometric functions are now established:

$$(\sin x)' = \cos x \qquad (\tan x)' = \sec^2 x \qquad (\sec x)' = \sec x \tan x$$
$$(\cos x)' = -\sin x \qquad (\cot x)' = -\csc^2 x \qquad (\csc x)' = -\csc x \cot x.$$

2.5 EXERCISES

Read-through questions

The derivatives of $\sin x \cos x$ and $1/\cos x$ and $\sin x/\cos x$ and $\tan^3 x$ come from the __a__ rule, __b__ rule, __c__ rule, and __d__ rule. The product of $\sin x$ times $\cos x$ has $(uv)' = uv' +$ __e__ $=$ __f__. The derivative of $1/v$ is __g__, so the slope of $\sec x$ is __h__. The derivative of u/v is __i__, so the slope of $\tan x$ is __j__. The derivative of $\tan^3 x$ is __k__. The slope of x^n is __l__ and the slope of $(u(x))^n$ is __m__. With $n = -1$ the derivative of $(\cos x)^{-1}$ is __n__, which agrees with the rule for $\sec x$.

Even simpler is the rule of __o__, which applies to $au(x) + bv(x)$. The derivative is __p__. The slope of $3\sin x + 4\cos x$ is __q__. The derivative of $(3\sin x + 4\cos x)^2$ is __r__. The derivative of __s__ is $4\sin^3 x \cos x$.

Find the derivatives of the functions in 1–26.

1 $(x+1)(x-1)$

2 $(x^2+1)(x^2-1)$

3 $\dfrac{1}{1+x} + \dfrac{1}{1+\sin x}$

4 $\dfrac{1}{1+x^2} + \dfrac{1}{1-\sin x}$

5 $(x-1)(x-2)(x-3)$

6 $(x-1)^2(x-2)^2$

7 $x^2\cos x+2x\sin x$

8 $x^{1/2}(x+\sin x)$

9 $\dfrac{x^3+1}{x+1}+\dfrac{\cos x}{\sin x}$

10 $\dfrac{x^2+1}{x^2-1}+\dfrac{\sin x}{\cos x}$

11 $x^{1/2}\sin^2 x+(\sin x)^{1/2}$

12 $x^{3/2}\sin^3 x+(\sin x)^{3/2}$

13 $x^4\cos x+x\cos^4 x$

14 $\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)$

15 $\frac{1}{2}x^2\sin x-x\cos x+\sin x$

16 $(x-6)^{10}+\sin^{10}x$

17 $\sec^2 x-\tan^2 x$

18 $\csc^2 x-\cot^2 x$

19 $\dfrac{4}{(x-5)^{2/3}}+\dfrac{4}{(5-x)^{2/3}}$

20 $\dfrac{\sin x-\cos x}{\sin x+\cos x}$

21 $(\sin x\cos x)^3+\sin 2x$

22 $x\cos x\csc x$

23 $u(x)v(x)w(x)z(x)$

24 $[u(x)]^2\,[v(x)]^2$

25 $\dfrac{1}{\tan x}-\dfrac{1}{\cot x}$

26 $x\sin x+\cos x$

27 A growing box has length t, width $1/(1+t)$, and height $\cos t$.

(a) What is the rate of change of the volume?

(b) What is the rate of change of the surface area?

28 With two applications of the product rule show that the derivative of uvw is $uvw'+uv'w+u'vw$. When a box with sides u,v,w grows by $\Delta u,\Delta v,\Delta w$, three slabs are added with volume $uv\,\Delta w$ and ______ and ______.

29 Find the velocity if the distance is $f(t)=$

$$5t^2 \text{ for } t\leqslant 10, \qquad 500+100\sqrt{t-10} \text{ for } t\geqslant 10.$$

30 A cylinder has radius $r=\dfrac{t^{3/2}}{1+t^{3/2}}$ and height $h=\dfrac{1}{1+t}$.

(a) What is the rate of change of its volume?

(b) What is the rate of change of its surface area (including top and base)?

31 The height of a model rocket is $f(t)=t^3/(1+t)$.

(a) What is the velocity $v(t)$?

(b) What is the acceleration dv/dt?

32 Apply the product rule to $u(x)u^2(x)$ to find the power rule for $u^3(x)$.

33 Find the *second* derivative of the product $u(x)v(x)$. Find the *third* derivative. Test your formulas on $u=v=x$.

34 Find functions $y(x)$ whose derivatives are

(a) x^3 (b) $1/x^3$ (c) $(1-x)^{3/2}$ (d) $\cos^2 x\sin x$

35 Find the distances $f(t)$, starting from $f(0)=0$, to match these velocities:

(a) $v(t)=\cos t\sin t$ (b) $v(t)=\tan t\sec^2 t$

(c) $v(t)=\sqrt{1+t}$

36 Apply the quotient rule to $(u(x))^3/(u(x))^2$ and $-v'/v^2$. The latter gives the second derivative of ______.

37 Draw a figure like 2.13 to explain the *square rule*.

38 Give an example where $u(x)/v(x)$ is increasing but $du/dx=dv/dx=1$.

39 ***True or false***, with a good reason:

(a) The derivative of x^{2n} is $2nx^{2n-1}$.

(b) By linearity the derivative of $a(x)u(x)+b(x)v(x)$ is $a(x)du/dx+b(x)dv/dx$.

(c) The derivative of $|x|^3$ is $3|x|^2$.

(d) $\tan^2 x$ and $\sec^2 x$ have the same derivative.

(e) $(uv)'=u'v'$ is true when $u(x)=1$.

40 The cost of u shares of stock at v dollars per share is uv dollars. Check dimensions of $d(uv)/dt$ and $u\,dv/dt$ and $v\,du/dt$.

41 If $u(x)/v(x)$ is a ratio of polynomials of degree n, what are the degrees for its derivative?

42 For $y=5x+3$, is $(dy/dx)^2$ the same as d^2y/dx^2?

43 If you change from $f(t)=t\cos t$ to its tangent line at $t=\pi/2$, find the two-part function df/dt.

44 Explain in your own words why the derivative of $u(x)v(x)$ has two terms.

45 A plane starts its descent from height $y=h$ at $x=-L$ to land at $(0,0)$. Choose a,b,c,d so its landing path $y=ax^3+bx^2+cx+d$ is **smooth**. With $dx/dt=V=$constant, find dy/dt and d^2y/dt^2 at $x=0$ and $x=-L$. (To keep d^2y/dt^2 small, a coast-to-coast plane starts down $L>100$ miles from the airport.)

2.6 Limits

You have seen enough limits to be ready for a definition. It is true that we have survived this far without one, and we could continue. But this seems a reasonable time to define limits more carefully. The goal is to achieve rigor without rigor mortis.

First you should know that limits of $\Delta y/\Delta x$ are by no means the only limits in mathematics. Here are five completely different examples. They involve $n \to \infty$, not $\Delta x \to 0$:

1. $a_n = (n-3)/(n+3)$ (for large n, ignore the 3's and find $a_n \to 1$)
2. $a_n = \frac{1}{2}a_{n-1} + 4$ (start with any a_1 and always $a \to 8$)
3. $a_n =$ probability of living to year n (unfortunately $a_n \to 0$)
4. $a_n =$ fraction of zeros among the first n digits of π ($a_n \to \frac{1}{10}$?)
5. $a_1 = .4, a_2 = .49, a_3 = .493, \ldots$ *No matter what the remaining decimals are, the a's converge to a limit*. Possibly $a_n \to .493000\ldots$, but not likely.

The problem is to say what the limit symbol $\to$ ***really means***.

A good starting point is to ask about convergence to *zero*. When does a sequence of positive numbers approach zero? What does it mean to write $a_n \to 0$? The numbers $a_1, a_2, a_3, \ldots$, must become "small," but that is too vague. We will propose four definitions of ***convergence to zero***, and I hope the right one will be clear.

1. *All the numbers a_n are below* 10^{-10}. That may be enough for practical purposes, but it certainly doesn't make the a_n approach zero.

2. *The sequence is getting closer to zero*—each a_{n+1} is smaller than the preceding a_n. This test is met by 1.1, 1.01, 1.001, ... which converges to 1 instead of 0.

3. *For any small number you think of, at least one of the a_n's is smaller*. That pushes something toward zero, but not necessarily the whole sequence. The condition would be satisfied by $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$, which does not approach zero.

4. *For any small number you think of, the a_n's eventually go below that number and* ***stay below***. This is the correct definition.

I want to repeat that. To test for convergence to zero, start with a small number—say 10^{-10}. The a_n's must go *below that number*. They may come back up and go below again—the first million terms make absolutely no difference. Neither do the next billion, but eventually all terms must go below 10^{-10}. After waiting longer (possibly a lot longer), all terms drop below 10^{-20}. The tail end of the sequence decides everything.

Question 1 Does the sequence $10^{-3}, 10^{-2}, 10^{-6}, 10^{-5}, 10^{-9}, 10^{-8}, \ldots$ approach 0?

Answer Yes, These up and down numbers eventually stay below any ε.

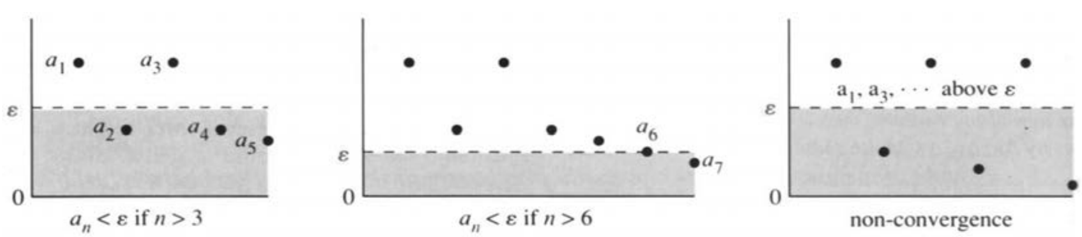


Fig. 2.17 Convergence means: Only a finite number of a's are outside any strip around L.

Question 2 Does $10^{-4}, 10^{-6}, 10^{-4}, 10^{-8}, 10^{-4}, 10^{-10}, \ldots$ approach zero ?
Answer No. This sequence goes below 10^{-4} but does not stay below.

There is a recognized symbol for "an arbitrarily small positive number." By worldwide agreement, it is the Greek letter ε (***epsilon***). Convergence to zero means that ***the sequence eventually goes below*** ε ***and stays there***. The smaller the ε, the tougher the test and the longer we wait. Think of ε as the tolerance, and keep reducing it.

To emphasize that ε comes from outside, Socrates can choose it. Whatever ε he proposes, the a's must eventually be smaller. *After some* a_N, *all the* a's *are below the tolerance* ε. Here is the exact statement:

for any ε ***there is an*** N ***such that*** $a_n < \varepsilon$ ***if*** $n > N$.

Once you see that idea, the rest is easy. Figure 2.17 has $N = 3$ and then $N = 6$.

EXAMPLE 1 The sequence $\frac{1}{2}, \frac{4}{4}, \frac{9}{8}, \ldots$ starts upward but goes to zero. Notice that $1, 4, 9, \ldots, 100, \ldots$ are squares, and $2, 4, 8, \ldots, 1024, \ldots$ are powers of 2. Eventually 2^n grows faster than n^2, as in $a_{10} = 100/1024$. The ratio goes below any ε.

EXAMPLE 2 $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots$ approaches zero. These a's do not decrease steadily (the mathematical word for steadily is "monotonically") but still their limit is zero. The choice $\varepsilon = 1/10$ produces the right response: *Beyond* a_{2001} *all terms are below* $1/1000$. So $N = 2001$ for that ε.

The sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ is much slower—but it also converges to zero.

Next we allow the numbers a_n to be *negative* as well as positive. They can converge upward toward zero, or they can come in from both sides. The test still requires the a_n to go inside any strip near zero (and stay there). But now the strip starts at $-\varepsilon$.

The distance from zero is the absolute value $|a_n|$. *Therefore* $a_n \to 0$ *means* $|a_n| \to 0$. The previous test can be applied to $|a_n|$:

for any ε ***there is an*** N ***such that*** $|a_n| < \varepsilon$ ***if*** $n > N$.

EXAMPLE 3 $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$ converges to zero because $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to zero.

It is a short step to limits other than zero. ***The limit is*** L ***if the numbers*** $a_n - L$ ***converge to zero***. Our final test applies to the absolute value $|a_n - L|$:

for any ε ***there is an*** N ***such that*** $|a_n - L| < \varepsilon$ ***if*** $n > N$.

This is the definition of convergence! Only a finite number of a's are outside any strip around L (Figure 2.18). We write $a_n \to L$ or $\lim a_n = L$ or $\lim_{n\to\infty} a_n = L$.

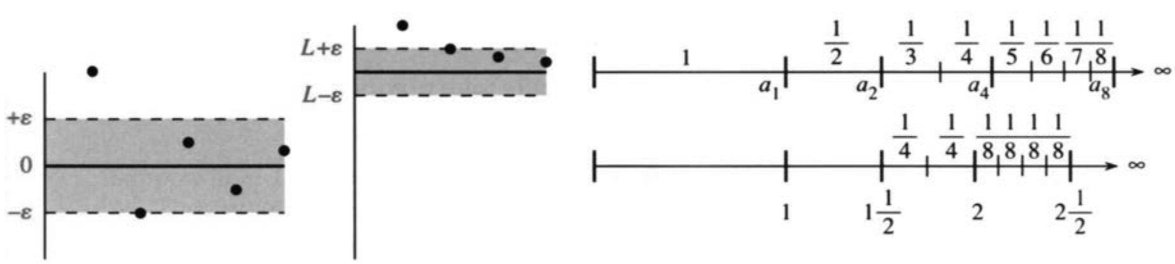


Fig. 2.18 $a_n \to 0$ in Example 3; $a_n \to 1$ in Example 4; $a_n \to \infty$ in Example 5 (but $a_{n+1} - a_n \to 0$).

EXAMPLE 4 The numbers $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots$ converge to $L = 1$. After subtracting 1 the differences $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$ converge to zero. Those difference are $|a_n - L|$.

EXAMPLE 5 ***The sequence*** $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \ldots$ ***fails to converge***.

The distance between terms is getting smaller. But those numbers $a_1, a_2, a_3, a_4, \ldots$ go past any proposed limit L. The second term is $1\frac{1}{2}$. The fourth term adds on $\frac{1}{3}+\frac{1}{4}$, so a_4 goes past 2. The eighth term has four new fractions $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$, totaling more than $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$. Therefore a_8 exceeds $2\frac{1}{2}$. Eight more terms will add more than 8 times $\frac{1}{16}$, so a_{16} is beyond 3. The lines in Figure 2.18c are infinitely long, not stopping at any L.

In the language of Chapter 10, the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ *does not converge*. The sum is infinite, because the "partial sums" a_n go beyond every limit L (a_{5000} is past $L = 9$). We will come back to infinite series, but this example makes a subtle point: The steps between the a_n can go to zero while still $a_n \to \infty$.

Thus the condition $a_{n+1} - a_n \to 0$ is ***not sufficient*** for convergence. However this condition is ***necessary***. If we do have convergence, then $a_{n+1} - a_n \to 0$. That is a good exercise in the logic of convergence, emphasizing the difference between "sufficient" and "necessary." We discuss this logic below, after proving that [statement A] implies [statement B]:

$$\textbf{\textit{If}}\ [a_n \text{ converges to } L]\ \textbf{\textit{then}}\ [a_{n+1} - a_n \text{ converges to zero}]. \tag{1}$$

Proof Because the a_n converge, there is a number N beyond which $|a_n - L| < \varepsilon$ and also $|a_{n+1} - L| < \varepsilon$. Since $a_{n+1} - a_n$ is the sum of $a_{n+1} - L$ and $L - a_n$, its absolute value cannot exceed $\varepsilon + \varepsilon = 2\varepsilon$. Therefore $a_{n+1} - a_n$ approaches zero.

Objection by Socrates: We only got below 2ε and he asked for ε. *Our reply*: If he particularly wants $|a_{n+1} - a_n| < 1/10$, we start with $\varepsilon = 1/20$. Then $2\varepsilon = 1/10$. But this juggling is not necessary. To stay below 2ε is just as convincing as to stay below ε.

THE LOGIC OF "IF" AND "ONLY IF"

The following page is inserted to help with the language of mathematics. In ordinary language we might say "I will come if you call." Or we might say "I will come only if you call." That is different! A mathematician might even say "I will come *if and only if* you call." Our goal is to think through the logic, because it is important and not so fami1iar.†

Statement A above implies statement B. Statement A is $a_n \to L$; statement B is $a_{n+1} - a_n \to 0$. Mathematics has at least five ways of writing down $A \Rightarrow B$, and I though you might like to see them together. It seems excessive to have so many expressions for the same idea, but authors get desperate for a little variety. Here are the five ways that come to mind:

$$A \Rightarrow B$$

A implies B

if A ***then*** B

A is a ***sufficient*** condition for B

B is true ***if*** A is true

† Logical thinking is much more important than ε and δ.

EXAMPLES ***If*** [positive numbers are decreasing] ***then*** [they converge to a limit].
If [sequences a_n and b_n converge] ***then*** [the sequence $a_n + b_n$ converges].
If [$f(x)$ is the integral of $v(x)$] ***then*** [$v(x)$ is the derivative of $f(x)$].

Those are all true, but not proved. A is the hypothesis, B is the conclusion.

Now we go in the other direction. (It is called the "converse," not the inverse.) *We exchange A and B*. Of course stating the converse does not make it true! B might imply A, or it might not. In the first two examples the converse was false—the a_n can converge without decreasing, and $a_n + b_n$ can converge when the separate sequences do not. The converse of the third statement is true—and there are five more ways to state it:

$$A \Leftarrow B$$

A is implied by B

if B ***then*** A

A is a ***necessary*** condition for B

B is true ***only if*** A is true

Those words "necessary" and "sufficient" are not always easy to master. The same is true of the deceptively short phrase "if and only if." The two statements $A \Rightarrow B$ and $A \Leftarrow B$ are completely different and *they both require proof*. That means two separate proofs. But they can be stated together for convenience (when both are true):

$$A \Leftrightarrow B$$

A implies B and B implies A

A is ***equivalent*** to B

A is a ***necessary and sufficient*** condition for B

A is true ***if and only if*** B is true

EXAMPLES $[a_n \to L] \Leftrightarrow [2a_n \to 2L] \Leftrightarrow [a_n + 1 \to L + 1] \Leftrightarrow [a_n - L \to 0]$.

RULES FOR LIMITS

Calculus needs a *definition of limits*, to define dy/dx. That derivative contains two limits: $\Delta x \to 0$ and $\Delta y/\Delta x \to dy/dx$. Calculus also needs *rules for limits*, to prove the sum rule and product rule for derivatives. We started on the definition, and now we start on the rules.

Given *two convergent sequences*, $a_n \to L$ and $b_n \to M$, other sequences also converge:

Addition: $a_n + b_n \to L + M$ Subtraction: $a_n - b_n \to L - M$

Multiplication: $a_n b_n \to LM$ Division: $a_n/b_n \to L/M$ (provided $M \neq 0$)

We check the multiplication rule, which uses a convenient identity:

$$a_n b_n - LM = (a_n - L)(b_n - M) + M(a_n - L) + L(b_n - M). \qquad (2)$$

Suppose $|a_n - L| < \varepsilon$ beyond some point N, and $|b_n - M| < \varepsilon$ beyond some other point N'. Then beyond the larger of N and N', the right side of (2) is small. It is less than $\varepsilon \cdot \varepsilon + M\varepsilon + L\varepsilon$. This proves that (2) gives $a_n b_n \to LM$.

An important special case is $ca_n \to cL$. (The sequence of b's is $c, c, c, c, \ldots$) Thus a constant can be brought "outside" the limit, to give $\lim\ ca_n = c\ \lim\ a_n$.

THE LIMIT OF $f(x)$ AS $x \to a$

The final step is to replace sequences by functions. Instead of $a_1, a_2, \ldots$ there is a continuum of values $f(x)$. The limit is taken as x approaches a specified point a (instead of $n \to \infty$). Example: As x approaches $a = 0$, the function $f(x) = 4 - x^2$ approaches $L = 4$. As x approaches $a = 2$, the function $5x$ approaches $L = 10$. Those statements are fairly obvious, but we have to say what they mean. Somehow it must be this:

***if** x **is close to** a **then** $f(x)$ **is close to** L.*

If $x - a$ is small, then $f(x) - L$ should be small. As before, the word *small* does not say everything. We really mean "arbitrarily small," or "below any ε." The difference $f(x) - L$ must become *as small as anyone wants*, when x gets near a. In that case $\lim_{x\to a} f(x) = L$. Or we write $f(x) \to L$ as $x \to a$.

The statement is awkward because it involves *two limits*. The limit $x \to a$ is forcing $f(x) \to L$. (Previously $n \to \infty$ forced $a \to L$.) But it is wrong to expect the same ε in both limits. We do not and cannot require that $|x - a| < \varepsilon$ produces) $|f(x) - L| < \varepsilon$. ***It may be necessary to push** x **extremely close to*** a (closer than ε). We must guarantee that if x is close enough to a, then $|f(x) - L| < \varepsilon$.

We have come to the "***epsilon-delta definition***" of limits. First, Socrates chooses ε. He has to be shown that $f(x)$ is within ε of L, for every x near a. Then somebody else (maybe Plato) replies with a number δ. That gives the meaning of "near a." Plato's goal is to get $f(x)$ within ε of L, by keeping x within δ of a:

$$\textbf{\textit{if}} \quad 0 < |x - a| < \delta \quad \textbf{\textit{then}} \quad |f(x) - L| < \varepsilon. \tag{3}$$

The input tolerance is δ (delta), the output tolerance is ε. When Plato can find a δ for every ε, Socrates concedes that the limit is L.

EXAMPLE Prove that $\lim\limits_{x\to 2} 5x = 10$. In this case $a = 2$ and $L = 10$.

Socrates asks for $|5x - 10| < \varepsilon$. Plato responds by requiring $|x - 2| < \delta$. What δ should he choose? In this case $|5x - 10|$ is exactly 5 times $|x - 2|$. So Plato picks δ below $\varepsilon/5$ (a smaller δ is always OK). Whenever $|x - 2| < \varepsilon/5$, multiplication by 5 shows that $|5x - 10| < \varepsilon$.

Remark* 1** In Figure 2.19, Socrates chooses the height of the box. It extends above and below L, by the small number ε. Second, Plato chooses the width. He must make the box narrow enough for the graph to go ***out the sides. Then $|f(x) - L| < \varepsilon$.

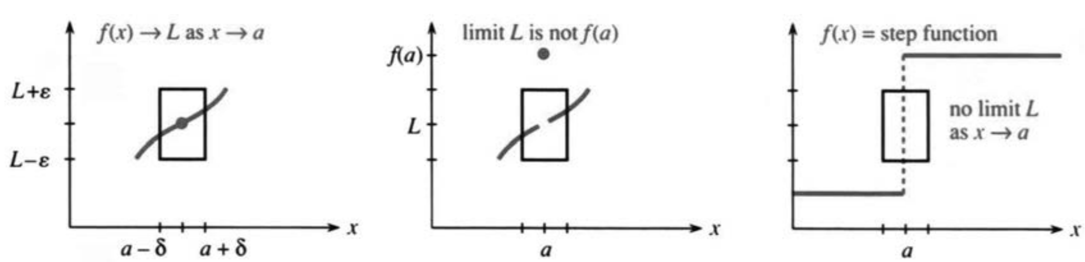


Fig. 2.19 S chooses height 2ε, then P chooses width 2δ. Graph must go out the sides.

When $f(x)$ has a jump, the box can't hold it. A step function has no limit as x approaches the jump, because the graph goes through the top or bottom—no matter how thin the box.

Remark* 2** The second figure has $f(x) \to L$, because in taking limits ***we ignore the final point $x = a$. The value $f(a)$ can be anything, with no effect on L. The first figure has more: $f(a)$ equals L. Then a special name applies—f is ***continuous***. The left figure shows a continuous function, the other figures do not.

We soon come back to continuous functions.

***Remark* 3** In the example with $f = 5x$ and $\delta = \varepsilon/5$, the number 5 was the *slope*. That choice barely kept the graph in the box—it goes out the corners. A little narrower, say $\delta = \varepsilon/10$, and the graph goes safely out the sides. *A reasonable choice is to divide* ε *by* $2|f'(a)|$. (We double the slope for safety.) I want to say why this δ works—even if the $\varepsilon - \delta$ test is seldom used in practice.

The ratio of $f(x) - L$ to $x - a$ is distance up over distance across. This is $\Delta f/\Delta x$, close to the slope $f'(a)$. When the distance across is δ, the distance up or down is near $\delta|f'(a)|$. That equals $\varepsilon/2$ for our "reasonable choice" of δ—so we are safely below ε. This choice solves most exercises. But Example 7 shows that a limit might exist even when the slope is infinite.

EXAMPLE 7 $\lim_{x \to 1^+} \sqrt{x-1} = 0$ (***a one-sided limit***).

Notice the plus sign in the symbol $x \to 1^+$. The number x approaches $a = 1$ *only from above*. An ordinary limit $x \to 1$ requires us to accept x on both sides of 1 (the exact value $x = 1$ is not considered). Since negative numbers are not allowed by the square root, we have a *one-sided limit*. It is $L = 0$.

Suppose ε is $1/10$. Then the response could be $\delta = 1/100$. A number below $1/100$ has a square root below $1/10$. In this case the box must be made extremely narrow, δ much smaller than ε, because the square root starts with infinite slope.

Those examples show the point of the $\varepsilon - \delta$ definition. (Given ε, look for δ. This came from Cauchy in France, not Socrates in Greece.) We also see its bad feature: The test is not convenient. Mathematicians do not go around proposing ε's and replying with δ's. We may live a strange life, but not that strange.

It is easier to establish once and for all that $5x$ approaches its obvious limit $5a$. The same is true for other familiar functions: $x^n \to a^n$ and $\sin x \to \sin a$ and $(1-x)^{-1} \to (1-a)^{-1}$—except at $a = 1$. ***The correct limit*** L ***comes by substituting*** $x = a$ ***into the function***. This is exactly the property of a "***continuous function***." Before the section on continuous functions, we prove the Squeeze Theorem using ε and δ.

> **2H** ***Squeeze Theorem*** Suppose $f(x) \leqslant g(x) \leqslant h(x)$ for x near a. If $f(x) \to L$ and $h(x) \to L$ as $x \to a$, then the limit of $g(x)$ is also L.

Proof $g(x)$ is squeezed between $f(x)$ and $h(x)$. After subtracting L, $g(x) - L$ is between $f(x) - L$ and $h(x) - L$. Therefore

$$|g(x) - L| < \varepsilon \quad \text{if} \quad |f(x) - L| < \varepsilon \quad \text{and} \quad |h(x) - L| < \varepsilon.$$

For any ε, the last two inequalities hold in some region $0 < |x - a| < \delta$. So the first one also holds. This proves that $g(x) \to L$. Values at $x = a$ are not involved—until we get to continuous functions.

2.6 EXERCISES

Read-through questions

The limit of $a_n = (\sin n)/n$ is __a__. The limit of $a_n = n^4/2^n$ is __b__. The limit of $a_n = (-1)^n$ is __c__. The meaning of $a_n \to 0$ is: Only __d__ of the numbers $|a_n|$ can be __e__. The meaning of $a_n \to L$ is: For every __f__ there is an __g__ such that __h__ if $n >$ __i__. The sequence $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \ldots$ is not __j__ because eventually those sums go past __k__.

The limit of $f(x) = \sin x$ as $x \to a$ is __l__. The limit of $f(x) = x/|x|$ as $x \to -2$ is __m__, but the limit as $x \to 0$ does not __n__. This function only has __o__ -sided limits. The meaning of $\lim_{x\to a} f(x) = L$ is: For every ε there is a δ such that $|f(x)-L| < \varepsilon$ whenever __p__.

Two rules for limits, when $a_n \to L$ and $b_n \to M$, are $a_n + b_n \to$ __q__ and $a_n b_n \to$ __r__. The corresponding rules for functions, when $f(x) \to L$ and $g(x) \to M$ as $x \to a$, are __s__ and __t__. In all limits, $|a_n - L|$ or $|f(x) - L|$ must eventually go below and __u__ any positive __v__.

$A \Rightarrow B$ means that A is a __w__ condition for B. Then B is true __x__ A is true. $A \Leftrightarrow B$ means that A is a __y__ condition for B. Then B is true __z__ A is true.

1 What is a_4 and what is the limit L? After which N is $|a_n - L| < \frac{1}{10}$? (Calculator allowed)

(a) $-1, +\frac{1}{2}, -\frac{1}{3}, \ldots$ (b) $\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{6}, \ldots$

(c) $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots a_n = n/2^n$ (d) $1.1, 1.11, 1.111, \ldots$

(e) $a_n = \sqrt[n]{n}$ (f) $a_n = \sqrt{n^2+n} - n$

(g) $1+1, (1+\frac{1}{2})^2, (1+\frac{1}{3})^3, \ldots$

2 Show by example that these statements are false:

(a) If $a_n \to L$ and $b_n \to L$ then $a_n/b_n \to 1$

(b) $a_n \to L$ if and only if $a_n^2 \to L^2$

(c) If $a_n < 0$ and $a_n \to L$ then $L < 0$

(d) If infinitely many a_n's are inside every strip around zero then $a_n \to 0$.

3 Which of these statements are equivalent to $B \Rightarrow A$?

(a) If A is true so is B

(b) A is true if and only if B is true

(c) B is a sufficient condition for A

(d) A is a necessary condition for B.

4 Decide whether $A \Rightarrow B$ or $B \Rightarrow A$ or neither or both:

(a) $A = [a_n \to 1]$ $B = [-a_n \to -1]$

(b) $A = [a_n \to 0]$ $B = [a_n - a_{n-1} \to 0]$

(c) $A = [a_n \leqslant n]$ $B = [a_n = n]$

(d) $A = [a_n \to 0]$ $B = [\sin a_n \to 0]$

(e) $A = [a_n \to 0]$ $B = [1/a_n \text{ fails to converge}]$

(f) $A = [a_n < n]$ $B = [a_n/n \text{ converges}]$

***5** If the sequence $a_1, a_2, a_3, \ldots$ approaches zero, prove that we can put those numbers in any order and the new sequence still approaches zero.

***6** Suppose $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Prove from the definitions that $f(x) + g(x) \to L + M$ as $x \to a$.

Find the limits 7–24 if they exist. An $\varepsilon - \delta$ test is not required.

7 $\displaystyle\lim_{t\to 2} \frac{t+3}{t^2-2}$ **8** $\displaystyle\lim_{t\to 2} \frac{t^2+3}{t-2}$

9 $\displaystyle\lim_{x\to 0} \frac{f(x+h)-f(x)}{h}$ (careful) **10** $\displaystyle\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

11 $\displaystyle\lim_{h\to 0} \frac{\sin^2 h \cos^2 h}{h^2}$ **12** $\displaystyle\lim_{x\to 0} \frac{2x \tan x}{\sin x}$

13 $\displaystyle\lim_{x\to 0^+} \frac{|x|}{x}$ (one-sided) **14** $\displaystyle\lim_{x\to 0^-} \frac{|x|}{x}$ (one-sided)

15 $\displaystyle\lim_{x\to 1} \frac{\sin x}{x}$ **16** $\displaystyle\lim_{c\to a} \frac{f(c)-f(a)}{c-a}$

17 $\displaystyle\lim_{x\to 5} \frac{x^2+25}{x-5}$ **18** $\displaystyle\lim_{x\to 5} \frac{x^2-25}{x-5}$

19 $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ (test $x = .01$) **20** $\displaystyle\lim_{x\to 2} \frac{\sqrt{4-x}}{\sqrt{6+x}}$

21 $\displaystyle\lim_{x\to a} [f(x)-f(a)]$ (?) **22** $\displaystyle\lim_{x\to \pi/2} (\sec x - \tan x)$

23 $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sin x/2}$ **24** $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1}$

25 Choose δ so that $|f(x)| < \frac{1}{100}$ if $0 < x < \delta$.

$f(x) = 10x$ $f(x) = \sqrt{x}$ $f(x) = \sin 2x$ $f(x) = x \sin x$

26 Which does the definition of a limit require?

(1) $|f(x)-L| < \varepsilon \Rightarrow 0 < |x-a| < \delta$.

(2) $|f(x)-L| < \varepsilon \Leftarrow 0 < |x-a| < \delta$.

(3) $|f(x)-L| < \varepsilon \Leftrightarrow 0 < |x-a| < \delta$.

27 The definition of "$f(x) \to L$ as $x \to \infty$" is this: For any ε there is an X such that ______ $< \varepsilon$ if $x > X$. Give an example in which $f(x) \to 4$ as $x \to \infty$.

28 Give a correct definition of "$f(x) \to 0$ as $x \to -\infty$."

29 The limit of $f(x) = (\sin x)/x$ as $x \to \infty$ is ______. For $\varepsilon = .01$ find a point X beyond which $|f(x)| < \varepsilon$.

30 The limit of $f(x) = 2x/(1+x)$ as $x \to \infty$ is $L = 2$. For $\varepsilon = .01$ find a point X beyond which $|f(x) - 2| < \varepsilon$.

31 The limit of $f(x) = \sin x$ as $x \to \infty$ does not exist. Explain why not.

32 (Calculator) Estimate the limit of $\left(1+\frac{1}{x}\right)^x$ as $x \to \infty$.

33 For the polynomial $f(x)=2x-5x^2+7x^3$ find

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to\infty} f(x)$

(c) $\lim_{x\to\infty} \frac{f(x)}{x^3}$ (d) $\lim_{x\to-\infty} \frac{f(x)}{x^3}$

34 For $f(x)=6x^3+1000x$ find

(a) $\lim_{x\to\infty} \frac{f(x)}{x}$ (b) $\lim_{x\to\infty} \frac{f(x)}{x^2}$

(c) $\lim_{x\to\infty} \frac{f(x)}{x^4}$ (d) $\lim_{x\to\infty} \frac{f(x)}{x^3+1}$

Important rule As $x\to\infty$ the ratio of polynomials $f(x)/g(x)$ has the same limit as the ratio of their *leading terms*. $f(x)=x^3-x+2$ has leading term x^3 and $g(x)=5x^6+x+1$ has leading term $5x^6$. Therefore $f(x)/g(x)$ behaves like $x^3/5x^6\to 0$, $g(x)/f(x)$ behaves like $5x^6/x^3\to\infty$, $(f(x))^2/g(x)$ behaves like $x^6/5x^6\to 1/5$.

35 Find the limit as $x\to\infty$ if it exists:

$$\frac{3x^2+2x+1}{3+2x+x^2} \quad \frac{x^4}{x^3+x^2} \quad \frac{x^2+1000}{x^3-1000} \quad x\sin\frac{1}{x}.$$

36 If a particular δ achieves $|f(x)-L|<\varepsilon$, why is it OK to choose a smaller δ?

37 The sum of $1+r+r^2+\cdots+r^{n-1}$ is $a_n=(1-r^n)/(1-r)$. What is the limit of a_n as $n\to\infty$? For which r does the limit exist?

38 If $a_n\to L$ prove that there is a number N with this property: If $n>N$ and $m>N$ then $|a_n-a_m|<2\varepsilon$. This is Cauchy's test for convergence.

39 No matter what decimals come later, $a_1=.4, a_2=.49$, $a_3=.493,\ldots$ approaches a limit L. How do we know (when we can't know L)? *Cauchy's test* is passed: the a's get closer to each other.

(a) From a_4 onwards we have $|a_n-a_m|<$ ______.

(b) After which a_N is $|a_m-a_n|<10^{-7}$?

40 Choose decimals in Problem 39 so the limit is $L=.494$. Choose decimals so that your professor can't find L.

41 If every decimal in $.abcde\cdots$ is picked at random from $0,1,\ldots,9$, what is the "average" limit L?

42 If every decimal is 0 or 1 (at random), what is the average limit L?

43 Suppose $a_n=\frac{1}{2}a_{n-1}+4$ and start from $a_1=10$. Find a_2 and a_3 and a connection between a_n-8 and $a_{n-1}-8$. Deduce that $a_n\to 8$.

44 "For every δ there is an ε such that $|f(x)|<\varepsilon$ if $|x|<\delta$." That test is twisted around. Find ε when $f(x)=\cos x$, which does *not* converge to zero.

45 Prove the Squeeze Theorem for sequences, using ε: If $a_n\to L$ and $c_n\to L$ and $a_n\leqslant b_n\leqslant c_n$ for $n>N$, then $b_n\to L$.

46 Explain in 110 words the difference between "we will get there if you hurry" and "we will get there only if you hurry" and "we will get there if and only if you hurry."

2.7 Continuous Functions

This will be a brief section. It was originally included with limits, but the combination was too long. We are still concerned with the limit of $f(x)$ as $x \to a$, but a new number is involved. That number is $f(a)$, *the value of* f *at* $x = a$. For a "limit," x approached a but never reached it—so $f(a)$ was ignored. For a "continuous function," this final number $f(a)$ must be right.

May I summarize the usual (good) situation as x approaches a?

1. The number $f(a)$ exists (f is defined at a)
2. The limit of $f(x)$ exists (it was called L)
3. The limit L equals $f(a)$ ($f(a)$ is the right value)

In such a case, $f(x)$ is ***continuous*** at $x = a$. These requirements are often written in a single line: $f(x) \to f(a)$ as $x \to a$. By way of contrast, start with four functions that are *not* continuous at $x = 0$.

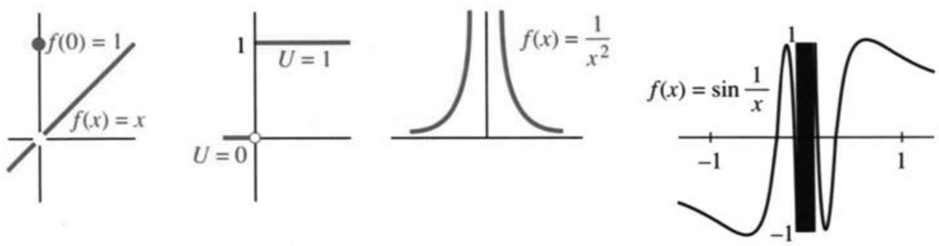


Fig. 2.20 Four types of discontinuity (others are possible) at $x = 0$.

In Figure 2.20, the first function would be continuous if it had $f(0) = 0$. But it has $f(0) = 1$. After changing $f(0)$ to the right value, the problem is gone. The discontinuity is *removable*. Examples $2, 3, 4$ are more important and more serious. There is no "correct" value for $f(0)$:

2. $f(x) =$ step function (jump from 0 to 1 at $x = 0$)
3. $f(x) = 1/x^2$ (infinite limit as $x \to 0$)
4. $f(x) = \sin(1/x)$ (infinite oscillation as $x \to 0$).

The graphs show how the limit fails to exist. The step function has a ***jump discontinuity***. It has ***one-sided limits***, from the left and right. It does not have an ordinary (two-sided) limit. The limit from the left ($x \to 0^-$) is 0. The limit from the right ($x \to 0^+$) is 1. Another step function is $x/|x|$, which jumps from -1 to 1.

In the graph of $1/x^2$, the only reasonable limit is $L = +\infty$. I cannot go on record as saying that this limit exists. Officially, it doesn't—but we often write it anyway: $1/x^2 \to \infty$ as $x \to 0$. This means that $1/x^2$ goes (and stays) above every L as $x \to 0$.

In the same unofficial way we write one-sided limits for $f(x) = 1/x$:

$$\text{From the left, } \lim_{x \to 0^-} \frac{1}{x} = -\infty. \quad \text{From the right, } \lim_{x \to 0^+} \frac{1}{x} = +\infty. \tag{1}$$

Remark $1/x$ has a "***pole***" at $x = 0$. So has $1/x^2$ (a double pole). The function $1/(x^2 - x)$ has poles at $x = 0$ and $x = 1$. In each case the denominator goes to zero and the function goes to $+\infty$ or $-\infty$. Similarly $1/\sin x$ has a pole at every multiple of π (where $\sin x$ is zero). Except for $1/x^2$ these poles are "*simple*"—the functions are completely smooth at $x=0$ when we multiply them by x:

$$(x)\left(\frac{1}{x}\right)=1 \text{ and } (x)\left(\frac{1}{x^2-x}\right)=\frac{1}{x-1} \text{ and } (x)\left(\frac{1}{\sin x}\right) \text{ are continuous at } x=0.$$

$1/x^2$ has a double pole, since it needs multiplication by x^2 (not just x). A ratio of polynomials $P(x)/Q(x)$ has poles where $Q=0$, provided any common factors like $(x+1)/(x+1)$ are removed first.

Jumps and poles are the most basic discontinuities, but others can occur. The fourth graph shows that $\sin(1/x)$ has no limit as $x\to 0$. This function does not blow up; the sine never exceeds 1. At $x=\frac{1}{3}$ and $\frac{1}{4}$ and $\frac{1}{1000}$ it equals $\sin 3$ and $\sin 4$ and $\sin 1000$. Those numbers are positive and negative and (?). As x gets small and $1/x$ gets large, the sine oscillates faster and faster. Its graph won't stay in a small box of height ε, no matter how narrow the box.

CONTINUOUS FUNCTIONS

DEFINITION f is "***continuous at*** $x=a$" if $f(a)$ is defined and $f(x)\to f(a)$ as $x\to a$. If f is continuous at every point where it is defined, it is a ***continuous function***.

Objection The definition makes $f(x)=1/x$ a continuous function! It is not defined at $x=0$, so its continuity can't fail. The logic requires us to accept this, but we don't have to like it. Certainly there is no $f(0)$ that would make $1/x$ continuous at $x=0$.

It is amazing but true that the definition of "continuous function" is still debated (*Mathematics Teacher*, May 1989). You see the reason—we speak about a discontinuity of $1/x$, and at the same time call it a continuous function. The definition misses the difference between $1/x$ and $(\sin x)/x$. *The function* $f(x)=(\sin x)/x$ *can be made continuous at all* x. Just set $f(0)=1$.

We call a function "***continuable***" if its definition can be extended *to all* x in a way that makes it continuous. Thus $(\sin x)/x$ and $\sqrt{x}$ are continuable. The functions $1/x$ and $\tan x$ are not continuable. This suggestion may not end the debate, but I hope it is helpful.

EXAMPLE 1 $\sin x$ and $\cos x$ and all polynomials $P(x)$ are continuous functions.

EXAMPLE 2 The absolute value $|x|$ is continuous. Its slope jumps (not continuable).

EXAMPLE 3 Any rational function $P(x)/Q(x)$ is continuous except where $Q=0$.

EXAMPLE 4 The function that jumps between 1 at fractions and 0 at non-fractions is ***discontinuous everywhere***. There is a fraction between every pair of non-fractions and vice versa. (Somehow there are many more non-fractions.)

EXAMPLE 5 The function 0^{x^2} is zero for every x, except that 0^0 is not defined. So define it as zero and this function is continuous. But see the next paragraph where 0^0 has to be 1.

We could fill the book with proofs of continuity, but usually the situation is clear. "A function is continuous if you can draw its graph without lifting up your pen." At a jump, or an infinite limit, or an infinite oscillation, there is no way across the discontinuity except to start again on the other side. The function x^n is continuous for $n > 0$. It is not continuable for $n < 0$. The function x^0 equals 1 for every x, except that 0^0 is not defined. This time continuity requires $0^0 = 1$.

The interesting examples are the close ones—we have seen two of them:

EXAMPLE 6 $\dfrac{\sin x}{x}$ and $\dfrac{1-\cos x}{x}$ are both continuable at $x = 0$.

Those were crucial for the slope of $\sin x$. The first approaches 1 and the second approaches 0. Strictly speaking we must give these functions the correct values (1 and 0) at the limiting point $x = 0$—which of course we do.

It is important to know what happens when the denominators change to x^2.

EXAMPLE 7 $\dfrac{\sin x}{x^2}$ blows up but $\dfrac{1-\cos x}{x^2}$ has the limit $\dfrac{1}{2}$ at $x = 0$.

Since $(\sin x)/x$ approaches 1, dividing by another x gives a function like $1/x$. There is a simple pole. It is an example of $0/0$, in which the zero from x^2 is reached more quickly than the zero from $\sin x$. The "***race to zero***" produces almost all interesting problems about limits.

For $1 - \cos x$ and x^2 the race is almost even. Their ratio is 1 to 2:

$$\frac{1-\cos x}{x^2} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} \to \frac{1}{1+1} \quad \text{as} \quad x \to 0.$$

This answer $\frac{1}{2}$ will be found again (more easily) by "l'Hôpital's rule." Here I emphasize not the answer but the problem. A central question of differential calculus is *to know how fast the limit is approached*. ***The speed of approach is exactly the information in the derivative***.

These three examples are all continuous at $x = 0$. The race is controlled by the slope—because $f(x) - f(0)$ is nearly $f'(0)$ times x:

derivative of $\sin x$ is 1	$\leftrightarrow$	$\sin x$ decreases like x
derivative of $\sin^2 x$ is 0	$\leftrightarrow$	$\sin^2 x$ decreases faster than x
derivative of $x^{1/3}$ is ∞	$\leftrightarrow$	$x^{1/3}$ decreases more slowly than x.

DIFFERENTIABLE FUNCTIONS

The absolute value $|x|$ is continuous at $x = 0$ but has no derivative. The same is true for $x^{1/3}$. ***Asking for a derivative is more than asking for continuity***. The reason is fundamental, and carries us back to the key definitions:

Continuous at x: $f(x+\Delta x) - f(x) \to 0$ as $\Delta x \to 0$

Derivative at x: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} \to f'(x)$ as $\Delta x \to 0$.

In the first case, Δf goes to zero (maybe slowly). In the second case, Δf goes to zero *as fast as* Δx (because $\Delta f/\Delta x$ has a limit). That requirement is stronger:

2I At a point where $f(x)$ has a derivative, the function must be continuous. But $f(x)$ can be continuous with no derivative.

Proof The limit of $\Delta f = (\Delta x)(\Delta f/\Delta x)$ is $(0)(df/dx) = 0$. So $f(x+\Delta x) - f(x) \to 0$.

The continuous function $x^{1/3}$ has no derivative at $x = 0$, because $\frac{1}{3}x^{-2/3}$ blows up. The absolute value $|x|$ has no derivative because its slope jumps. The remarkable function $\frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \cdots$ is continuous at *all points* and has a derivative at *no points*. You can draw its graph without lifting your pen (but not easily—it turns at every point). To most people, it belongs with space-filling curves and unmeasurable areas—in a box of curiosities. Fractals used to go into the same box! They are beautiful shapes, with boundaries that have no tangents. The theory of fractals is very alive, for good mathematical reasons, and we touch on it in Section 3.7.

I hope you have a clear idea of these basic definitions of calculus:

1 ***Limit*** $(n \to \infty$ or $x \to a)$ **2** ***Continuity*** (at $x = a$) **3** ***Derivative*** (at $x = a$).

Those go back to ε and δ, but it is seldom necessary to follow them so far. In the same way that economics describes many transactions, or history describes many events, a function comes from many values $f(x)$. A few points may be special, like market crashes or wars or discontinuities. At other points df/dx is the best guide to the function.

This chapter ends with two essential facts about ***a continuous function on a closed interval***. The interval is $a \leqslant x \leqslant b$, written simply as $[a,b]$.† At the endpoints a and b we require $f(x)$ to approach $f(a)$ and $f(b)$.

Extreme Value Property A continuous function on the finite interval $[a,b]$ has a maximum value M and a minimum value m. There are points $x_{\max}$ and $x_{\min}$ in $[a,b]$ where it reaches those values:

$$f(x_{\max}) = M \geqslant f(x) \geqslant f(x_{\min}) = m \quad \text{for all } x \text{ in } [a,b].$$

Intermediate Value Property If the number F is between $f(a)$ and $f(b)$, there is a point c between a and b where $f(c) = F$. Thus if F is between the minimum m and the maximum M, there is a point c between $x_{\min}$ and $x_{\max}$ where $f(c) = F$.

Examples show why we require closed intervals and continuous functions. For $0 < x \leqslant 1$ the function $f(x) = x$ never reaches its minimum (zero). If we close the interval by defining $f(0) = 3$ (discontinuous) the minimum is still not reached. Because of the jump, the intermediate value $F = 2$ is also not reached. The idea of continuity was inescapable, after Cauchy defined the idea of a limit.

†The interval $[a,b]$ is ***closed*** (endpoints included). The interval (a, b) is ***open*** (a and b left out). The infinite interval $[0, \infty)$ contains all $x \geqslant 0$.

2.7 EXERCISES

Read-through questions

Continuity requires the __a__ of $f(x)$ to exist as $x \to a$ and to agree with __b__. The reason that $x/|x|$ is not continuous at $x=0$ is __c__. This function does have __d__ limits. The reason that $1/\cos x$ is discontinuous at __e__ is __f__. The reason that $\cos(1/x)$ is discontinuous at $x=0$ is __g__. The function $f(x)=$ __h__ has a simple pole at $x=3$, where f^2 has a __i__ pole.

The power x^n is continuous at all x provided n is __j__. It has no derivative at $x=0$ when n is __k__. $f(x)=\sin(-x)/x$ approaches __l__ as $x \to 0$, so this is a __m__ function provided we define $f(0)=$ __n__. A "continuous function" must be continuous at all __o__. A "continuable function" can be extended to every point x so that __p__.

If f has a derivative at $x=a$ then f is necessarily __q__ at $x=a$. The derivative controls the speed at which $f(x)$ approaches __r__. On a closed interval $[a,b]$, a continuous f has the __s__ value property and the __t__ value property. It reaches its __u__ M and its __v__ m, and it takes on every value __w__.

In Problems 1–20, find the numbers c that make $f(x)$ into (A) a continuous function and (B) a differentiable function. In one case $f(x) \to f(a)$ at every point, in the other case $\Delta f/\Delta x$ has a limit at every point.

1 $f(x)=\begin{cases} \sin x & x<1 \\ c & x \geqslant 1 \end{cases}$

2 $f(x)=\begin{cases} \cos^3 x & x \neq \pi \\ c & x=\pi \end{cases}$

3 $f(x)=\begin{cases} cx & x<0 \\ 2cx & x \geqslant 0 \end{cases}$

4 $f(x)=\begin{cases} cx & x<1 \\ 2cx & x \geqslant 1 \end{cases}$

5 $f(x)=\begin{cases} c+x & x<0 \\ c^2+x^2 & x \geqslant 0 \end{cases}$

6 $f(x)=\begin{cases} x^3 & x \neq c \\ -8 & x=c \end{cases}$

7 $f(x)=\begin{cases} 2x & x<c \\ x+1 & x \geqslant c \end{cases}$

8 $f(x)=\begin{cases} x^c & x \neq 0 \\ 0 & x=0 \end{cases}$

9 $f(x)=\begin{cases} (\sin x)/x^2 & x \neq 0 \\ c & x=0 \end{cases}$

10 $f(x)=\begin{cases} x+c & x \leqslant c \\ 1 & x>c \end{cases}$

11 $f(x)=\begin{cases} c & x \neq 4 \\ 1/x^3 & x=4 \end{cases}$

12 $f(x)=\begin{cases} c & x \leqslant 0 \\ \sec x & x \geqslant 0 \end{cases}$

13 $f(x)=\begin{cases} \dfrac{x^2+c}{x-1} & x \neq 1 \\ 2 & x=1 \end{cases}$

14 $f(x)=\begin{cases} \dfrac{x^2-1}{x-c} & x \neq c \\ 2c & x=c \end{cases}$

15 $f(x)=\begin{cases} (\tan x)/x & x \neq 0 \\ c & x=0 \end{cases}$

16 $f(x)=\begin{cases} x^2 & x \leqslant c \\ 2x & x>c \end{cases}$

17 $f(x)=\begin{cases} (c+\cos x)/x & x \neq 0 \\ 0 & x=0 \end{cases}$

18 $f(x)=|x+c|$

19 $f(x)=\begin{cases} (\sin x-x)/x^c & x \neq 0 \\ 0 & x=0 \end{cases}$

20 $f(x)=|x^2+c^2|$

Construct your own $f(x)$ with these discontinuities at $x=1$.

21 Removable discontinuity

22 Infinite oscillation

23 Limit for $x \to 1^+$, no limit for $x \to 1^-$

24 A double pole

25 $\lim_{x \to 1^-} f(x) = 4 + \lim_{x \to 1^+} f(x)$

26 $\lim_{x \to 1} f(x) = \infty$ but $\lim_{x \to 1} (x-1) f(x) = 0$

27 $\lim_{x \to 1} (x-1) f(x) = 5$

28 The statement "$3x \to 7$ as $x \to 1$" is false. Choose an ε for which no δ can be found. The statement "$3x \to 3$ as $x \to 1$" is true. For $\varepsilon = \frac{1}{2}$ choose a suitable δ.

29 How many derivatives $f', f'', \ldots$ are continuable functions?

(a) $f=x^{3/2}$ (b) $f=x^{3/2}\sin x$ (c) $f=(\sin x)^{5/2}$

30 Find one-sided limits at points where there is no two-sided limit. Give a 3-part formula for function (c).

(a) $\dfrac{|x|}{7x}$ (b) $\sin |x|$ (c) $\dfrac{d}{dx}|x^2-1|$

31 Let $f(1)=1$ and $f(-1)=1$ and $f(x)=(x^2-x)/(x^2-1)$ otherwise. Decide whether f is continuous at

(a) $x=1$ (b) $x=0$ (c) $x=-1$

*__32__ Let $f(x)=x^2 \sin 1/x$ for $x \neq 0$ and $f(0)=0$. If the limits exist, find

(a) $\lim_{x \to 0} f(x)$ (b) df/dx at $x=0$ (c) $\lim_{x \to 0} f'(x)$

33 If $f(0)=0$ and $f'(0)=3$, rank these functions from smallest to largest as x decreases to zero:

$$f(x), \quad x, \quad xf(x), \quad f(x)+2x, \quad 2(f(x)-x), \quad (f(x))^2.$$

34 Create a discontinuous function $f(x)$ for which $f^2(x)$ is continuous.

35 *True or false*, with an example to illustrate:

(a) If $f(x)$ is continuous at all x, it has a maximum value M.

(b) If $f(x) \leqslant 7$ for all x, then f reaches its maximum.

(c) If $f(1)=1$ and $f(2)=-2$, then somewhere $f(x)=0$.

(d) If $f(1)=1$ and $f(2)=-2$ and f is continuous on $[1,2]$, then somewhere on that interval $f(x)=0$.

36 The functions $\cos x$ and $2x$ are continuous. Show from the ______ property that $\cos x = 2x$ at some point between 0 and 1.

37 Show by example that these statements are false:

(a) If a function reaches its maximum and minimum then the function is continuous.

(b) If $f(x)$ reaches its maximum and minimum and all values between $f(0)$ and $f(1)$, it is continuous at $x = 0$.

(c) (mostly for instructors) If $f(x)$ has the intermediate value property between all points a and b, it must be continuous.

38 Explain with words and a graph why $f(x) = x\sin(1/x)$ is continuous but has no derivative at $x = 0$. Set $f(0) = 0$.

39 Which of these functions are *continuable*, and why?

$$f_1(x) = \begin{cases} \sin x & x < 0 \\ \cos x & x > 1 \end{cases} \qquad f_2(x) = \begin{cases} \sin 1/x & x < 0 \\ \cos 1/x & x > 1 \end{cases}$$

$$f_3(x) = \frac{x}{\sin x} \text{ when } \sin x \neq 0 \qquad f_4(x) = x^0 + 0^{x^2}$$

40 Explain the difference between a continuous function and a continuable function. Are continuous functions always continuable?

*__41__ $f(x)$ is any continuous function with $f(0) = f(1)$.

(a) Draw a typical $f(x)$. Mark where $f(x) = f(x + \frac{1}{2})$.

(b) Explain why $g(x) = f(x + \frac{1}{2}) - f(x)$ has $g(\frac{1}{2}) = -g(0)$.

(c) Deduce from (b) that (a) is always possible: There *must* be a point where $g(x) = 0$ and $f(x) = f(x + \frac{1}{2})$.

42 Create an $f(x)$ that is continuous only at $x = 0$.

43 If $f(x)$ is continuous and $0 \leqslant f(x) \leqslant 1$ for all x, then there is a point where $f(x^*) = x^*$. Explain with a graph and prove with the intermediate value theorem.

44 In the ε–δ definition of a limit, change $0 < |x - a| < \delta$ to $|x - a| < \delta$. Why is $f(x)$ now *continuous* at $x = a$?

45 A function has a ______ at $x = 0$ if and only if $(f(x) - f(0))/x$ is ______ at $x = 0$.

Resource: Calculus
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