PROFESSOR: All right. So, this homework that is due on Friday contains some questions on the harmonic oscillator. And the harmonic oscillator is awfully important. I gave you notes on that. And I want to use about half of the lecture, perhaps a little less, to go over some of those points in the notes concerning the harmonic oscillator.

After that, we're going to begin, essentially, our study of dynamics. And we will give the revision, today, of the Schrodinger equation. It's the way Dirac, in his textbook on quantum mechanics, presents the Schrodinger equation.

I think it's actually, extremely insightful. It's probably not the way you should see it the first time in your life. But it's a good way to think about it. And it will give you a nice feeling that this Schrodinger equation is something so fundamental and so basic that it would be very hard to change or do anything to it and tinker with it. It's a rather complete theory and quite beautiful [? idea. ?]

So we begin with the harmonic oscillator. And this will be a bit quick. I won't go over every detail. You have the notes. I think that's pretty much all you need to know. So we'll leave it at that.

So the harmonic oscillator is a quantum system. And as quantum systems go, they're inspired by classical systems. And the classical system is very famous here.

It's the system in which, for example, you have a mass and a spring. And it does an oscillation for which the energy is written as p squared over 2m plus 1/2 m, omega squared, x squared. And m omega squared is sometimes called k squared, the spring constant.

And you are supposed to do quantum mechanics with this. So nobody can tell you this is what the harmonic oscillators in quantum mechanics. You have to define it. But since there's only one logical way to define the quantum system, everybody agrees on what the harmonic oscillator quantum system is.

Basically, you use the inspiration of the classical system and declare, well, energy will be the Hamiltonian operator. p will be the momentum operator. And x will be the position operator.

And given that these are operators, will have a basic commutation relation between x and p being equal to i h-bar. And that's it. This is your quantum system. Hamiltonian is-- the set of operators that are relevant for this are the x the p, and the energy operator that will control the dynamics.

You know also you should specify a vector space, the vector space where this acts. And this will be complex functions on the real line. So this will act in wave functions that define the vector space, sometimes called Hilbert space.

It will be the set of integrable functions on the real line, so complex functions on the real line. These are your wave functions, a set of states of the theory. All these complex functions on the real line work. I won't try to be more precise. You could say they're square integrable. That for sure is necessary. And we'll leave it at that.

Now you have to solve this problem. And in 804, we discussed this by using the differential equation and then through the creation annihilation operators. And we're going do it, today, just through creation and annihilation operators.

But we want to emphasize something about this Hamiltonian and something very general, which is that you can right the Hamiltonian as say 1/2m, omega squared, x squared. And then you have plus p squared, m squared, omega squared.

And a great solution to the problem of solving the Hamiltonian-- and it's the best you could ever hope-- is what is called the factorization of the Hamiltonian, in which you would manage to write this Hamiltonian as some operator times the dagger operator.

So this is the ideal situation. It's just wonderful, as you will see, if you can manage to do that. If you could manage to do this factorization, you would know immediately what is the ground state energy, how low can it go, something about the Hamiltonian. You're way on your way of solving the problem. If you could just factorize it.

Yes?

AUDIENCE: [INAUDIBLE] if you could just factorize it in terms of v and v instead of v dagger and v?

PROFESSOR: You want to factorize in which way instead of that?

AUDIENCE: Would it be helpful, if it were possible, to factor it in terms of v times v instead of v dagger?

PROFESSOR: No, no, I want, really, v dagger. I don't want v v. That that's not so good. I want that this factorization has a v dagger there. It will make things much, much better.

So how can you achieve that? Well, it almost looks possible. If you have something like this, like a squared plus b squared, you write it as a minus ib times a plus ib. And that works out.

So you try here, 1/2 m, omega squared, x minus ip over m omega, x plus ip over m omega.

And beware that's not quite right. Because here, you have cross terms that cancel. You have aib b and minus iba. And they would only cancel if a and b commute. And here they don't commute.

So it's almost perfect. But if you expand this out, you get the x squared for sure. You get this term. But then you get an extra term coming from the cross terms. And please calculate it. Happily, it's just a number, because the commutator of x and b is just a number.

So the answer for this thing is that you get, here, x squared plus this is equal to this, plus h-bar over m omega, times the unit operator. So here is what you could call v dagger. And this is what we'd call v.

So what is your Hamiltonian? Your Hamiltonian has become 1/2 m, omega squared, v dagger v, plus, if you multiply out, H omega times the identity. So we basically succeeded. And it's as good as what we could hope or want, actually.

I multiply this out, so h-bar omega was the only thing that was left. And there's your Hamiltonian. Now, in order to see what this tells you, just sandwich it between any two states.

Well, this is 1/2 m, omega squared, psi, v dagger, v, psi, plus 1/2 half h, omega. And assume it's a normalized state, so it just gives you that. So this thing is the norm of the state, v psi.

You'd think it's dagger and it's this. So this is the norm squared of v psi. And therefore that's positive. So H, between any normalized state, is greater than or equal to 1/2 h-bar omega.

In particular, if psi is an energy eigenstate, so that H psi is equal to E psi. If psi is an energy eigenstate, then you have this. And back here, you get that the energy must be greater than or equal to 1/2 h omega, because H and psi gives you an E. The E goes out. And you're left with psi, psi, which is 1. So you already know that the energy is at least greater than or equal to 1/2 h omega.

So this factorization has been very powerful. It has taught you something extremely nontrivial about the spectrum of the Hamiltonian. All energy eigenstates must be greater than or equal to 1/2 h omega.

In fact, this is so good that people try to do this for almost any problem. Any Hamiltonian, probably the first thing you can try is to establish a factorization of this kind. For the hydrogen atom, that factorization is also possible.

There will be some homework sometime later on. It's less well known and doesn't lead to useful creation and annihilation operators. But you can get the ground state energy in a proof that you kind of go below that energy very quickly.

So a few things are done now to clean up this system. And basically, here I have the definition of v and v dagger. Then you define a to be square root of m omega over 2 h-bar, v. And a dagger must be m omega over 2 h-bar v dagger.

And I have not written for you the commutator of v and v dagger. We might as well do the commutator of a and a dagger. And that commutator turns out to be extremely simple. a with a dagger is just equal to 1.

Now things that are useful, relations that are useful is-- just write what v is in here so that you have a formula for a and a dagger in terms of x and p. So I will not bother writing it. But it's here already. Maybe I'll do the first one.

m omega over 2 h-bar. v is here would be x, plus ip over m omega. And you can write the other one there. So you have an expression for a and a dagger in terms of x and p.

And that can be inverted as well. And it's pretty useful. And it's an example of formulas that you don't need to know by heart. And they would be in any formula sheet. And the units and all those constants make it hard to remember. But here they are.

So you should know that x is a plus a dagger up to a constant. And p is a dagger minus a. Now p is Hermitian, that's why there is an i here. So that this, this anti-Hermitian, the i becomes a Hermitian operator. x is manifestly Hermitian, because a plus a dagger is.

Finally, you want to write the Hamiltonian. And the Hamiltonian is given by the following formula. You know you just have to put the v and v dagger, what they are in terms of the creation, annihilation operators. So v dagger, you substitute a dagger. v, you go back here and just calculate it.

And these calculations really should be done. It's something that is good practice and make sure you don't make silly mistakes. So this operator is so important it has been given a name.

It's called the number operator, N. And its eigenvalues are numbers, 0, 1, 2, 3, all these things. And the good thing about it is that, once you are with a's and a daggers, all this m omega, h-bar are all gone.

This is all that is happening here. The basic energy is h-bar omega. Ground state energies, what we'll see is 1/2 h-bar omega. And this is the number operator. So this is written as h-bar omega, number operator-- probably with a hat-- like that.

So when you're talking about eigenvalues, as we will talk soon, or states for which these thing's are numbers, saying that you have a state that is an eigenstate of the Hamiltonian is exactly the same thing as saying that it's an eigenstate of the number operator.

Because that's the only thing that is an operator here. There's this plus this number. So this number causes no problem. Any state multiplied by a number is proportional to itself. But it's not true that every state multiplied by a dagger a is proportional to itself.

So being an eigenstate of N means that acting on a state, N, gives you a number. But then H is just N times the number. So H is also an eigenstate. So eigenstates of N or eigenstates of H are exactly the same thing.

Now there's a couple more properties that maybe need to be mentioned. So I wanted to talk in terms of eigenvalues. I would just simply write the energy eigenvalue is therefore equal h-bar omega, the number eigenvalue-- so the operator is with a hat-- plus 1/2.

So in terms of eigenvalues, you have that. From here, the energy is greater than 1/2 h omega. So the number must be greater or equal than 0 on any state. And that's also clear from the definition of this operator. On any state, the expectation value of this operator has to be positive. And therefore, you have this.

So two more properties that are crucial here are that the Hamiltonian commuted with a is equal to minus h omega a and that the Hamiltonian committed with a dagger is plus h omega a dagger.

Now there is a reasonably precise way of going through the whole spectrum of the harmonic oscillator without solving differential equations, almost to any degree, and trying to be just very logical about it.

It's possible to deduce the properties of the spectrum. So I will do that right now. And we begin with the following statement. We assume there is some energy eigenstate. So assume there is a state E such that the Hamiltonian-- for some reason in the notes apparently I put hats on the Hamiltonian, so I'll start putting hats here-- so that the states are labeled by the energy.

And this begins a tiny bit of confusion about the notation. Many times you want to label the states by the energy. We'll end up labeling them with the number operator. And then, I said, it will turn out, when the number operator is 0, we'll put a 0 in here. And that doesn't mean 0 energy. It means energy equal 1/2 h-bar omega.

So if you assume there is an energy eigenstate, that's the first step in the construction. You assume there is one. And what does that mean? It means that this is a good state.

So it may be normalized. It may not be normalized. In any case, it should be positive. I put first the equal, but I shouldn't put the equal. Because we know in a complex vector space, if a state has 0 norm, it's 0.

And I want to say that there's really some state that is non-0, that has this energy. If the state would be 0, this would become a triviality. So this state is good. It's all good.

Now with this state, you can define, now, two other states, acting with the creation, annihilation operators. I didn't mention that name. But a dagger is going to be called the creation operator. And this is the destruction or annihilation operator.

And we built two states, E plus is a dagger acting on E. And E minus is a acting on

E. Now you could fairly ask a this moment and say, well, how do you know these states are good? How do you know they even exist? How do you know that if you act with this, don't you get an inconsistent state? How do you know this makes sense?

And these are perfectly good questions. And in fact, this is exactly what you have to understand. This procedure can give some funny things. And we want to discuss algebraically why some things are safe and why some things may not quite be safe.

And adding an a dagger, we will see it's safe. While adding a's to the state could be fairly unsafe. So what can be bad about the state? It could be a 0 state, or it could be an inconsistent state.

And what this an inconsistent state? Well, all our states are represented by wave functions. And they should be normalizable. And therefore they have norms that are positive, norms squared that are positive.

Well you may find, here, that you have states that have norms that are negative, norm squareds that are negative. So this thing that should be positive, algebraically you may show that actually you can get into trouble.

And trouble, of course, is very interesting. So I want to skip this calculation and state something that you probably checked in 804, several times, that this state has more energy than E and, in fact, has as much energy as E plus h-bar omega.

Because a dagger, the creation operator, adds energy, h-bar omega. And this subtracts energy, h-bar omega. This state has an energy, E plus, which is equal to E plus h-bar omega. And E minus is equal to E minus h-bar omega.

Now how do you check that? You're supposed to act with a Hamiltonian on this, use the commutation relation that we wrote up there, and prove that those are the energy eigenvalues.

So at this moment, you can do the following. So these states have energies, they have number operators, they have number eigenvalues. So we can test, if these states are good, by computing their norms.

So let's compute the norm, a dagger on E, a dagger on E for the first one. And we'll compute a E, a E. We'll do this computation. We just want to see what this is.

Now remember how you do this. An operator acting here goes with a dagger into the other side. So this is equal to E a, a dagger, E.

Now a, a dagger is not quite perfect. It differs from the one that we know is an eigenvalue for this state, which is the number operator. So what is a, a dagger in terms of N? Well, a, a dagger-- it's something you will use many, many times-- is equal to a commutator with a dagger plus a dagger a. So that's 1 plus the number operator. So this thing is E 1 plus the number operator acting on the state E.

Well, the 1 is clear what it is. And the number operate is clear. If this has some energy E, well, I can now what is the eigenvalue of the number operator because the energy on the number eigenvalues are related that way. So I will simply call it the number of E and leave it at that. Times EE.

So in here, the computation is easier because it's just E a dagger a E. That's the number, so that's just NE times EE. OK, so these are the key equations we're going to be using to understand the spectrum quickly. And let me say a couple of things about them. So I'll repeat what we have there, a dagger E a dagger E is equal to 1 plus NE EE.

On the other hand, 888 aE aE is equal to NE EE. OK, so here it goes. Here is the main thing that you have to think about.

Suppose this state was good, which means this state has a good norm here. And moreover, we've already learned that the energy is greater than some value. So the number operator of this state could be 0-- could take eigenvalue 0. But it could be bigger than 0, so that's all good.

Now, at this stage, we have that-- for example, this state, a dagger E has number one higher than this one, than the state E because it has an extra factor of the a dagger which adds an energy of h omega. Which means that it adds number of 1,

So if this state has some number, this state has a number which is bigger. So suppose you keep adding. Now, look at the norm of this state. The norm of this state is pretty good because this is positive and this is positive.

If you keep adding a daggers here, you always have that this state, the state with two a daggers, you could use that to find its norm. You could use this formula, put in the states with one a dagger here. But the states with one a dagger already has a good norm. So this state with two a daggers would have also good norm. So you can go on step by step using this equation to show that as long as you keep adding a daggers, all these states will have positive norms.

And they have positive norms because their number eigenvalue is bigger and bigger. And therefore, the recursion says that when you add one a dagger, you don't change the sign of this norm because this is positive and this is positive, and this keeps happening.

On the other hand, this is an equation that's a lot more dangerous. Because this says that in this equation, a lowers the number. So if this has some number, NE, this has NE minus 1.

And if you added another a here, you would use this equation again and try to find, what is the norm of things with two a's here? And put in the one with one a here and the number of that state.

But eventually, the number can turn into a negative number. And as soon as the number turns negative, you run into trouble. So this is the equation that is problematic and the equation that you need to understand. So let me do it in two stages.

Here are the numbers. And here is 5 4, 3, 2, 1, 0. Possibly minus 1, minus 2, and all these numbers.

Now, suppose you start with a number that is an integer. Well, you go with this equation. This has number 4. Well, you put an a. Now it's a state with number 3, but its norm is given 4 times that. So it's good.

Now you go down another 1, you have a state with number 3, with number 2, with number 1, with number 0. And then if you keep lowering, you will get minus 1, which is not so good. We'll see what happens.

Well, here you go on and you start producing the states-- the state with number 4, state with number 3, state with number 2, state with number 1. And state here, let's call it has an energy E prime. And it has number equal 0. Number of E prime equals 0.

So you look at this equation and it says aE prime times aE prime is equal N E prime times E prime E prime.

Well, you obtain this state at E prime, and it was a good state because it came from a state that was good before. And therefore, when you did the last step, you had the state at 1 here, with n equals to 1, and then that was the norm of this state. So this E E prime is a fine number positive. But the number E prime is 0. So this equation says that aE prime aE prime is equal to 0.

And if that's equal to 0, the state aE prime must be equal to 0. And 0 doesn't mean the vacuum state or anything. It's just not there. There's no such state. You can't create it.

You see, aE prime would be a state here with number minus 1. And everything suggests to us that that's not possible. It's an inconsistent state. The number must be less than 1. And we avoided the inconsistency because this procedure said that as you go ahead and do these things, you eventually run into this state E prime at 0 number. But then, you get that the next state is 0 and there's no inconsistency. Now, that's one possibility.

The other possibility that could happen is that there are energy eigenstates that have numbers which are not-- well, I'll put it here. That are not integer.

So maybe you have a state here with some number E which is not an integer. It doesn't belong to the integers. OK, so what happens now?

Well, this number is positive. So you can lower it and you can put another state with number 1 less. Also, not integer and it has good norm. And this thing has number 2.5, say.

Well, if I use the equation again, I put the 2.5 state with its number 2.5 and now I get the state with number 1.5 and it still has positive norm.

Do it again, you find the state with 0.5 number and still positive norm. And looking at this, you start with a state with 0.5, with 0.5 here. And oops, you get a state that minus 0.5. And it seems to be good, positive norm.

But then, if this is possible, you could also build another state acting with another a. And this state is now very bad because the N for this state was minus 1/2. And therefore, if you put that state, that state at the minus 1/2, you get the norm of the next one that has one less. And this state now is inconsistent.

So you run into a difficulty. So what are the ways in which this difficulty could be avoided? What are the escape hatches?

There are two possibilities. Well, the simplest one would be that the assumption is bad. There's no state with fractional number because it leads to inconsistent states. You can build them and they should be good, but they're bad.

The other possibility is that just like this one sort of terminated, and when you hit 0-- boom, the state became 0. Maybe this one with a fractional one, before you run into trouble you hit a 0 and the state becomes 0. So basically, what you really need to know now on the algebraic method cannot tell you is how many states are killed by a.

If maybe the state of 1/2 is also killed by a, then we would have trouble. Now, as we will see now, that's a simple problem. And it's the only place where it's interesting to solve some equation. So the equation that we want to solve is the equation a on some state is equal to 0.

Now, that equation already says that this possibility is not going to happen. Why? Because from this equation, you can put an a dagger on this. And therefore, you get that NE is equal to 0. This is the number operator, so the eigenvalue of the number operator, we call it NE. So in order to be killed by a, you have to have NE equals 0.

So in the fractional case, no state will be killed and you would arrive to an inconsistency. So the only possibility is that there's no fractional states. So it's still interesting to figure out this differential equation, what it gives you. And why do we call it a differential equation?

Because a is this operator over there. It has x and ip. So the equation is x a E equals 0, which is square root of m omega over 2 h bar x x plus ip over m omega on E equals 0.

And you've translated these kind of things. The first term is an x multiplying the wave function. We can call it psi E of x. The next term, the coefficient in front is something you don't have to worry, of course. It's just multiplying everything, so it's just irrelevant. So have i over m omega. And p, as you remember, is h bar over i d dx of psi E of x zero.

So it's so simple differential equation, x plus h bar over m omega d dx on psi E of x is equal to 0. Just one solution up to a constant is the Gaussian that you know represents a simple harmonic oscillator.

So that's pretty much the end of it. This ground state wave function is a number times the exponential of minus m omega over 2 h bar x squared. And that's that.

This is called the ground state. It has N equals 0 represented as a state. We say this number is N equals 0. So this state is the thing that represents this psi E. In other words, psi E of x is x with 0. And that 0 is a little confusing.

Some people think it's the 0 vector. That's not good. This is not the 0 vector. The 0 vector is not a state. It's not in the Hilbert space. This is the ground state.

Then, the worst confusion is to think it's the 0 vector. The next confusion is to think it's 0 energy. That's not 0 energy, it's number equals 0. The energy is, therefore, 1/2 h bar omega.

And now, given our discussion, we can start building states with more oscillators. So we build a state with number equal 1, which is constructed by an a dagger on the vacuum. This has energy 1 h bar omega more. It has number equal to 1. And that's sometimes useful to just make sure you understand why N on a dagger on the vacuum is a dagger a a dagger on the vacuum.

Now, a kills the vacuum, so this can be replaced by the commutator, which is 1. And therefore, you're left with a dagger on the vacuum. And that means that the eigenvalue of n hat is 1 for this state.

Moreover, this state is where normalized 1 with 1 actually gives you a good normalization if 0 is well-normalized. So we'll take 0 with 0 to be 1, the number 1. And that requires fixing that N0 over here.

Now, these are things that you've mostly seen, so I don't want to say much more about them. I'd rather go through the Schrodinger thing that we have later. So let me conclude by just listing the general states, and then leaving for you to read what is left there in the notes so that you can just get an appreciation of how you use it. And with the practice problems, you'll be done.

So here it is. Here is the answer. The n state is given by 1 over square root of n factorial a dagger to the n acting on the vacuum.

And these n states are such that m with n is delta mn. So here we're using all kinds of things.

First, you should check this is well normalized, or read it and do the calculations. And these are, in fact, orthogonal unless they have the same number of creation operators are the same number. Now, that had to be expected.

These are eigenstates of a Hermitian operator. The N operator is Hermitian.

Eigenstates of a Hermitian operator with different eigenvalues are always orthogonal to each other. If you have eigenstates of a Hermitian operator with the same eigenvalue, if you have a degeneracy, you can always arrange them to make them orthogonal. But if the eigenvalues are different, they are orthogonal. And there's no degeneracies in this spectrum whatsoever.

You will, in fact, argue that because there's no degeneracy in the ground state, there cannot be degeneracy anywhere else. So this result, this orthonormality is really a consequence of all the theorems we've proven. And you could check it by doing the algebra and you would start moving a and a daggers. And you would be left with either some a's or some a daggers. If you're left with some a's, they would kill the thing on the right. If you're left with some a daggers, it would kill the thing on the left. So this can be proven. But this is just a consequence that these are eigenstates of the Hermitian operator n that have different eigenvalues. And therefore, you've succeeded in constructing a full decomposition of the state space of the harmonic oscillator.

We spoke about the Hilbert space. Are now very precisely, see we can say this is u0 plus u1 plus u2 where uk is the states of the form alpha k, where N on k-- maybe I should put n here. It looks nicer. n. Where N n equal n n. So every one-dimensional subspace is spanned by that state of number n. So you have the states of number 0, states of number 1, states of number 2. These are all orthogonal subspaces. They add up to form everything.

It's a nice description. So the general state in this system is a complex number times the state with number 0 plus the complex number states of number 1, complex number, and that. Things couldn't have been easier in a sense.

The other thing that you already know from 804 is that if you try to compute expectation values, most of the times you want to use a's and a daggers. So the typical thing that one wants to compete is on the state n, what is the uncertainty in x on the state n? How much is it? What is the uncertainty of momentum on the energy eigenstate of number n?

These are relatively straightforward calculations. If you have to do the integrals, each one-- by the time you organize all your constants-- half an hour, maybe 20 minutes. If you do it with a and a daggers, this computation should be five minutes, or something like that. We'll see that done on the notes. You can also do them yourselves. You probably have played with them a bit.

So this was a brief review and discussion of them spectrum. It was a little detailed. We had to argue things carefully to make sure we don't assume things. And this is the way we'll do also with angular momentum in a few weeks from now.

But now I want to leave that, so I'm going to take questions. If there are any questions on this logic, please ask. Yes.

AUDIENCE: [INAUDIBLE] for how you got a dagger, a, a dagger, 0, 2 dagger, 0?

PROFESSOR: Yes, that calculation. So let me do at the step that I did in words. So at this place-- so the question was, how did I do this computation?

Here I just copied what N is. So I just copied that. Then, the next step was to say, since a kills this, this is equal to a dagger times a a dagger minus a dagger a. Because a kills it. And I can add this, it doesn't cost me anything.

Now, I added something that is convenient, so that this is a dagger commutator of a with a dagger on 0. This is 1, so you get that.

It's a little more interesting when you have, for example, the state 2, which is 1 over square root of 2 a dagger a dagger on 0. I advise you to try to calculate n on that. And in general, convince yourselves that n is a number operator, which means counts the number of a daggers.

You'll have to use that property if you have N with AB. It's N with A B and then A N with B. The derivative property of the bracket has to be used all the time. So Schrodinger dynamics, let's spend the last 20 minutes of our lecture on this.

So basically, it's a postulate of how evolution occurs in quantum mechanics. So we'll state it as follows. What is time in quantum mechanics?

Well, you have a state space. And you see the state space, you've seen it in the harmonic oscillator is this sum of vectors. And these vectors were wave functions, if you wish. There's no time anywhere there. There's no time on this vector space. This vector space is an abstract vector space of functions or states, but time comes because you have clocks.

And then you can ask, where is my state? And that's that vector on that state space. And you ask the question a littler later and the state has moved. It's another vector. So these are vectors and the vectors change in time. And that's all the dynamics is in quantum mechanics. The time is sort of auxiliary to all this. So we must have a picture of that.

And the way we do this is to imagine that we have a vector space H. And here is a vector. And that H is for Hilbert space. We used to call it in our math part of the course V, the complex vector space. And this state is the state of the system. And we sometimes put the time here to indicate that's what it is. At time t0, that's it.

Well, at time t, some arbitrary later time, it could be here. And the state moves. But one thing is clear. If it's a state of a system, if we normalize it, it should be of unit length. And we can think of a sphere in which this unit sphere is the set of all the tips of the vectors that have unit norm. And this vector will move here in time, trace a trajectory, and reach this one.

And it should do it preserving the length of the vector. And in fact, if you don't use a normalized vector, it has a norm of 3. Well, it should preserve that 3 because you'd normalize the state once and forever.

So we proved in our math part of the subject that an operator that always preserves the length of all vectors is a unitary operator. So this is the fundamental thing that we want. And the idea of quantum mechanics is that psi at time t is obtained by the action of a unitary operator from the state psi at time t0. And this is for all t and t0. And this being unitary.

Now, I want to make sure this is clear. It can be misinterpreted, this equation. Here, psi at t0 is an arbitrary state. If you had another state, psi prime of t0, it would also evolve with this formula. And this U is the same. So the postulate of unitary time evolution is that there is this magical U operator that can evolve any state.

Any state that you give me at time equal 0, any possible state in the Hilbert space, you plug it in here. And by acting with this unitary operator, you get the state at the later time.

Now, you've slipped an extraordinary amount of physics into that statement. If you've bought it, you've bought the Schrodinger equation already. That is going to come out by just doing a little calculation from this. So the Schrodinger equation is really fundamentally, at the end of the day, the statement that this unitary time evolution, which is to mean there's a unitary operator that evolves any physical state.

So let's try to discuss this. Are there any questions? Yes.

AUDIENCE: So you mentioned at first that in the current formulation [INAUDIBLE]?

PROFESSOR: A little louder. We do what in our current formulation?

AUDIENCE: So if you don't include time [INAUDIBLE].

PROFESSOR: That's right. There's no start of the vector space.

AUDIENCE: Right. So is it possible to consider a vector space with time?

PROFESSOR: Unclear. I don't think so. It's just nowhere there. What would it mean, even, to add time to the vector space?

I think you would have a hard time even imagining what it means.

Now, people try to change quantum mechanics in all kinds of ways. Nobody has succeeded in changing quantum mechanics. That should not be a deterrent for you to try, but should give you a little caution that is not likely to be easy. So we'll not try to do that.

Now, let me follow on this and see what it gives us. Well, a few things.

This operator is unique. If it exists, it's unique. If there's another operator that evolves states the same way, it must be the same as that one. Easy to prove.

Two operators that attack the same way on every state are the same, so that's it. Unitary, what does it mean that u t, t0 dagger times u t, t0 is equal to 1?

Now, here these parentheses are a little cumbersome. This is very clear, you take this operator and you dagger it. But it's cumbersome, so we write it like this. This means the dagger of the whole operator. So this is just the same thing. OK, what else?

u of t0, t0, it's the unit operator. If the times are the same, you get the unit operator for all t0 because you're getting psi of t0 here and psi of t0 here. And the only operator that leaves all states the same is the unit operator. So this unitary operator must become the unit operator, in fact, for the two arguments being equal.

Composition. If you have psi t2, that can be obtained as U of t2, t1 times the psi of t1. And it can be obtained as u of t2, t1, u of t1, t0, psi of t0. So what do we learn from here?

That this state itself is u of t2, t0 on the original state. So u of t2, t0 is u of t2, t1 times u of t1, t0. It's like time composition is like matrix multiplication. You go from t0 to t1, then from t1 to t2. It's like the second index of this matrix. In the first index of this matrix, you are multiplying them and you get this thing. So that's composition.

And then, you have inverses as well. And here are the inverses. In that equation, you take t2 equal to t0. So the left-hand side becomes 1. And t1 equal to t, so you get u of t0, t be times u of t, t0 is equal to 1, which makes sense. You propagate from t0 to t. And then from t to t0, you get nothing.

Or if it's to say that the inverse of an operator-- the inverse of this operator is this one. So to take the inverse of a u, you flip the arguments. So I'll write it like that, the inverse minus 1 of t, t0. You just flip the arguments. It's u of t0, t.

And since the operator is Hermitian, the dagger is equal to the inverse. So the inverse of an operator is equal to the dagger. so t, t0 as well. So this one we got here. And Hermiticity says that the dagger is equal to the inverse. Inverse and dagger are the same.

So basically, you can delete the word "inverse" by flipping the order of the arguments. And since dagger is the same as inverse, you can delete the dagger by flipping the order of the arguments.

All right, so let's try to find the Schrodinger equation. So how c we c the Schrodinger equation?

Well, we try obtaining the differential equation using that time evolution over there. So the time evolution is over there. Let's try to find what is d dt of psi t.

So d dt of psi of t is just the d dt of this operator u of t, t0 psi of t0. And I should only differentiate that operate.

Now, I want an equation for psi of t. So I have here psi of t0. So I can write this as du of t, t0 dt. And now put a psi at t. And then, I could put a u from t to t0.

Now, this u of t and t0 just brings it back to time t0. And this is all good now, I have this complicated operator here. But there's nothing too complicated about it. Especially if I reverse the order here, I'll have du dt of t, t0 and u dagger of t, t0. And I reverse the order there in order that this operator is the same as that, the one that is being [INAUDIBLE] that has the same order of arguments, t and t0.

So I've got something now. And I'll call this lambda of t and t0. So what have I learned?

That d dt of psi and t is equal to lambda of t, t0 psi of t. Questions? I don't want to loose you in their derivation. Look at it. Anything-- you got lost, notation, anything. It's a good time to ask. Yes.

AUDIENCE: Just to make sure when you differentiated the state by t, the reason that you don't put that in the derivative because it doesn't have a time [INAUDIBLE] necessarily, or because-- oh, because you're using the value at t0.

PROFESSOR: Right. Here I looked at that equation and the only part that has anything to do with time t is the operator, not the state. Any other comments or questions?

OK, so what have we learned? We want to know some important things about this operator lambda because somehow, it's almost looking like a Schrodinger equation. So we want to see a couple of things about it.

So the first thing that I will show to you is that lambda is, in fact, anti-Hermitian.

Here is lambda. I could figure out, what is lambda dagger?

Well, lambda dagger is you take the dagger of this. You have to think when you take the dagger of this thing. It looks a little worrisome, but this is an operator. This is another operator, which is a time derivative. So you take the dagger by doing the reverse operators and daggers. So the first factor is clearly u of t, t0. And then the dagger of this.

Now, dagger doesn't interfere at all with time derivatives. Think of the time derivative-- operator at one time, operator at another slightly different time. Subtract it. You take the dagger and the dagger goes through the derivative. So this is d u dagger t, t0 dt. So I wrote here what lambda dagger is. You have here what lambda is.

And the claim is that one is minus the other one. It doesn't look obvious because it's supposed to be anti-Hermitian. But you can show it is true by doing the following-- u of t, t0 u dagger of t, t0 is a unitary operator. So this is 1. And now you differentiate with respect to t.

If you differentiate with respect to t, you get du dt of t, t0 u dagger of t, t0 plus u of t, t0 du dagger of t, t0 equals 0 because the right-hand side is 1. And this term is lambda. And the second term is lambda dagger. And they add up to 0, so lambda dagger is minus lambda. Lambda is, therefore, anti-Hermitian as claimed.

Now, look. This is starting to look pretty good. This lambda depends on t and t0. That's a little nasty though. Why?

Here is t. What is t0 doing here?

It better not be there. So what I want to show to you is that even though this looks like it has a t0 in there, there's no t0. So we want to show this operator is actually independent of t0. So I will show that if you have lambda of t, t0, it's actually equal to lambda of t, t1 for any t1. We'll show that.

Sorry.

[LAUGHTER]

PROFESSOR: So this will show that you could take t1 to be t0 plus epsilon. And take the limit and say the derivative of this with respect of t0 is 0. Or take this to mean that it's just absolutely independent of t0 and t0 is really not there.

So if you take t1 equal t dot plus epsilon, you could just conclude from these that this lambda with respect to t0 is 0. No dependence on t0. So how do we do that? Let's go a little quick.

This is du t, t0 dt times u dagger of t, t0. Complete set of states said add something. We want to put the t1 here. So let's add something that will help us do that. So let's add t, t0 and put here a u of t0, t1 and a u dagger of t0, t1. This thing is 1, and I've put the u dagger of t, t0 here. OK, look at this.

T0 and t1 here and t dot t1 there like that. So actually, we'll do it the following way. Think of this whole thing, this d dt is acting just on this factor. But since it's time, it might as well be acting on all of this factor because this has no time. So this is d dt on u t, t0 u t0, t1.

And this thing is u of t1m t0. The dagger can be compensated by this. And this dagger is u of t0, t. This at a t and that's a comma. t0, t. Yes.

OK, so should I go there? Yes. We're almost there.

You see that the first derivative is already d dt of u of t, t1. And the second operator by compensation is u of t1, t, which is the same as u dagger of t, t1. And then, du of t, t1 u dagger of t, t1 is lambda of t, t1. So it's a little sneaky, the proof, but it's totally rigorous. And I don't think there's any step you should be worried there. They're all very logical and reasonable.

So we have two things. First of all, that this quantity, even though it looks like it depends on t0, we finally realized that it does not depend on t0. So I will rewrite this equation as lambda of t. And lambda of t is anti-Hermitian, so we will multiply by an i to make it Hermitian. And in fact, lambda has units of 1 over time. Unitary operators have no units. They're like numbers, like 1 or e to the i phi, or something like that-- have no units. So this has units of 1 over time.

So if I take i h bar lambda of t, this goes from lambda being anti-Hermitian-- this operator is now Hermitian. This goes from lambda having units of 1 over time to this thing having units of energy. So this is a Hermitian operator with units of energy.

Well, I guess not much more needs to be said. If that's a Hermitian operator with units of energy, we will give it a name called H, or Hamiltonian. i h bar lambda of t.

Take this equation and multiply by i h bar to get i h bar d dt of psi is equal to this i h bar lambda, which is h of t psi of t. Schrodinger equation.

So we really got it. That's the Schrodinger equation. That's the question that must be satisfied by any system governed by unitary time evolution. There's not more information in the Schrodinger equation than unitary time evolution. But it allows you to turn the problem around.

You see, when you went to invent a quantum system, you don't quite know how to find this operator u. If you knew u, you know how to evolve anything. And you don't have any more questions. All your questions in life have been answered by that. You know how to find the future. You can invest in the stock market. You can do anything now.

Anyway, but the unitary operator then gives you the Hamiltonian. So if somebody tells you, here's my unitary operator. And they ask you, what is the Hamiltonian?

You go here and calculate I h bar lambda, where lambda is this derivative. And that's the Hamiltonian. And we conversely, if you are lucky-- and that's what we're going to do next time. If you have a Hamiltonian, you try to find the unitary time evolution. That's all you want to know.

But that's a harder problem because you have a differential equation. You have h, which is here , and you are to find u. So it's a first-order matrix differential equation. So it's not a simple problem. But why do we like Hamiltonians?

Because Hamiltonians have to do with energy. And we can get inspired and write quantum systems because we know the energy functional of systems. So we invent a Hamiltonian and typically try to find the unitary time operator.

But logically speaking, there's not more and no less in the Schrodinger equation than the postulate of unitary time evolution. All right, we'll see you next week. In fact--

[APPLAUSE]

Thank you.