PROFESSOR: OK so we're going to do this thing of the hydrogen atom and the algebraic solution. And I think it's not that long stuff so we can take it easy as we go along.

I want to remind you of a couple of facts that will play a role. One result that is very general about the addition of angular momentum that you should of course know is that if you have a j1 times j2. What does this mean? You have some states of-- first angular momentum J1 that so you have a whole multiplet with J1 equals little j1. Which means the states in that multiplet have J1 squared, giving you h squared. Little j1 times little j1 plus 1. That's having a j1 multiplet.

You have a j2 multiplet. And these are two independent commuting angular momenta acting on different degrees of freedom of the same particle or different particles. And what you've learned is that this can be written as a sum of representations. As a direct sum of angular momenta, which goes from j1 plus j2 plus j1 plus j2 minus 1 all the way up to the representation with j1 minus j2. And these are all representations or multiplets that live in the tense or product, but they are multiplets of J equals j1 plus j2.

These states here can be reorganized into these multiplets, and that's our main result for the addition of angular momentum. Mathematically, this formula summarizes it all. These states, when you write them as a basis here-- you take a basis state here times a basis state here-- these are called the coupled bases. And then you reorganize, you form linear combinations that you have been playing with, and then they get reorganized into these states. So these are called the coupled bases in which we're talking about states of the sum of angular momentum. So that's one fact we've learned about.

Now as far as hydrogen is concerned, we're going to try today to understand the spectrum. And for that let me remind you what the spectrum was. The way we organized it was with an L versus energy levels. And we would put an L equals 0 state here-- well maybe-- there's color, so why not using color. Let's see if it works. Yeah, it's OK. L equals 0.

And this was called n equals 1. There's an n equals 2 that has an L equals 0 state and an L equals 1 state. There's an n equals 3 state, set of states that come within L equals 0 and L equals 1 and an L equals 2. And it just goes on and on.

With the energy levels En equals minus e squared over 2a0. That combination is familiar for energy, Bohr radius, charge of electron, with a 1 over n squared. And the fact is that for any level, for each n, L goes from 0, 1, 2, up to n minus 1.

And for each n there's a total of n squared states. And you see it here, you have n equals 2, n equals 1, one state. n equals 2, you have L equals 0, one state. L equals 1 is 3 states. So it's 4. Here we'll have 4 plus 5. So 9. And maybe you can do it, it's a famous thing, there's n squared states at every level.

So this pattern that of course continues and-- it's a little difficult to do a nice diagram of the hydrogen atom in scale because it's all pushed towards the zero energy with 1 over n squared, but that's how it goes. For n equals 4, you have 1, 2, 3, 4 for example.

And this is what we want to understand. So in order to do that, let's return to this Hamiltonian, which is p squared over 2m minus e squared over r. And to the Runge-Lenz vector that we talked about in lecture and you've been playing with. So this Runge-Lenz vector, r, is defined to be 1 over 2me squared p cross L minus L cross p minus r over r. And it has no units.

It's a vector that you've learned has interpretation of a constant vector that points in this direction, r. And it just stays fixed wherever the particle is going. Classically this is a constant vector that points in the direction of the major axis of the ellipse. With respect to this vector, this vector is Hermitian. And you may recall that when we did the classical vector, you had just p cross L and no 2 here.

There are now two terms here. And they are necessary because we want to have a Hermitian operator, and this is the simplest way to construct the Hermitian operator, r. And the way is that you add to this this term, that if L and p commuted as they do in classical mechanics, that the term is identical to this. And you get back to the conventional thing that you had in classical mechanics.

But in quantum mechanics, of course, they don't commute, so it's a little bit different. And moreover this thing, r, is Hermitian. L and p are Hermitian but when you take the Hermitian conjugate, L goes to the other side of p. And since they don't commute, that's not the same thing.

So actually the Hermitian conjugate of this term is this. There's an extra minus sign in hermiticity when you have a cross product. So this is the Hermitian conjugate of this, this is the Hermitian conjugate of this second term, here's the first and therefore this is actually a Hermitian operator. And you can work with it.

Moreover, in the case of classical mechanics, it was conserved. In the case of quantum mechanics this statement of conservation quantum mechanics is something that in one of the exercises that you were asked to try to do this computation so these computations are challenging. They're not all that trivial and are good exercises. So this is one of them. This is practice.

OK this is the vector r. What about it? A few more things about it that are interesting. Because of the hermiticity condition or in-- a way you can check this directly in fact was one of the exercises for you to do, was p cross L-- you did it long time ago, I think-- is equal to minus L cross b plus 2ih bar p.

This is an identity. And this identity helps you write this kind of term in a way in which you have just one order of products and a little extra term, rather than having two complicated terms. So the r can be written as 1 over me squared alone, p cross L minus ihp minus r over r. For example. By writing this term as another p cross L minus that thing gives you that expression for r.

You have an alternative expression in which you solve for the other one. So it's 1 over me squared minus L cross p plus ih bar p. Now, r-- we need to understand r better. That's really the challenge of this whole derivation. So we have one thing that is conserved. Angular momentum is conserved. It commutes with the Hamiltonian. We have another thing that is conserved, this r. But we have to understand better what it is.

So one thing that you can ask is, well, r is conserved, so r squared is conserved as well. So r squared, if I can simplify it-- if I can do the algebra and simplify it-- it should not be that complicated. So again a practice problem was given to do that computation. And I think these forms are useful for that, to work less.

And the computation gives a very nice result, where r squared is equal to 1 plus 2 H over me to the fourth L squared plus h bar squared. Kind of a strange result if you think about it. People that want to do this classically first would find that there's no h squared. And here, this h, is that whole h that we have here. It's a complicated thing.

So this right hand side is quite substantial. You don't have to worry that h is in this side or whether it's on the other side because h commutes with L. L is conserved. So h appears like that. And this, again, is the result of another computation.

So we've learned something. r-- oh, I'm sorry, this is r squared. Apologies. r is conserved. r squared must be conserved because if h commutes with r it commutes with r squared as well. And therefore whatever you see on the right hand side, the whole thing must be conserved. And h is conserved, of course. And L squared is conserved.

Now we need one more property of a relation-- you see, you have to do these things. Even if you probably don't have an inspiration at this moment how you're going to try to understand this, there are things that just curiosity should tell that you should do. We have L, we do L squared. It's an important operator. OK. We had r, we did r squared, which is an important operator.

But one thing we can do is L dot r. It's a good question what L dot r is. So what is L dot r? So-- or r dot L. What is it? Well a few things that are important to note are that you did show before that you know that r dot L, little r dot L, is 0. And little p dot L is 0. These are obvious classically, because L is perpendicular to both r and p.

But quantum mechanically they take a little more work. They're not complicated, but you've shown those two. So if you have r dot L, you would have, for example, here-- r dot L, you would have to do and think of this whole r and put an L on the right.

Well this little r dotted with the L on the right would be 0. That p dotted with L on the right would be 0. And we're almost there, but p cross L dot L, well, what is that? Let me talk about it here. P cross L dot L-- so this is part of the computation of this r dot L. We've already seen this term will give nothing, this term will give nothing. But this term could give something.

So when you face something like that, maybe you say, well, I don't know any identities I should be using here. So you just do it. Then you say, this is i-th component of this vector times the i-th of that.

So it's epsilon ijkpjLkLi. And then you say look this looks a little-- you could say many things that are wrong and get the right answer. So you could say, oh, ki symmetric and ki anti-symmetric. But that's wrong, because these k and i are not symmetric really because these operators don't commute.

So the answer will be zero, but for a more complicated reason. So what do you have in here? ki. Let's move the i to the end of the epsilon, so jkipjLkLi. And now you see this part is pj L cross cross L j. Is the cross product of this.

But what is L cross L? You probably remember. This is ih bar L L cross L, that's the computation in relation of angular momentum. In case you kind of don't remember it was ih bar L. Like that. So now p dot L is anyway 0. So this is 0.

So it's kind of-- it's a little delicate to do these computations. But so since that term is zero, this thing is zero. Now you may as well-- r dot L is 0. Is L dot r also 0? It's not all that obvious you can even do that. Well in a second we'll see that that's true as well. L dot r and r dot L, capital R, are 0.

Let's remember-- let's continue-- let's see, I wanted to number some of these equations. We're going to need them. So this will be equation one. This will be equation two, it's an important one.

Now what was-- let me remind you of a notation we also had about vectors and their rotations. Vector under rotations. So what was a vector, Vi, under rotations was something that you had LiVj equals ih bar epsilon ijkvk.

So there is a way to write this with cross products that is useful in some cases. So I will do it. You probably have seen that in the notes, but let me remind you. Consider this product, L cross V plus V cross L and the i-th component of it. i-th component of this product.

So this is epsilon ijk and you have LjVk plus VjLk. Now in this term you can do something nice. If you think of it like expanded out, you have the second term has epsilon ijkVjk.

Change j for k. If you change j for k, this will be VkLj. And these would have the opposite order. But this order can be changed up to the cost of a minus sign.

So I claim this is ijk-- first term is the same-- minus VkLj. So in the second term, for this term alone, we've done for this term, multiplied with this of course, we've done j and k relabeling.

But this is nothing else than the commutator of L with V. So this is epsilon ijkLjVk. That's epsilon ijk, and this is epsilon jkp or LVL.

Now, 2 epsilons with 2 common indices is something that it simple. It's a commutator dealt on the other indices. Now it's better if they are sort of all aligned in the same way, but they kind of are because this L, without paying a price, can be put as the first index. So you have jk as the second and third and-- jk as the second and third-- once L has been moved to the first position.

So this thing is 2 times delta iL. And there's an h bar, ih bar I forgot here. ih bar. 2 delta ik ih bar iL ih bar VL. So this is 2 ih bar Vi.

So this whole thing the i-th component of this thing, using this commutation relation is this. So what we've learned is that L cross V plus V cross L you see go to 2 ih bar V. And that's a statement as a vector relation of the fact that V is a vector in the rotations.

So for V to be a vector in the rotations means this. And if you wish, it means this thing as well. It's just another thing of what it means.

Now R is a vector in the rotations. This capital R. Why? You've shown that if you have a vector in the rotations and you multiply it by another vector in the rotations under the cross product, it's still a vector in the rotations. So this is a vector in the rotations, this is, and this is a vector in the rotations. R is a vector in the rotations.

So this capital R is a vector on the rotations, which means two things. It means it satisfies this kind of equation. So r cross-- or L cross R plus R cross L is equal to ih bar R.

So R is a vector in the rotation. It's a fact beyond doubt. And that means that we now know the commutation relations between L and R. So we're starting to put together this picture in which we get familiar with R and the commutators that are possible.

So I can summarize it here. L dot R LiRj is ih bar epsilon ijkRk. That's the same statement as this one but in components.

And now you see why R dot L is equal to L dot R. Because actually if you put the same two indices here, i and i, you get zero. So when you have R dot L you have R1L1 plus R2L2 plus R3L3. And each of these two commute when the two indices are the same. Because of the epsilon.

So R dot L is 0. And now you also appreciate that L dot R is also 0, too.

OK. Now comes, in a sense, the most difficult of all calculations. Even if this seemed a little easy. But you can get quite far with it.

So what do you do with Ls? You computed L commutators and you got the algebra of angular momentum. Over here. This is the algebra for angular momentum. And this kind of nontrivial calculation, you did it by building results. You knew how R was a vector in the rotation or how p was a vector in the rotation. You multiplied the two of them, and it was not so difficult.

But the calculation that you really need to do now is the calculation of the commutator say of Ri with Rj. And that looks like a little bit of a nightmare. You have to commute this whole thing with itself. Lots of p's, L's, R's. 1 over R's, those don't commute with p. You remember that.

So this kind of calculation done by brute force. You're talking a day, probably. I think so. And probably it becomes a mess, but. You'll find a little trick helps to organize it better. It's less of a mess, but still you don't get it and-- try several times.

So what we're going to do is try to think of what the answer could be by some arguments. And then once we know what the answer can be, there's still one calculation to be done. That I will probably put in the notes, but it's not a difficult one. And the answer just pops out.

So the question is what is R cross R. R cross R is really what we have when we have this commutator. So we need to know what R cross R is, just like L cross L.

Now R is not likely to be an angular momentum. It's a vector but it's not an angular momentum. Has nothing to do with it. It's more complicated. So what is R cross R quantum-mechanically?

Classically, of course, it would be zero. So first thing is you think of what this should be. We have a vector, because the cross product of two vectors. Now I want to emphasize one other thing, that it should be this thing-- R cross R-- is tantamount to this thing. What is this thing?

It should be actually proportional to some conserved quantity. And the reason is quite interesting. So this is a small aside here. If some operator is conserved, it commutes with the Hamiltonian. Say if S1 and S2 are symmetries, that means that S1 with h is equal to S2 with h is equal to zero.

Then the claim is that the commutator of this S1 and S2 claim S1 commutator with S2 is also a symmetry. So the reason is because commutator of S1 S2 commutator with h is equal actually to zero.

And why would it be equal to zero? It's because of the so-called Jacobi identity for commutators. You'll remember when you have three things like that, this term is equal to 1-- this term plus 1, in which you cycle them. And plus another one where you cycle them again is equal to zero. That's a Jacobi identity. And in those cyclings you get an h with S2, for example, that is zero.

And then an h with S1, which is zero. So you use these things here and you prove that. So I write here, by Jacobi. So if you have a conserved-- this is the great thing about conserved quantities, if you have one conserved quantity, it's OK. But if you have two, you're in business. Because you can then take the commutator of these two and you get another conserved quantity.

And then more commutators and you keep taking commutators and if you're lucky you get all of the conserved quantities. So here R cross R refers to this commutator. So whatever is on the right should be a vector and should be conserved.

And what are our conserved vectors? Well our conserved vectors-- candidates here-- are L, R itself, and L cross R. That's pretty much it. L and R are our only conserved things, so it better be that.

Still this is far too much. So there could be a term proportional to L, a term proportional to R, a term proportional to L dot R.

So this kind of analysis is based by something that Julian Schwinger did. This same guy that actually did quantum electrodynamics along with Feynman and Tomonaga. And he's the one of those who invented the trick of using three-dimensional angular momentum for the two-dimensional oscillator. And had lots of bags of tricks.

So actually this whole discussion of the hydrogen atom-- most books just say, well, these calculations are hopeless. Let me give you the answers. Schwinger, on the other hand, in his book on quantum mechanics-- which is kind of interesting but very idiosyncratic-- finds a trick to do every calculation.

So you never get into a big mess. He's absolutely elegant and keeps pulling tricks from the bag. So this is one of those tricks. Basically he goes through the following analysis now and says, look, suppose I have the vector R and I do a parity transformation. I change it for minus R.

What happens under those circumstances? Well the momentum is the rate of change of R, should also change sign. Quantum mechanically this is consistent, because a commutation between R and p should give you h bar. And if R changes, p should change sign.

But now when you do this, L, which is R cross p, just goes to L. And R, however, changes sign because L doesn't change sign but p does and R does. So under these changes-- so this is the originator, the troublemaker and then everybody else follows-- R also changes sign.

So this is extremely powerful because if you imagine this being equal to something, well it should be consistent with the symmetries. So as I change R to minus R, capital R changes sign but the left hand side doesn't change sign.

Therefore the right hand side should not change sign. And R changes sign and L cross R changes sign. So computation kind of finished because the only thing you can get on the right is L.

This is the kind of thing that you do and probably if you were writing a paper on that you would anyway do the calculation. The silly way, the- the right way. But this is quite save of times. So actually what you have learned is that R cross R is equal to some scalar conserved quantity, which is something that is conserved that could be like an h, for example, here. But it's a scalar. And, L.

Well once you know that much, it doesn't take much work to do this and to calculate what it is. But I will skip that calculation. This is the sort of thoughtful part of it. And R cross R turns out to be ih bar minus 2 h again. h shows up in several places, like here, so it tends-- it has a tendency to show up. me to the fourth L.

So this is our equation for-- and in a sense, all the hard work has been done. Because now you have a complete understanding of these two vectors, L and R. You know what L squared is, what R squared is, what L dot R is. And you know all the commutators, you know the commutation of L with L, L with R, and R with R. You've done all the algebraic work.

And the question is, how do we proceed from now to solve the hydrogen atom. So the way we proceed is kind of interesting. We're going to try to build from this L that is an angular momentum. And this R that is not an angular momentum. Two sets of angular momenta. You have two vectors.

So somehow we want to try to combine them in such a way that we can invent two angular momenta. Just like the angular momentum in the two-dimensional harmonic oscillator. It was not directly through angular momentum, but was mathematical angular momentum. These two angular momenta we're going to build, one of them is going to be recognizable. The other one is going to be a little unfamiliar. But now I have to do something that-- it may sound a little unusual, but is necessary to simplify our life.

I want to say some words that will allow me to think of this h here as a number. And would allow me to think of this h as a number. So here's what we're going to say. It's an assumption-- it's no assumption, but it sounds like an assumption. But there's no assumption whatsoever. We say the following: this hydrogen atom is going to have some states.

So let's assume there is one state, and it has some energy. If I have that state with some energy, well, that would be the end of the story. But in fact, the thing that they want to allow the possibility for is that at that the energy there are more states. One state would be OK, maybe sometimes it happens. But in general there are more states at that energy.

So I don't-- I'm not making any physical assumption to state that there is a subspace of degenerate states. And in that subspace of degenerate states, there may be just one state, there are two states, there are three states, but there's subspace of degenerate states that have some energy. And I'm going to work in that subspace.

And all the operators that I have are going to be acting in that subspace. And I'm going to analyze subspace by subspace of different energies. So we're going to work with one subspace of degenerate energies.

And if I have, for example, the operator R squared acting on any state of that subspace, since h commutes with L squared, h can go here, acts on this thing, becomes a number. So you might as well put a number here. You might as well put a number here as well. It has to be stated like that. Carefully. We're going to work on a degenerate subspace of some energy. But then we can treat the h as a number.

So let me say it here. We'll work in a degenerate subspace with eigenvalues of h equal to h prime, for h prime. Now I want to write some numbers here to simplify my algebra. So without loss of generality we put what this dimensionless-- this is dimensionless. I'm sorry, this is not dimensionless. This one has units of energy.

This is roughly the right energy, with this one would be the right energy for the ground state. Now we don't know the energies and this is going to give us the energies as well. So without solving the differential equation, we're going to get the energies. So if I say, well that's the energies you would say, come on, you're cheating.

So I'll put one over nu squared where nu can be anything. Nu is real. And that's just a way to write things in order to simplify the algebra. I don't know what nu is. How you say-- you don't know, but you have this in mind and it's going to be an integer, sure.

That's what good notation is all about. You write things and then, you know, it's nu. You don't call it N. Because you don't know it's an integer. You call it nu, and you proceed.

So once you have called it nu, you see here that, well, that's what we call h really. h will be-- this h prime is kind of not necessary. This is what-- where h becomes in every formula. So from here you get that minus 2h over me to the fourth is 1 over h squared nu squared.

I have a minus here, I'm sorry. 2h minus me to the fourth down is h squared nu squared. So we can substitute that in our nice formulas that hme to the fourth so our formulas four and five have become-- I'm going to use this blackboard. Any blackboard where I don't have a formula boxed can be erased. So I will continue here.

And so what do we have? R cross R, from that formula, well this thing is over there minus 2h over me to the fourth, you substitute it in here. So it's i over h bar, one over nu squared L. Doesn't look that bad.

And, R squared is equal to 1 minus 1 over h bar nu squared. Like this. L squared plus h squared. 2h, that's minus h squared nu squared. Yeah. So these are nice formulas. These are already quite clean. We'll call them five, equation five.

I still want to rewrite them in a way that perhaps is a little more understandable or suggestive. I will put an h bar nu together with each R. So h nu R cross h nu R is equal to ih bar L. Makes it look nice.

Then for this one you'll put h squared nu squared R squared is equal to h squared nu squared minus 1 minus L squared. It's sort of trivial algebra. You multiply by h squared nu squared, you get this. You get h squared nu squared minus L squared because it's all multiplied minus h squared.

So these two equations, five, have become six. So five and six are really the same equations. Nothing much has been done.

And if you wish, in terms of commutators this equation says that the commutator h nu Ri with h nu Rj is equal to ih bar epsilon ijkLk. H cross this thing h nu R, h nu R cross equal iHL in components means this. That is not totally obvious. It requires a small computation, but is the same computation that shows that this thing is really LiLj equal ih bar epsilon ijkLk. In which these L's have now become R's.

OK so, we've cleaned up everything. We've made great progress even though at this moment it still looks like we haven't solved the problem at all. But we're very close. So are there any questions about what we've done so far? Have I lost you in the algebra, or any goals here? Yes.

AUDIENCE: Why is R cross R not a commutation? Why would we expect that to not be a commutation?

PROFESSOR: In general, it's the same thing as here. L cross L is this. The commutator of two Hermitian operators is anti-Hermitian. So there's always an i over there. Other questions? It's good, you have to-- you should worry about those things. Are the units right, or the right number of i's on the right hand side. That's a great way to catch mistakes.

OK so we're there. And now it should really almost look reasonable to do what we're going to do. h nu R with h nu R gives you like L. So you have L with L, form angular momentum. L and R are vectors in their angular momentum.

Now R cross R is L. And with these units, h nu R and h nu R looks like it has the units of angular momentum. So h nu R can be added to angular momentum to form more angular momentum. So that's exactly what we're going to do.

So here it comes. Key step. J1-- I'm going to define two angular momenta. Well, we hope that they are angular momenta. L plus h nu R. And J2, one half L minus h nu R. These are definitions. It's just defining two operators. We hope something good happens with these operators, but at this moment you don't know.

It's a good suggestion because of the units match and all that stuff. So this is going to be our definitions, seven. And from these of course follows that L, the quantity we know, is J1 plus J2. And R, or h nu R, is J1 minus J2. You solve in the other way.

Now my first claim is that J1 and J2 commute. Commute with each other. So these are nice, commuting angular momenta. Now this computation has to be done-- let me-- yeah, we can do it.

J1i with J2J. It's one half and one half gives you one quarter of Li plus h nu Ri with LJ minus h nu RJ. Now the question is where do I-- I think I can erase most of this blackboard. I can leave this formula. It's kind of the only very much needed one.

So I'll continue with this computation here. This gives me one quarter-- and we have a big parentheses-- ih bar epsilon iJkLk. For the commutator of these two. And then you have the commutator of the cross terms. So what do they look like? They look like minus h nu Li with RJ, and minus h nu Ri with-- no.

So I have minus h nu Li with RJ, and now I have a plus of this term. But I will write this as a minus h nu of LJ with Ri. Those are the two cross products.

And then finally we have this thing, the h nu with h nu Rijk. So I have minus h nu squared, and you have then RiRJk. No, I'll do it this way. I'm sorry.

You have minus over there, and I have this thing so it's minus ih bar epsilon iJkLk from the last two commutators. So this one you use essentially equation six.

Now look. This thing and this thing cancels. And these two terms, they actually cancel as well. Because here you get an epsilon iJR. And here there's an epsilon Ji something. So these two terms actually add up to zero. And this is zero. So indeed, J1i and J2i-- 2J-- is zero. And these are commuting things.

I wanted to say commuting angular momentum, but not quite yet. Haven't shown their angular momenta. So how do we show their angular momenta? We have to try it and see if they really do form an algebra of angular momentum.

So again, for saving room, I'm going to erase this formula. It will reappear in lecture notes. But now it should go.

So the next computation is something that I want to do. J1 cross J1 or the J2 cross J2, to see if they form angular momenta. And I want to do them simultaneously, so I will do one quarter of J1 cross J2 would be L plus minus h nu R cross L plus minus h nu R.

OK that doesn't look bad at all, especially because we have all these formulas for products. So look, you have L cross L, which we know. Then you have L cross R plus R cross L that is conveniently here. And finally, you have R cross R which is here.

So it's all sort of done in a way that the composition should be easy. So indeed 1 over 4 L cross L gives you an ih bar L. From L cross L. From these ones, you get plus minus with plus minus. It's always plus but you get another ihL. So you get another ihL.

And then you get plus minus L cross h nu R plus h nu R cross L. So here you get one quarter of 2 ihL. And look at this formula, just put an h nu here and h nu here and an h nu here. So you get plus minus 2 ih from here and an h nu R.

OK so the twos and the fours and the iH's go out and then you get ih times one half times L plus minus h nu R, which is either J1 or J2.

So, very nicely, we've shown that J1 cross J1 is ih bar J1 and J2 cross J2 is ih bar J2. And now finally you can say that you've discovered two independent angular momenta in the hydrogen atom.

You did have an angular momentum on an R vector, and all of our work has gone into showing now that you have two angular momenta. Pretty much we're at the end of this because, after we do one more little thing, we're there. So let me do it here. I will not need these equations anymore. Except this one I will need. So

L dot R is zero. So from L dot R equals zero, this time you get J1 plus J2 is equal to-- no, times-- J1 minus J2 is equal to zero. Now J1 and J2 commute. So the cross terms vanish. J1 and J2 commute.

So this implies that J1 squared is equal to J2 squared. Now this is a very surprising thing. These two angular momenta have the same length squared.

Let's look a little more at the length squared of it. So let's, for example, square J1. Well, if I square J1, I have one fourth L squared plus h squared nu squared R squared. No L dot R term, because L dot R is 0. And h squared nu squared R squared is here.

So this is good news. This is one fourth L squared plus h squared nu squared minus 1 minus L squared. The L squared cancels. And you've got that J1 equals to J2 squared. And it's equal to one fourth of h squared nu squared minus 1.

OK. Well the problem has been solved, even if you don't notice at this moment. It's all solved. Why?

You've been talking a degenerate subspace with angular momentum with equal energies. And there's two angular momenta there. And their squares equal to the same thing. So these two angular momenta, our squares are the same and the square is precisely what we call h squared J times J plus 1, where j J is quantized. It can be zero, one half, one, all of this. So here comes a quantization. J squared being nu squared, we didn't know what nu squared is, but it's now equal to these things.

So at this moment, things have been quantized. And let's look into a little more detail what has happened and confirm that we got everything we wanted. So let me write that equation again here. J1 squared is equal J2 squared is equal to one quarter h squared nu squared minus 1, which is h squared J times J plus 1.

So cancel the h squares and solve for nu squared. Nu squared would be 1 plus 4J times J plus 1, which is 4J squared plus 4J plus 1, which is 2J plus 1 squared.

That's pretty neat. Why is it so neat? Because as J is equal to zero, all the possible values of angular momentum-- three halves, all these things-- nu, which is 2J plus 1, will be equal to 1, 2, 3, 4-- all the integers. And what was nu? It was the values of the energies.

So actually you've proven the spectrum. Nu has come out to be either 1, 2, 3, but you have all representations of angular momentum. You have the singlet, the spin one half-- where are the spins here? Nowhere. There was an electron, a proton, we never put spin for the hydrogen atom. But it all shows up as these representations in which they come along. Even more is true, as we will see right away and confirm that everything really shows up the right way.

So what happened now? We have two independent, equal angular momentum. So what is this degenerate subspace we were inventing? Is the space J, which is J1 and m1 tensor product with J, which is J2 but has the same value because the squares are the same, m2.

So this is an uncoupled basis. Uncoupled basis of states in the degenerate subspace. And now, you know, it's all a little surreal because these don't look like our states at all. But this is the way algebraically they show up. We choose our value of J, we have then that nu is equal to this and for that value of J there's some values of m's. And therefore, this must be the degenerate subspace.

So this is nothing but the tensor product of a J multiplet with a J multiplet. Where J is that integer here. And what is the tensor product of a J multiplet? First, J is for J1. The second J is for J2.

So at this moment of course we're calling this N for the quantum number. But what is this thing? This is 2J plus 2J minus 1 plus-- all the way up to the singlet.

But what are these representations of? Well here we have J1 and here is J2. These must be the ones of the sum. But who is the sum, L? So these are the L representations that you get. L is your angular momentum. L representations. And if 2J plus 1 is N, you got a representation with L equals N minus 1, because 2J plus 1 is N, L equals N minus 2, all the way up to L equals 0.

Therefore, you get precisely this whole structure. So, just in time as we get to 2 o'clock, we've finished the quantization of the hydrogen atom. We've finished 805. I hope you enjoyed. I did a lot. [INAUDIBLE] and Will did, too. Good luck and we'll see you soon.