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BARTON

ZWIEBACH:

Let's begin. So today's lecture will deal with the subject of squeeze states and photon states. And it all builds up from the ideas of coherent states that we were talking about last time. So let me begin by reminding you about the few facts that we had about coherent states.

So a coherent state was born by taking the ground state of the harmonic oscillator and displacing it with a translation operator some distance, x_0 . And then we let it go, and we saw that this sort of wave function would just move from the left and to the right coherently, without spreading out, without changing shape. It would move in a nice way.

Now, this was obtained with a translation operator, which was an exponential that had on the exponent the momentum operator. But we realized eventually that the reason it all works out is because its exponential of something that depends on creation or annihilation operators, and we could do something more general, which was to use a complex number, α , and define a displacement operator, a more general one, that is some linear combination of a and a^\dagger with α and the complex conjugate of α . So this is only proportional to the momentum if α is real, but if α is not real, that operator in the exponent is not quite the momentum. It's something that has a bit of position as well.

So this is a more general operator, but on the other hand, it's clear that it's anti-Hermitian, because if you take the dagger of this thing, this term becomes that and that term becomes this one, each one with a change of sign. So you're really with an anti-Hermitian operator.

Therefore, the whole operator is unitary and you're acting with a unitary operator on

the vacuum. And therefore, this state is also well normalized and represents a state with some expectation value of the position. Just like a coherent state, we moved it to the right and it had some expectation value of the position. But this one also has some expectation value of the momentum.

So in fact, we realize that the real part of alpha in this axis was related to the expectation value of the position divided by $\sqrt{2} d_0$. So if you produce a coherent state with this value of alpha in the complex alpha plane, well, you go down and that's the expectation value of the position. You go horizontally, well, that's the expectation value of the momentum scaled because this alpha is a pure number, has no units. So this x over square root of $2 d_0$ and $p d_0$ over \hbar have no units, and that's how it should be.

So we learned also that the annihilation operator acting on this coherent state was alpha times the coherent state. So it's a very simple property. That number, alpha, is the eigenvalue of the destruction operator.

Now, that's a one line computation based on this, or a two line computation maybe. But it should be a computation that is easy for you to do. So make sure you know how to get this very quickly from this definition.

So that's a coherent state. And then the thing we finished the lecture with was with the time evolution of this coherent state. And the time evolution was that as the state, alpha, in time becomes the state alpha at time t , it remains a coherent state but the value of alpha is changed. In fact, the value of alpha is changed in such a way that you can just imagine this thing rotating, and rotating with angular velocity, omega.

So this thing was the coherent state, $e^{-i\omega t}$ alpha. So this whole complex number, instead of being alpha, is just this. There's no comma t . This is the time development of the state. And there was a phase here, $e^{-i\omega t}$, that was not very relevant. But that's what the state was doing in time.

So basically, that's where we got last time. Before I push on, do you have any

questions? I did post the notes associated to coherent states about half an hour ago, so you have them. You do have two problems on coherent states in this homework, so the notes should help you. But any questions about this picture?

OK. So I want to develop it a little more before starting with squeeze states. So here's what I want to tell you. And this is an intuition that people have about these states. Alpha is a complex number. And you know, it's a well-defined complex number.

But you know, this is a coherent state, so it's not a position eigenstate. It's not a momentum eigenstate. It's not an energy eigenstate. It has all kinds of uncertainties. It has uncertainties in position, in momentum, and in energy. Yes?

AUDIENCE: On that complex plane where you have the x and the p , are those expectation values of the position of momentum?

BARTON
ZWIEBACH: Yes. I think I wrote them with expectation values of the position of momentum last time. Yes. So given alpha, that number you get is the expectation value of position or expectation value of momentum. Correct.

So actually, this is the expectation value of the position, but the position is a little bit rounded. Yeah, this is the expectation value of the position. It's a number. But intuitively, this is spread out a little. The position is not just one point. This is a coherent state. It looks like a Gaussian wave function.

The momentum is also spread out a little. So in some sense, many people draw this as a little blob. And that blob represents your intuition that yes, the expectation value is this and the expectation value is this, but the position is, well, somewhere around this thing and somewhere around that stuff.

You can complain, this is very hand wavy, but it's useful. It's good to have that physical picture that the state really is some sort of blob here, not that the expectation values are not well defined, but rather that it's something like this.

And I want to relate it to an idea that comes along with waves, and it's important for

what we're going to be doing later today. If you have a wave with energy e , and suppose your wave is a light wave. Light wave with energy e . And it's described by, say, $A \cos(\omega t)$. That's some component of the electric field or the magnetic field for this wave. It is like that.

Well, we've been talking about energy time uncertainty, and we know that unless we make things very precise. It's easy to get things wrong. So I will first do something fairly imprecise to give you a feeling of what people talk about, and then we'll use this picture to do it more precisely.

So if you have this wave, the phase of this wave-- we'll call it ϕ -- is ωt . And if we are naive there, the phase is divided by ω is the error in time.

Now, this wave has energy E . It has some number of photons. So the energy, E , is the number of photons times $\hbar \omega$. N is equal to number of photons. And again, we could say ΔE is $\Delta N \hbar \omega$, and then substitute these two relations into this to see what we get.

Well, ΔE is $\Delta N \hbar \omega$. Δt is $\Delta \phi / \omega$. This should be $\hbar / 2$. The ω s cancel, the \hbar cancels, and you get $\Delta N \Delta \phi$ is about 1, or 1 over 2 or 1 over square root of 2.

And that's, in fact, the relation that people do take somewhat seriously, if you have a wave in quantum optics and say, well, the uncertainty in the number of photons and the uncertainty in the coherence of these photons, the phases, if they're out of phase, they're not coherent, there's a relation of this kind.

And this derivation is certainly pretty bad. It's just not precise because even we started with this that is not precise unless you really explain what you mean by Δt . So let's see if we can make some sense of this picture here. So here we go. I want to do a small calculation first. So let's see what we have.

In this coherent state, what is the expectation value of the number operator? Expectation value of the number operator in α . Well, the number operator in α , you would do this $a^\dagger a$ in α .

These are easy to do because a on α is α times α , and if you dagger that equation, a^\dagger on α is α^* . So you get-- I'll go slowly-- $\alpha \alpha^*$ here, and then you get $\alpha \alpha$. And α has unit norm. These are numbers, so this is equal to length of α squared. So if this is a harmonic oscillator, the expectation value of the number operator, in fact, is the length squared of this vector.

Now, how about N^2 ? N^2 is a little more work because you have $\alpha a^\dagger a^\dagger \alpha$. This one gives me the factor you know, this one gives me the factor you know. And therefore, we already have α^2 times $\alpha a^\dagger \alpha$.

And here, the a 's and the a^\dagger 's are kind of in the wrong order because I know what a is on a ket, but a is now on the bra. And I know what a^\dagger is on the bra, but now a^\dagger is on the ket. But the answer is simple. You replace this by the commutator plus the reverse order.

So this is equal to the commutator, which is 1, plus the thing in the reverse order. And this is 1 plus α^2 . So you have α^2 times 1 plus α^2 , and that's the expectation value of N^2 .

All that, because we're actually interested in what is ΔN , the uncertainty in N in the coherent state. And that would be this, square root of this, which is α to the fourth plus α^2 minus the square of the expectation value, which is α^2 to the fourth. And this is length of α .

So the uncertainty in N is just length of α . It happens to be the square root of the expectation value of N . So in fact, if you think of this picture, you're tempted to say, oh, this represents the number of excited states that you have. This length represents the expectation value of N . No. The expectation value of N is this length squared. This length represents ΔN in the picture.

So what else can we say? Well, this picture is useful because now, I can be a little more precise here. This thing is rotating. That is time evolution of your coherent

state. Now, this thing this rotating, but I can ask now how wide this is.

So what is the uncertainty in x in a coherent state? Well, the uncertainty in x in a coherent state is the same as the uncertainty of the ground state because you just moved it. Uncertainty doesn't change. So the uncertainty, Δx , is in fact this quantity that we call d_0 over square root of 2. That's the uncertainty Δx , and the uncertainty Δp is \hbar over d_0 square root of 2.

These are not hard to remember. d_0 is the length scale of the harmonic oscillator, so that's typically what the uncertainty should be. The square root of 2, yes, it's hard to remember. But Δp is this one. And then the other thing that you know is that the product should be \hbar over 2, so that is correct.

Now, look at this. How big is this thing? If the uncertainty in x is d_0 over square root of 2, this width is about how much, roughly? Nobody? This is the uncertainty in x , d_0 over square root of 2 in these units, if you move the expectation value of x plus the uncertainty of x over 2 and the other uncertainty of x roughly.

This thing is d_0 over square root of 2, so it represents basically $1/2$, because you change the expectation value of x by this amount, and then this thing moves $1/2$. The size of this is $1/2$, roughly. Could be $1/4$ or could be 2, but it's roughly $1/2$.

And the vertical one corresponds to the uncertainty in momentum. So intuitively, this is \hbar over square root v_0 , so if you plug it in there, this amount, p plus Δp , you'll get $1/2$ as well. So in this plot-- yes?

AUDIENCE: Wouldn't the width be 1 because the uncertainty is the width in one direction [INAUDIBLE]?

BARTON
ZWIEBACH: Well, the uncertainty is neither the width in one direction or not. It's a Gaussian, so I don't know where it stops. This picture is not very precise when I talk about this, so let me leave it with $1/2$ or something like that. I don't think we can do better.

Now, there's also $1/2$ here. So finally, we get to something that is kind of interesting. If really the state in some sense, in terms of x and p , is spread here, and this is

moving around, the face is a little ambiguous. Because you would say, well, the face is this one, but well, you could go the whole uncertainty that you go here.

The uncertainty in where the coherent state is, we could call the face here $\Delta\phi$ in this picture. We don't know where this state is because it's a little blob. We know the expectation values where they are, but the state itself is a little imprecise. So there's an angle here in this diagram that represents the face because this is going with frequency ωt . So this is the face as this goes around, so this angle, $\Delta\phi$, is how much.

Well, if this is $1/2$ and this is $1/2$, I'm going to assume that this is $1/2$ as well, or 1, or something like that. So it's 1 over this length. That's the uncertainty.

But ΔN , we calculated. This is roughly. And ΔN we calculated, and it's exactly α . So $\Delta\phi \Delta N$ is about 1 correctly. And here, there is at least a picture of what the face uncertainty is and why it originates. Yes?

AUDIENCE: Can you tell me again how the Gaussian relates to the uncertainty?

BARTON One second. Let me see. I've got one question first.

ZWIEBACH:

AUDIENCE: Yes. Can you explain one more time where the $1/2$'s come from? [INAUDIBLE] the graph. I'm not sure why.

BARTON Yes.

ZWIEBACH:

AUDIENCE: Are you saying that the width is $1/2$, or is that how high it is?

BARTON The width of this little ball.

ZWIEBACH:

AUDIENCE: So how does that follow from the graph?

BARTON It's a little hand wavy, but I'll say it like this. I expect the position, if measured, to be

ZWIEBACH: between expectation value of x plus minus Δx . So if I'm going to measure the position of something in a state, the most likely measurement that I will get is the expectation value of x , statistically, after I repeat this many times. But if I just measure, I'm probably going to get some number between this and this.

So if you think of this diagram not as the expectation value of x in here, but whatever you got for x as you measured, if you do 1,000 measurements, you're going to get points all over here in some region because you measure x in one case, then you measure the momentum, you get a plot of data, and you measure them all. And then suppose you're doing it with x first. You measure x and you say, well, I get all kinds of values. I don't know what the momentum is, but I get all kinds of values. They're going to run all over here between these two positions.

So when I add to the expectation value of x this thing, when I want to see in this graph, what it is, I must divide by square root of $2d$. So I divide by $1/\sqrt{2}$ to see how it goes and how I plot it in this graph because these are the units in this graph. So if Δx is $d/\sqrt{2}$, I'm going to get some values that go from the expectation value of x up to $1/2$ more and $1/2$ less.

AUDIENCE: So it's not actually $1/2$. It's actually $1/2$ times whatever that set amount.

BARTON Well, if I say this, that you obtain between this and that, then I should say it's 1.

ZWIEBACH: Maybe I had in mind that you sort of get most things between $1/2$ of Δx . It's not terribly precise, but it's roughly this is the picture. You measure the position, you're going to get that.

Similarly, you decide to measure momentum. You don't measure position, measure momentum, and you're going to get roughly the expectation value, but you're going to get a little plus minus uncertainty. So you're going to all points here in your measurement. So this dashed thing is your histogram after doing lots of experiments. You have lots of dots in here. And roughly, this is how it comes about.

It's not terribly precise because I cannot put a point either here, because if I say the measurement was this, I'm suggesting that I also measured the momentum on that

state, or I could only measure the position. But it's a rough idea, rough picture, of how big the spread is here.

There is a mathematical theory to do this more precisely, although physically not much clearer, which are called Wigner distributions. I don't think it helps too much to understand it, but the rough picture is relatively clear. So if you divide by $1/\sqrt{2}$, this quantity that was equal to $d/\sqrt{2}$, you get, in this scale plus $1/2$ and minus $1/2$, so $1, 1, 1$, and this value there. There was a question there. Yes?

AUDIENCE: Can you explain again how the Gaussian relates to the uncertainty [INAUDIBLE]?

BARTON
ZWIEBACH: So I don't know how the Gaussian relates to uncertainty in x . So basically, we computed the uncertainty in x for the ground state, and I claimed that for a coherent state, the uncertainty in x cannot change because you just took the state and you moved it away.

And the uncertainty of x doesn't talk about what the expectation value of x is. That changes when you move a state. But just how much it's spread and how much the state is spread is not changed by a translation. So this is the old result for the ground state uncertainty, ground state uncertainty, and neither is changed.

Let's go now into our squeezed states. So what are going to be squeezed states? They're going to be pretty useful states. They have lots of applications nowadays. They've been constructed over the last 10 years more and more often, and people are now able to construct what is called squeezed states experimentally.

And the way we're going to motivate it is by imagining that you have a harmonic oscillator, a particle m , and some spring, k , or an ω . And there's a Hamiltonian. H is equal to $p^2/2m + 1/2 m \omega^2 x^2$. But this Hamiltonian is going to be the Hamiltonian of a particle that has mass m_1 , and the oscillator has frequency ω_1 , and that's what we're going to call the first Hamiltonian.

After a little while, you observe this Hamiltonian. I will erase this thing. We don't need them anymore. We observe this thing and this has an uncertainty, Δx ,

proportional to-- we know this v over square root of 2, so \hbar over $2 m_1 \omega$.
And an uncertainty in p , which is Δp equals square root of $\hbar m_1 \omega$ over 2.

And again, they saturate the bound if you have your ground state. So ground state is here. These two uncertainties, the bound is saturated, all is good. Nevertheless, suddenly, at time equals 0, this state, this particle, is in the ground state.

The Hamiltonian changes. There's an abrupt change in the physics. Maybe the temperature was changed and the spring constant changed, or the particle, a drop was added to it and its mass changed, but the Hamiltonian has changed all of a sudden. So this Hamiltonian, H_1 , is valid for $H < 0$ and a particle in the ground state.

So the particle's in the ground state, the Hamiltonian is fine there, but suddenly, the Hamiltonian changes. The particle identity has not changed. The particle is there, but it is the Hamiltonian that changes. So there's an H_2 , p^2 over $2 m_2$ plus $1/2 m_2 \omega^2 x^2$. The picture is physically clean. The particle is sitting there in the ground state, and suddenly, the parameters of the system change.

So this particle was having a good time, it was at the ground state, relaxed. Then suddenly, the wave function didn't change at time equals 0. It was spread over some distance. No measurement was done, nothing. And suddenly, this particle finds itself with some wave function but in another Hamiltonian. From now on, its time evolution is going to be governed by the second Hamiltonian.

Now, since the second Hamiltonian is different from the first Hamiltonian, this particle is not going to be any more in the ground state. Even though it was in the ground state of the first Hamiltonian, it's not anymore in the ground state of the second Hamiltonian as soon as the thing gets turned on. So for t greater than 0, this Hamiltonian is there.

So actually, the wave function does not change, so let me write Δx , and I'll write

it the following way, $\hbar / 2 m_2 \omega_1$. But you say, no, Δx didn't change. Correct. So I'll put the factor $m_2 \omega_2$ over $m_1 \omega_1$, and now it's the same Δx . Similarly, for Δp , I will write that this is square root of $\hbar m_2 \omega_2$ over 2, and put the factor $m_1 \omega_1$ over $m_2 \omega_2$ in front in such a way that it is the same Δx and the same Δp .

Now, Δx times Δp multiply to be $\hbar / 2$, and they still multiply to that number because I didn't change them. But this is equal-- I'll call this number e to the minus gamma. I'll go to another blackboard. Δx is e to the minus gamma times square root of $\hbar / 2 m_2 \omega_2$. And Δp is e to the gamma, because it's the inverse factor on the one that we call gamma, square root of $\hbar m_2 \omega_2$ over 2, where e to the gamma is the square root of $m_1 \omega_1$ over $m_2 \omega_2$.

Look, we've done very simple things. We haven't done really much. But already, we start to see what's happening. From the viewpoint of the second Hamiltonian, these uncertainties are not right. They are not the uncertainties of the ground state, because from the viewpoint of the second Hamiltonian, the ground state uncertainty is this and the ground state uncertainty is this.

And indeed, this particle was in the ground state, it had some Gaussian, but that's not the right Gaussian for the second Hamiltonian. It's the right Gaussian for the first Hamiltonian. So it's not in the ground state of the second Hamiltonian, but it's in a particular state in which, if gamma is positive, the uncertainty in x is squeezed from the lowest uncertainty that you get in an energy eigenstate. And the uncertainty and the momentum will be stretched in that direction.

So you see, in the ground state of the harmonic oscillator, you get that uncertainty, and that's a canonical uncertainty. But this uncertainty is squeezed because it's different from what it should be, and this is squeezed. So from the viewpoint of the second Hamiltonian, the ground state of the first Hamiltonian is a squeezed state. It's a state whose uncertainties have been squeezed.

And those states exist, and the purpose of what we're going to do now is try to

determine them, find them, see what they are, how they behave. Any questions?
Yes, Nicholas?

AUDIENCE: I'm a little confused why we can say that Δx [INAUDIBLE] these new ones are just related to the old ones by this factor.

BARTON OK. You see, what I assumed is that before time equals 0, you had a Gaussian.

ZWIEBACH: That was the original Gaussian. That was the original wave function, and you had some Δx and some Δp that were given by this one [INAUDIBLE].

Now, I didn't do anything except rewrite the same quantities here, because what I said next was that even though at time equals 0, the Hamiltonian changes, at time equals 0, the wave function doesn't change. The wave function remains the same. After that time, it's going to start changing because the new Hamiltonian kicks in. But this Δx 's are the same as that I wrote, and here are the same. But here, you see clearly that this Δx with respect to the second Hamiltonian is not the one that it would be if it would be a ground state, nor the Δp . Yes?

AUDIENCE: Just at the instant you change the Hamiltonian, because they might have all [INAUDIBLE], the uncertainties would change.

BARTON Sorry?

ZWIEBACH:

AUDIENCE: Is this just at the instant where we change the Hamiltonian, because after some time, the wave function might change [INAUDIBLE].

BARTON That's right. This is just after I change the Hamiltonian. The time evolution of this

ZWIEBACH: state is something that we have to figure out later. But after I've changed the Hamiltonian, the state looks squeezed.

So how can we calculate and understand these things? So the way to think of this is the following. You see, you have this system of two Hamiltonians. There's an x and a p operator, and the second Hamiltonian has an x and a p operator. These are the properties of the particles.

Therefore, what that I'm going to think of is that the x and the p operators are the operators that describe the particle. They are unchanged because we're talking about this same object, same particle. So if I have the x operator, which is equal to this formula, $\frac{\hbar}{2m_1\omega_1} (a_1^\dagger + a_1)$ like this. From the first Hamiltonian, the x 's are related to a_1 's and a_1^\dagger 's, but this is the same x describing the same position as you would do in the second Hamiltonian. So $\frac{\hbar}{2m_2\omega_2} (a_2^\dagger + a_2)$.

It's a very strong physical assumption I'm making here. It's an assumption that's so strong that in many ways, you could almost say, well, I'll buy it, but we'll see if it gives something reasonable. I'm saying the x operator is really the same thing, and you could view it as constructed from ingredients of the first Hamiltonian or the second Hamiltonian. So is the p operator. p , which is-- well, I have a formula here-- $-i\sqrt{\frac{\hbar m_1\omega_1}{2}} (a_1 - a_1^\dagger)$ -- should be the same as $-i\sqrt{\frac{\hbar m_2\omega_2}{2}} (a_2 - a_2^\dagger)$.

So x and p are not changing. We're not talking about two particles that have an x_1 and a p_1 , and the second particle, an x_2 and a p_2 . It's just one particle has an x and a p is what you observe when you measure position and you observe when you measure momentum. Nevertheless, x and p are related in this way to the creation and annihilation operators. So we're going to find from this some very strange relation between the creation operators, the annihilation operators of the first system and the second system.

So what do we get, in fact? Well, the constants disappear from the first equation roughly, and you get a_1^\dagger is equal to-- you get the ratio of $m_1\omega_1$ over $m_2\omega_2$, so you get $e^{\gamma} a_2 + a_2^\dagger$. From the bottom one, $a_1 - a_1^\dagger$ is equal to $e^{-\gamma} (a_2 - a_2^\dagger)$. These two equations give you that. It should be clear. You just cancel the constants and remember the definition of e to the gamma.

And now we can solve for a_1 and a_1^\dagger in terms of a_2 and a_2^\dagger . And what do we find? a_1 is equal to $a_2 \cosh \gamma + a_2^\dagger \sinh \gamma$, and

the dagger is what you would imagine. So a_1 dagger is equal to a_2 dagger $\cosh \gamma$ plus a_2 $\sinh \gamma$.

The second equation that you can calculate is the dagger of the first. It should be that. And now you've found the scrambling of the creation, annihilation operators. The old annihilation operator is a mixture of the new annihilation operator and a creation operator. They're mixed. It's a very strange thing that has happened, a mixture between creation and annihilation operators.

This is so famous in physics, it has a name. It's called the Bogoliubov transformation. It appears in the analysis of black hole radiation. There's a Bogoliubov transformation between the fields far away of the black hole and the fields near the black hole. It appears everywhere. And here it has appeared, so we're going to try to understand what it does for us.

Similarly, you can find what a_2 is in terms of a_1 's by the symmetry of these equations. This corresponds to actually letting γ go to minus γ , because if you pass these γ 's to the other side, the equations are of the same form. By letting 1 become 2, 2 becomes 1 and γ goes to minus γ . So we don't need it right now, but in case you want to find the other ones, the 2's in terms of the 1's, you would just change the sign of γ and it would work out.

So this relation is the key to allow you to calculate things. So what do we want to calculate? Well, here is what I would like to calculate. The ground state of the first oscillator is this thing we had. It's the thing that has the wave function. But I want to express it as a superposition of states of the second oscillator because the second oscillator is what gives you the new Hamiltonian and what's going to tell you how the state is going to evolve later.

So presumably, this state is some number times the ground state of the second oscillator, plus maybe some creation operator on the second vacuum as well with a constant. Now, this wave function of the ground state is even, and I would expect that it's a superposition of even eigenstates of the second oscillator as well.

And even eigenstates are things that have even occupation numbers. Those are the even Hermite polynomials. So presumably, it goes like this and things with four oscillators and things like that. So what that after is this sort of expression of the original state in terms of energy eigenstates in terms of anything of the second oscillator.

So how can we do that? Well, one thing we know about this state is that a_1 [?] is equal to 0. It's killed by a_1 , but that a_1 is an interesting thing. It's $a_2 \cosh \gamma + a_2 \sinh \gamma$, and that thing must kill that state.

So I could at least, if I had infinite time, put a few terms and try to calculate more or less what kind of state is killed by this strange combination of creation and annihilation operators. You see, we know a ground state is killed by the normal annihilation operator. That's what this is. But this operator, now we know it's given by this formula over there, and then it must kill all that.

So we're faced with a problem that is in principle fairly difficult, and you could not hope for an exact solution unless there's something very nice going on. Happily, squeezed states are still very nice and tractable states, so let's see what we can do.

Well, what I'm going to do is to put an ansatz for this state based on this expansion that I had there. I would say, look, there's going to be a normalization constant, but at the end of the day, we have things acting on the vacuum, so there's going to be something very messy acting on the vacuum of 2. And what is that going to be?

Well, we've learned about coherent states that are exponentials of oscillators, exponentials of a 's and a daggers added. So here, we're going to attempt something a little more general. I'll put an exponential minus $1/2$, and what should I put?

Well, let's try to be simple minded still. It seems to go in even power, so if we're very lucky, maybe we can put just an $a_2 \sinh \gamma + a_2 \cosh \gamma$ here, an exponential something quadratic in oscillators. And I don't know what the coefficient is in front, and it may depend on γ because I have to solve an equation with γ . So

I'll put minus $1/2$ of γ times that. And we'll see if we can solve this.

So what does it mean to solve it? Well, it means that it must be annihilated by this operator. So our computations with the creation and annihilation operators are becoming more and more complicated. They look more and more complicated. They're really not harder. Let's see what happens.

So I need now that $a^2 \cosh \gamma + a^2 \text{dagger} \sinh \gamma$ kill this state. So the N is going to go outside. It's a number. So acting on $e^{-1/2 \gamma}$ of $a^2 \text{dagger} a^2 \text{dagger}$ on the vacuum ψ_0 , that must be 0.

How does one solve this? Well, let's see what we have. Let's see this term. $a^2 \text{dagger}$, good. $a^2 \text{dagger}$ commutes with $a^2 \text{dagger}$, so I can bring the $a^2 \text{dagger}$ all the way to the right and it doesn't kill the vacuum, so I don't gain anything. Can be to the right or to the left because it commutes with this whole thing, so I haven't gained anything if I move it, so false start. I don't want to move that one. This one, I want to leave it here, and this one somehow must produce something that cancels this one.

Now, a^2 , on the other hand, is the kind of thing that always should be dealt with because this is an annihilator and that does kill that. So as it moves along, it encounters obstacles, but obstacles are opportunities because an obstacle means we're going to get something that maybe cancels that. So if it also went through and killed the vacuum, we're finished. This doesn't kill the vacuum. Happily, it gets stuck here. Now the thing that we have to hope is that we can disentangle that commutator.

Now, here is a universal thing. How do I want to write this? I'm going to write it like this. I have an a^2 , a number, I don't care about the number, and a complicated thing, and a vacuum. Whenever you have an a , any operator, and a vacuum, this is equal to a commutator with the operator on the vacuum. That should be second nature because this is even given to that minus oa , but oa , the a is near to the vacuum and it kills it. So whenever you have an a or vacuum, you can put the commutator, so I'll do that here.

So I put a^2 , the cosh gamma, I take it out. I put this whole thing minus $1/2 f a$ dagger a dagger 2 . This whole thing and the vacuum. That's the first term. And the second term, I have to just copy it. Sinh gamma a^2 dagger e to the minus $1/2 f a$ squared dagger on the vacuum. All that should be 0.

So what do we get? Is that commutator doable or undoable? It's happily a simple commutator, even if it doesn't look like it, because whenever you see a commutator like that, you think A to the B , and then you know if you're in luck, this is just $AB e$ to the B , and this is true if AB commutes with B . So that's what you must think whenever you see these things. Do mind this lucky situation.

Yes, you are, because with this commutator, one a will kill an a dagger, so you will be left with an dagger. But a dagger commutes with b , which is a dagger a dagger. So AB , A with B is just add an a dagger up to a function or a number, and then a dagger commutes with B so you are in good shape. This is true.

So what do we get here? We get cosh gamma, and then we just get the commutator of a^2 with minus $1/2 f a^2$ dagger a^2 dagger times the whole exponential-- I won't write it-- times the vacuum plus sinh times a^2 dagger times the whole exponential times the vacuum.

We have to do this commutator, but the f doesn't matter. It's a constant. It's a function. No operator in there. a^2 with a^2 daggers are 1. There are two of them, so you get a 2, and the $1/2$ cancels this, so you get minus cosh gamma $f a^2$ dagger times the exponential plus, from the other term, sinh gamma a^2 dagger times the exponential on the vacuum equals 0.

And, as promised, we were good. We get an a^2 dagger, a^2 dagger. These two terms cancel if f is equal to tan hyperbolic of gamma, which is sine over cosine so that these two things cancel. I can write this, of course, as minus cosh gamma f plus sinh gamma a^2 dagger, the exponential, and the vacuum, equals 0.

So it's just a simple relation, but there we go. Tanh gamma is the thing. Tanh gamma gives you the answer, and let me write this state so that you enjoy it. Let's

see. The state is just a fairly interesting thing, this 01 expressed in the new Hilbert space of the second oscillator is some n of γ times the exponential of minus $1/2$ tangent hyperbolic of γa^2 dagger, a^2 dagger on the vacuum sub 2.

And you need the normalization, n of γ , and it will be done. Now, the normalization, you may say well, look, normalizations are good things. Sometimes, you work without normalizations and you're OK, but it turns out that these normalizations are pretty useful, and unless you get them, some calculations are kind of undoable.

So it's a little bit of a challenge to get that normalization. You can try in several ways. The most naive way is to say, well, this must have unit norms, so n squared, and then I take the bra of this and the ket of that, so it would be a vacuum, an exponential of minus $1/2$ tangent a , and an exponential of minus $1/2$ tangent a dagger, a dagger. Must be 1. n squared times that.

The problem is that I've never been able to compute this. At least it takes a long time and you get it by indirect methods, but getting a number out of this is painful.

So there's one way of getting the normalization here that is not so bad. It's a little surprising what you do. You do the following. You declare, I'm going to compute the overlap of 2, the vacuum of 2, with the vacuum of 1. And now, what is this, n γ vacuum of 2 here, e to the minus $1/2$ tanh γa^2 dagger a^2 dagger vacuum of 2. How difficult is it to compute this inner product?

AUDIENCE: [INAUDIBLE].

BARTON Sorry?

ZWIEBACH:

AUDIENCE: Not difficult.

BARTON Not difficult. What is it?

ZWIEBACH:

AUDIENCE: [INAUDIBLE]?

BARTON Yeah, that thing.

ZWIEBACH:

AUDIENCE: e to the negative $1/2 \tanh \gamma$. It's 1.

BARTON Sorry?

ZWIEBACH:

AUDIENCE: I mean, you multiply the a^2 dagger right across to the left hand side of the ket.

BARTON Yeah, you're saying it, indeed. Look, this thing is as simple as can be. This is just 1.

ZWIEBACH: Why is that so? You expand the exponential, and you have 1 plus things, but all the things have a daggers. Now, a daggers don't kill this 1, but they killed the other 1 on the left, and there's nothing obstructing them from reaching the left, so this is 1.

It's completely different from this one because if you expand this one, the a daggers kill the thing but there's lots of a's to the left. And the a's want to get here, but there's lots of a daggers to the right, so this is hard, but this is easy. So n of γ is $0 \ 2 \ 0 \ 1$. But what is that? If you introduce a complete set of position states, z_x , This is $0 \ 2 \ x \ x \ 0 \ 1$.

This one is the ground state wave function of the first Hamiltonian, and this is the start of the ground state wave function of the second Hamiltonian. And those you know because you know m, ω . You know the ground state wave functions, so this integral can be done. So this whole normalization is given by this integral, and this integral gives you 1 over square root of $\cosh \gamma$. That interval takes a few lines to make, but the end result is there.

So you got your coherent states. You got now the squeezed state completely normalized, so let's write it out. $0 \ 1$ is equal to 1 over square root of $\cosh \gamma$ exponential of minus $1/2 \tanh \gamma a^2 \text{ dagger } a^2 \text{ dagger}$ on the vacuum sub 2.

Wow. That's it. That's a squeeze state that has been squeezed in such a way that the squeezing parameter appears here in the exponential.

Now, this is the way we got to it, but now I wanted to just think of it independently, just from the beginning. If you had a Hamiltonian, this is an interesting state all in itself because it is a squeezed state. It's a Gaussian, but of the wrong shape for this system. This is a Gaussian of the right shape for system two. But once you put all these oscillators, it's not anymore a Gaussian of the right type. It's a squeezed Gaussian.

So if we forget about this system one, let me write this thing from the beginning and say like this. We have a Hamiltonian, we have a ground state, we have m and ω , and we have a and a^\dagger . Let's just define what we call the squeezed vacuum, vacuum sub γ , to be precisely this thing.

$\frac{1}{\sqrt{2 \cosh \gamma}}$ exponential of minus $\frac{1}{2} \tanh \gamma a^\dagger a^\dagger$, not 2 anymore because we have just a single system. A single system, the ground state, and now we've defined this state, which is what we had there before, but we don't think of it anymore as, oh, it came from some other Hamiltonian, but rather, this is a state on its own. It's a squeezed vacuum state.

And from the computations that we did here, the Δx for this state would be $e^{-\gamma} \frac{\hbar}{2m\omega}$. and $m\omega$ over here. So these are these, and you don't need to know what γ is. That's a number that somebody chose for you. Any number that you want is γ , and therefore, you use it to squeeze the state. And that's what you've achieved.

So you have a Hamiltonian of a harmonic oscillator. You can construct the vacuum. You know how to construct coherent states by acting on the vacuum. Now you know how to construct squeezed states, states in which the expectation values do those things.

We had a very nice formula where we began the lecture today in which the coherent state was just a unitary operator acting on the vacuum. Now, we made sure to normalize this, so we did check in this calculation that $\langle 0 | 0 \rangle$ is equal to 1.

So this thing must come from the action of some unitary operator acting on the vacuum. Which is that unitary operator that acts on the vacuum and gives you that? Not so easy to find. All the computations here are a little challenging, as you've seen. But here's the answer. Cosh gamma e to the exponential of minus 1/2 tanh gamma a dagger a dagger should be something like an e to the what? Should be something like e to the a dagger a dagger minus aa acting on the vacuum.

Why? Because certainly, the aa's are going to disappear, and you're going to get products of this one squared. And this is anti-Hermitian, so that operator is unitary, but I now must put the gamma somewhere there. So what should I put here in order to get that to work?

Well, it's maybe something you can try by assuming gamma is very small and expanding both sides, or finding a differential equation, or doing things, but the answer is incredibly simple. It's e to the minus just gamma over 2. That's it.

Gamma appears here, and by the time you reorder this quadratic form-- you see, what you have to do here is expand, and then you have powers of these, and then you have to bring all the annihilators to the right and kill them. And then you have a power series in squares of this thing. That will reassemble into this exponential.

It's almost a miracle that something like that could happen, but it does happen. And it's a very interesting calculation, actually, to do that. We don't do it in the course. I may post some pages that I did once this computation.

And that is a nice operator. We call it the squeezing operator. So s of gamma is a unitary operator, s of gamma. The squeezed state of 0 gamma is equal to s of gamma on the vacuum where s of gamma is equal to e to the minus gamma over 2 a dagger a dagger minus aa, that operator. It's a unitary operator and it does the squeezing.

Actually, once you have squeezed states, you can do more things, and you can squeeze and then translate. Those are the most general states that people use in quantum optics. So you take a vacuum, you squeeze it with s of gamma, and then

you translate it with v of α . And this is the state, α gamma, squeeze factor, translation factor.

One picture of that is in our α plane. You take the vacuum that is some spherical ball here in the x expectation value, p expectation value. You squeeze it. You might decide, I don't want to have too much Δx , so you squeeze it and you produce something like this.

That's a squeezed vacuum by the time you apply this. And then you do the α , and you translate it out, and this state is now going to start rotating and doing all kinds of motion. It's pretty practical stuff.

Actually, some of you are taking junior lab, and the person that works a lot there in junior lab is Nergis Mavalvala, and she does gravity wave detection, and squeezed states has been exactly what she's been working. In order to minimize displacements in the gravity wave detectors, they have a squeeze vacuum state injected into the detector to make the harmonic oscillator that represents the mirror stabilize its uncertainty in position to the maximum possible. There's a whole fabulous technique that people use with the squeezed states.

Now, the squeezed states allow you to construct some states that seemed to us that they were pretty strange and that we never had good formulas for them. So that's how I want to conclude the lecture. I will leave photon states for next time, but I want to discuss one more application of the squeezed states, and this comes from limits.

So here is your squeezed state, $e^{-\frac{1}{2}\gamma}$. So let's squeeze the state to the end. Take γ to go to infinity. What happens to the squeezed state? So you're narrowing out the ground state in position space to the maximum possible. What happens to the state?

Well, it goes a little singular, but not terribly singular. γ is going to infinity, so \cosh is going to infinity as well. So the state is going kind of to 0, but 0 sub infinity. It's proportional, but the exponential is good. Exponential of $-\frac{1}{2}\gamma$ as γ goes to infinity is just 1. And this is a dagger a dagger on the

vacuum.

This state is in almost terrible danger to be infinite. If you try to find its wave function, you're not going to be able to normalize it. You've reached the end of the road kind of thing because of this. Gamma goes to infinity. This is going to be infinite here because this state, if you compute its overlap with itself, is blowing up.

And here, you see the niceness of this. It also suggests that gamma can go from plus infinity to minus infinity, and that's a natural thing. Nevertheless, here, it goes from plus 1 to minus 1. If you had a number 3 here, this is a state that blows much worse than the worst delta function or derivative or square that you've ever had. It's just unbelievably divergent because it just can't exist, this state. You're going beyond infinity here to go behind this thing. So it's just pretty much impossible.

So the limit is states are reasonable as long as this quadratic form goes from minus 1 to 1. And when you go to 1, you get this, and what should this be? This should be the wave function I associated to a delta function. This would be the position state, x equals 0. Roughly, it's a delta function. And indeed, if you act with x on it, x , remember, is a plus a dagger. Act on this exponential.

Now, do you remember how to do that? This a dagger doesn't do anything but the a goes here, and it's a trivial commutator. You get minus a dagger. So it actually kills this and gives you 0. So in fact, the exposition operator acting on here gives you 0. It looks like it is really the state x equals 0.

If you go the other way around, and you take gamma to be minus infinity, the only thing that changes here is the sign. So this is like the delta of x , or the x equals 0 state. And if you take 0 minus infinity, goes like x minus $1/2$ plus $1/2$ a dagger a dagger on the vacuum.

And this state is a delta function in momentum. It's the momentum state p equals 0. Why? Because gamma is going to minus infinity. The uncertainty of momentum is going to 0. And therefore, indeed, if you act with the momentum operator on this state, it's like acting with a minus a dagger, and you've changed the sign of this, but

you've changed the sign here, so it also kills this state.

So it looks like we can really construct position and momentum eigenstates now with squeezed states, and that's what they are supposed to be. A squeezed state is something that has been squeezed enough that you can get a delta function. So how do you finish that construction?

Here is the claim. Square root of $2 m \omega$ over \hbar times x a dagger minus $1/2$ a dagger a dagger acting on the vacuum. This is the claim, that this is the x position state. So basically, you have to squeeze first and then translate this thing to the exposition.

So how do you check this? Well, you should check that the x operator, which is something times a plus a dagger acting on this thing gives you little x . So you should have that x operator on this thing gives you x . And that is going to work out because the a dagger is going to sit here and it's going to get canceled with the a with this, but the a with this part is just going to bring down an x with the right factor.

So this state, which is a squeezed state and a little bit of a coherent state as well, is producing the position eigenstate. In the harmonic oscillator, you can really construct the position eigenstate and you can calculate the normalization. The normalization comes out to be a rather simple thing. So at the end of the day, the position eigenstate is $(m \omega / \pi \hbar)^{1/4} e^{-m \omega x^2 / 2 \hbar}$. And this whole exponential, square root of $2 m \omega$ over \hbar , x a dagger minus $1/2$ a dagger a dagger acting on the vacuum.

So your basis of creation and annihilation operators on the harmonic oscillator is flexible enough to allow for a concrete description of your position eigenstates, and a tractable one as well. And that's the extreme limit of squeezing, together with some little bit of coherent displacement. Next time, we'll do our photon states and we'll illustrate the ideas of both coherent and squeezed states at the same time.