PROFESSOR: It's good to be back. I really want to thank both Aaron and Will who took my teaching duties over last week. You've been receiving updates of the lecture notes, and, in particular, as I don't want to go back over some things, I would like you to read some of the material you have there. In particular, the part on projectors has been developed further. We will meet projectors a lot in this space, because in quantum mechanics, whenever you do a measurement, the effect of a measurement is to act on a stage with a projector.

So projectors are absolutely important. And orthogonal projectors are the ones that we're going to use-- are the ones that are relevant in quantum mechanics. There's a property, for example, of projectors that is quite neat that is used in maximization and fitting problems. And you will see that in the PSET. In the PSET, the last problem has to do with using a projector to find best approximations to some functions using polynomials. So there's lots of things to say about projectors, and we'll find them along when we go and do later stuff in the course. So please read that part on projectors.

The other thing is that much of what we're going to do uses the notation that we have for describing inner products-- for example, u, v. And then, as we've mentioned, and this is in the notes-- this, in the bracket notation, becomes something like this. And the bracket notation of quantum mechanics is fairly nice for many things, and it's used sometimes for some applications. Everybody uses the bracket notation for some applications. I hope to get to one of those today.

So much of the theory we've developed is done with this as the inner product. Nevertheless, the translation to the language of bras and kets is very quick. So the way the notes are going to be structured-- and we're still working on the notes, and they're going to chang a bit-- is that everything regarding the math is being developed more in this notation, but then we turn into bras and kets and just go quickly over all you've seen, just how it looks with bras and kets so that you're familiar. Then, in the later part of the course, we'll use sometimes bras and kets, and sometimes this. And sometimes some physicisists use this notation with parentheses. So for example, Weinberg's recent book on quantum mechanics uses this notation. It doesn't use bras and kets I think at all. So you have to be ready to work with any notation. The bra and ket notation has some nice properties that make it very fast to do things with it. It is very efficient. Nevertheless, in some ways this notation is a little clearer. So many of the things we'll develop is with this notation.

So today I'm going to develop the idea of the Hermitian conjugator for an operator, or the adjoint of an operator. And this idea is generally a little subtle, a little hard to understand. But we'll just go at it slowly and try to make it very clear. So adjoints or Hermitian operators, or Hermitian conjugates-- adjoints or Hermitian conjugates. So the idea of Adjoints, or Hermition conjugates, really begins with some necessary background on what they're called-- linear functionals. It sounds complicated, but it's not. What is a linear functional? A linear functional on V-- on a vector space V-- is a linear map from V to the numbers F. We've always been calling F the numbers. So it's just that, something that, once you have a vector, you get a number and it's linear. So a linear function of Phi, if it's a linear functional, Phi on v belongs to F. Phi acts on a vector, v, that belongs to the vector space and gives you a number. So "linear" means Phi of v1 plus v2 is Phi of v1 plus Phi of v2. And Phi of av, for a is number, is a Phi of v.

So seems simple, and indeed it is. And we can construct examples of linear functionals, some trivial ones, for example. Let Phi be a map that takes the vector space, reals in three dimensions, to the real numbers. So how does it act? Phi acts on a vector, which is x1, x2, and x3-- three components. And it must give a numbers, so it could be 3x1 minus x2 plus 7x3, as simple as that. It's linear. x1, x2, and x3 are the coordinates of a single vector.

And whenever you have this vector, that is, this triplet-- now, I could have written it like this-- Phi of x1, x2, and x3, as a vector. It looks like that. But it's easier to use horizontal notation, so we'll write it like that. And, if you have an inner product on this space-- on this three dimensional vector space-- there's something you can say. Actually this Phi is equal-- and this we call the vector V-- is actually equal to u, inner product with v, where u is the vector that has components 3, minus 1, and 7, because if you take the inner product of this vector with this vector, in three dimensions real vector spaces-- inner product is a dot product. And then we make the dot product of u with the vector V.

Maybe I should have called it v1, v2, v3. I'll change that-- v1, v2, v3 here are components of the vector-- v1, v2, and v3, not to be confused with three vectors. This whole thing is a vector V. So this linear functional, that, given a vector gives me a number. The clever thing is that the inner product is this thing that gives you numbers out of vectors. So you've reconstructed this linear functional as the inner product of some vector with the vector you're acting on, so, where u is given by that.

The most important result about linear functionals is that this is not an accident. This kind be that very generally. So any time you give me a linear functional, I can find a vector that, using the inner product, acts on the vector you're acting on the same way as the linear function of thus. The most general linear functional is just some most general vector acting this way. So let's state that and prove it. So this is a theorem, it's not a definition or anything like that. Let Phi be a linear functional on v. Then there is a unique vector u belonging to the vector space such that Phi acting on v is equal to u, v.

Since this is such a canonical thing, you could even invent a notation. Call this the linear functional created by u, acting on v. Everybody doesn't use this, but you could call it like that. This is a linear functional acting on v, but it's labeled by u, which is the vector that you've use there. This is important enough that we better understand why it works. So I'll prove it. We're going to use an orthonormal basis, say e1 up to en is an orthonormal, O-N, basis.

AUDIENCE: That means we're assuming v is finite dimensional here?

PROFESSOR: Sorry?

AUDIENCE: We're assuming V is finite dimensional, correct?

PROFESSOR: Yeah, it's finite dimensional I'm going to prove it using a finite basis like that. Is true finite dimensional? I presume yes.

AUDIENCE: If it's not [INAUDIBLE].

PROFESSOR: What hypothesis?

AUDIENCE: You say continuous when you're talking [INAUDIBLE].

PROFESSOR: OK, I'll check. But let's just prove this one finite dimensional like this. Let's take that. And now write the vector as a superposition of these vectors. Now we know how to do that. We just have the components of v along each basis vector. For example, the component of v along e1 is precisely e1, v. So then you go on like that until you go en, v, en. I think you've derived this a couple of times already, but this is a statement you can review, and let's take it to be correct.

Now let's consider what is Phi acting on a v like that. Well, it's a linear map, so it takes on a sum of vectors by acting on the vectors, each one. So it should act on from this plus that, plus that, plus that. Now, it acts on this vector. Well, this is a number. The number goes out. It's a linear function. So this is e1, v, Phi of e1, all the way up to en, v Phi of en.

Now this is a number, so let's bring it into the inner product. Now, if you brought it in on the side of V as a number it would go in just like the number. If you bring it into the left side, remember it's conjugate homogeneous, so this enters as a complex number. So this would be e1, Phi of e1 star times V plus en, Phi of en star, v. And then we have our result that this Phi of v has been written now. The left input is different on each of these terms, but the right input is the same. So at this moment linearity on the first input says that you can put here e1, Phi of e1 star plus up to en, Phi of en star, v. And this is the vector you were looking for, the vector U. Kind of simple, at the end of the day you just used the basis and made it clearer. It can always be constructed. Basically, the vector you want is e1 times Phi of u1 star plus en up to Phi of en star. So if you know what the linear map does to the basis vectors, you construct the vector this way. Vector is done.

The only thing to be proven is that it's unique. Uniqueness is rather easy to prove at this stage. Suppose you know that u with v works and gives you the right answer. Well, you ask, is there a u prime that also gives the right answer for all v? Well, pass it to the other side, and you would have u minus u prime, would have zero inner product with v for all v. Pass to the other side, take the difference, and it's that. So u minus u prime is a vector that has zero inner product with any vector. And any such thing as always zero. And perhaps the easiest way to show that, in case you haven't seen that before, if x with v equals 0 for all for all v. What can you say about x? Well, take v is the value for any v. So take v equal x. So you take x, x is equal to 0. And by the axioms of the inner product, if a vector has 0 inner product with itself, it's 0. So at this stage, you go u minus u prime equals 0, and u is equal to u prime. So it's definitely unique, you can't find another one that works.

So we have this thing. This theorem is proven. And now let's use to define this the adjoint, which is a very interesting thing. So the adjoing, or Hermitian conjugate, sometimes called adjoint-- physicists use the name Hermitian conjugate, which is more appropriate. Well, I don't know if it's more appropriate. It's more pictorial if you have a complex vector space. And if you're accustomed with linear algebra about Hermition matrices, and what they are, and that will show up a little later, although with a very curious twist. So given an operator T belonging to the set of linear operators on a vector space, you can define T dagger, also belonging to l of v. So this is the aim-- constructing an operator called the Hermitian conjugate.

Now the way we're going to do it is going to be defining something that is a T star. Well, I said "T star" because mathematicians in fact call it star. And most mathematicians, they complex conjugate if a number is not z star but z bar. So that's why we call it T star and I may make this mistake a few times today. We're going to use dagger. And so I will make a definition that will tell you what T dagger is supposed to be, acting on things. But it might not be obvious, at least at first sight, that it's a linear operator.

So let's see how does this go. Here is the claim. Consider the following thing-- u, T, v-- this inner product of u with T, v. And think of it as a linear functional. Well, it's certainly a linear functional of v. It's a linear functional because if you put a times v the a goes out. And if you put v1 plus v2 you get it's linear. So it's linear, but it's not the usual one's that we've been building, in which the linear functional looks like u with v. I just put an operator there. So by this theorem, there must be some vector that this can be represented as this acting with that vector inside here, because any linear operator is some vector acting on the vector-- on the vector v. Any linear functional, I'm sorry-- not linear operator. Any linear functional-- this is a linear functional. And every linear function can be written as some vector acting on v. So there must be a vector here. Now this vector surely will depend on what u is. So we'll give it a name. It's a vector that depends on U. I'll write it as T dagger u. At this moment, T dagger is just a map from v to v. We said that this thing that we must put here depends on u, and it must be a vector. So it's some thing that takes u and produces another vector called T dagger on u. But we don't know what T dagger is, and we don't even know that it's linear. So at this moment it's just a map, and it's a definition. This defines what T dagger u is, because some vector-- it could be calculated exactly the same way we calculated the other ones.

So let's try to see why it is linear. Claim T dagger belongs to the linear operators in v. So how do we do that? Well, we can say the following. Consider u1 plus u1 acting on Tv. Well, by definition, this would be the T dagger of u1 plus u2, some function on u1 plus u2, because whatever is here gets acted by T dagger times v. On the other hand, this thing is equal to u1, Tv plus u2, Tv, which is equal to T dagger u1, v plus T dagger u2, v. And, by linearity, here you get equal to T dagger u1 plus T dagger on u2. And then comparing this too-- and this is true for arbitrary v-- you find that T dagger, acting on this sum of vectors, is the same as this thing.

And similarly, how about au, Tv? Well, this is equal to T dagger on au, v. Now, T dagger on au, do you think the a goes out as a or as a bar? Sorry? a or a-bar?

What do you think T dagger and au is supposed to be? a, because it's supposed to be a linear operator, so no dagger here. You see-- well, I didn't show it here. Any linear operator, T on av, is supposed to be a T of v. And we're saying T dagger is also a linear operator in the vector space. So this should be with an a. We'll see what we get. Well, the a can go out here, and it becomes a star u1, Tv, which is equal. I'm going through the left side. By definition, a bar T dagger of u, v. And now the constant can go in, and it goes back as a, T dagger u, v. So this must be equal to that, and you get what we're claiming here, which is T dagger on au, is equal to a T dagger of u. So the operator is linear. So we've defined something this way, and it's linear, and it's doing all the right things.

Now, you really feel proud at this stage. This is still not all that intuitive. What does this all do? So we're going to do an example, and we're going to do one more property. Let me do one more property and then stop for a second. So here is one property-- ST dagger is supposed to be T dagger S dagger. So how do you get that? Not hard-- u, STv. Well, STv is really the same as S acting on Tv. Now the first S can be brought to the other side by the definition that you can bring something to the other side. Put in a dagger. So the S is brought there, and you get S dagger on u, T on v. And then the T can be brought here and act on this one, and you get T dagger S dagger u, v. So this thing is the dagger of this thing, and that's the statement here.

There's yet one more simple property, that the dagger of S dagger is S. You take dagger twice and you're back to the same operator. Nothing has changed. So how do you do that? Take, for example, this-- take u, put S dagger here, and put v. Now, by definition, this is equal to-- you put the operator on the other side, adding a dagger. So that's why we put that one like this. The operator gets daggers, so now you've got the double dagger.

So at this moment, however, you have to do something to simplify this. The easiest thing to do is probably the following-- to just flip these two, which you can do the order by putting a star. So this is equal. The left hand side is equal to this. And now this S dagger can be moved here and becomes an S. So this is u, Sv, and you still have the star. And now reverse this by eliminating the star, so you have S-- I'm sorry, I have this notation completely wrong. Sv-- this is u. The u's v's are easily confused. So this is v, and this is u. I move the S, and then finally I have Su, v without a star. I flipped it again. So then you compare these two, and you get the desired result.

OK, so we've gone through this thing, which is the main result of daggers, and I would like to see if there are questions. Anything that has been unclear as we've gone along here? And question? OK. No questions. So let's do a simple example, and it's good because it's useful to practice with explicit things. So here's an example. There's a vector space V, which is three complex numbers, three component vectors-- complex vectors.

So a v is equal to v1, v2, v3-- three numbers are all the vi. Each one belongs to the complex number. So three complex numbers makes a vector space like this. So somebody comes along and gives you the following linear map-- T on a vector, v1, v1, v3, gives you another vector. It's a linear map. So what is it? It's 0 times v1 plus 2v2 plus iv3 for the first component. The first component of the new vector-- I put the 0v1 just so you see that it just depends on v2 and v3. The second component is v1 minus iv2 plus 0v3.

Those are not vectors. These are components. These are numbers. So this is just a complex number. This is another complex number, as it should be. Acting on three complex numbers gives you, linearly, three other ones. And then the third component-- they don't have space there, so I'll put it here-- 3iv1 plus v2 plus 7v3.

And the question is two questions. Find T dagger, and write the matrix representations of T and T dagger. Write the matrices T and T dagger using the standard basis in which the three basis vectors are 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, and 0, 0, 1. These are the three basis vectors-- e1, e2, and e3. You know, to write the matrix you need the basis vectors. So that's a problem. It's a good problem in order to practice, to see that you understand how to turn an operator into a matrix. And you don't get confused. Is it a row? Is it a column? How does it go?

So let's do this. So first we're going to try to find the rules for T dagger. So we have the following. You see, you use the basic property. u on Tv is equal to T dagger u on v. So let's try to compute the left hand side, and then look at it and try to see if we could derive the right hand side.

So what is u supposed to be a three component vector? So for that use, u equals u1, u2, u3. OK, now implicit in all that is that when somebody tells you-- OK, you've got a three dimensional complex vector space what is the inner product? The inner product is complex conjugate of the first component. That's first component of the second, plus complex conjugate of the second times star, times star. So it's just a generalization of the dot product, but you complex conjugate the first entries.

So what is this? I should take the complex conjugate of the first term here-- u1-- times the first one. So I have 2v2 plus iv3. This is the left hand side, plus the complex conjugate of the second component-- there's the second component-- so u2 times v1 minus iv2 plus-- well, 0v3-- his time I won't write it-- plus u3 bar times the last vector, which is 3iv1 plus v2 plus 7v3. OK, that's the left hand side.

I think I'm going to use this blackboard here, because otherwise the numbers are going to be hard to see from one side to the other. So this information, those two little proofs, are to be deleted. And now we have this left hand side. Now, somehow when you say, OK, now I'm going to try to figure out this right hand side your head goes and looks in there and says well, in the left hand side the u's are sort of the ones that are alone, and the v's are acted upon. Here the v's must be alone.

So what I should do is collect along v. So let's collect along v. So let's put "something" times v1 plus "something" like v2 plus "something" like v3. And then I will know what is the vector T star this. So let's do that. So v1, let's collect. So you get u2 bar for this v1, and 3iu3 bar. v2 will have 2u1 bar minus iu2 bar plus u3 bar. I think I got them right. OK. And then v3, let's collect-- iu1 bar, nothing here, and v3 7u3 bar.

OK, and now I must say, OK, this is the inner product of T dagger u times v3. So actually, T dagger on u, which is u1, u2, u3, must be this vector with three components for which this thing is the inner product of this vector with the vector V.

So I look at this I say, well, what was the formula for the inner product? Well, you complex conjugate the first entry of this and multiply by the first entry of that. Complex conjugate the second entry. So here I should put u2 minus 3iu3, because the complex conjugate of that is that as multiplied by v1. So here I continue-- 2u1 plus iu2 plus u3. And, finally, minus iu1 plus 7u3. And that's the answer for this operator. So the operator is there for you.

The only thing we haven't done is the matrices. Let me do a little piece of one, and you try to compute the rest. Make sure you understand it. So suppose you get T on the basis vector e1. It's easier than what it looks. I'm going to have to write some things in order to give you a few components, but then once you get a little practice, or you look what it means, it will become clear. So what is T on e1? Well, it's T on the vector 1, 0, 0. T on the vector 1, 0, 0-- look at the top formula ther3-- is equal to 0, 1, and 3i. Top formula-- the v1 is 1, and all others are 0. And this is e2 plus 3ie3.

So how do you read, now, matrix elements? You remember the formula that T on ei is supposed to be Tkiek-- sum over k. So this thing is supposed to be equal to T11e1 plus T21e2 plus T31e3. Your sum over the first index, T of e1, is there for that. So then I read this, and I see that T21 is equal to 1. This is equal to 3i. And this is equal to 0.

So you've got a piece of the matrix, and the rest I will just tell you how you see it. But you should check it. You don't have to write that much after you have a little practice with this. But, the matrix T-- what you've learned is that you have 0, 1, and 3i. So 0, 1, and 3i are these numbers, in fact-- 0, 1, and 3i. And they go vertical. So 2, minus i, and 1 is the next column. 2, minus i, and 1 is the next column, and the third one would be i-- look at the v3 there. It has an i for the first entry, a 0 for the second, and a 7. So this is the matrix.

How about the matrix T dagger? Same thing-- once you've done one, don't worry. Don't do the one. So this you look for the first column. It's going to be a 0-- no u1 here-- a 2, and a minus i. 0, 2, and a minus i, then 1, i, and 0, minus 3i, 1, and 7.

And those are it.

And look how nice. The second one is in fact the Hermitian conjugate of the other. Transpose and complex conjugate gives it to you. So that example suggests that that, of course, is not an accident. So what do you need for that to happen?

Nobody said that what you're supposed to do to find T dagger is transpose some complex conjugate, but somehow that's what you do once you have the matrix, or at least what it seems that you do when you have the matrix. So let's see if we can get that more generally.

So end of example. Look at T dagger u, v is equal to u, Tv. We know this is the key equation. Everything comes from this. Now take u and v to be orthonormal vectors, so u equal ei, and v equal ej. And these are orthonormal.

The e's are going to be orthonormal each time we say basis vectors-- e, orthonormal. So put them here, so you get T dagger on ei times ej is equal to ei, Tej. Now use the matrix action on these operators. So T dagger on ei is supposed to be T dagger kiek. The equation is something worth knowing by heart. What is the matrix representation? If the index of the vector goes here, the sum index goes like that. So then you have ej here, and here you have ei, and you have Tkjek.

So now this basis orthonormal. This is a number, and this is the basis. The number goes out. T dagger ki-- remember, it's on the left side, so it should go out with a star. And then you have ekej. That's orthonormal, so it's delta kej. The number here goes out as well, and the inner product gives delta ik. So what do we get? T dagger ji star is equal to Tij. First, change i for j, so it looks more familiar. So then you have T dagger ij star is equal to Tji. And then take complex conjugate, so that finally you have T dagger ij is equal to Tji star. And that shows that, as long as you have an orthonormal basis you can see the Hermitian conjugate of the operator by taking the matrix, and then what you usually call the Hermitian conjugate of the matrix. But I want to emphasize that, if you didn't have an orthonormal basis-- if you have your operator, and you want to calculate the dagger of it, and you find its matrix representation. You take the Hermitian conjugate of the matrix. It would be wrong if your basis vectors are not orthonormal. It just fails. So what would happen if the basis vectors are not orthonormal? Instead of having ei with ej giving you delta iej, you have that ei with ej is some number. And you can call it aij, or alpha iej, or gij, I think, is maybe a better name. So if the basis is not orthonormal, then ei with ej is some sort of gij. And then you go back here. And, instead of having deltas here, you would have g's. So you would have the T dagger star ki with gkj is equal to Tkj, gik.

And there's no such simple thing as saying, oh, well you just take the matrix and complex conjugate and transpose. That's not the dagger. It's more complicated than that. If this matrix should be invertible, you could pass this to the other side using the inverse of this matrix. And you can find a formula for the dagger in terms of the g matrix, its inverses and multiplications. So what do you learn from here? You learn a fundamental fact, that the statement that an operator-- for example, you have T. And you can find T dagger as the adjoint. The adjoint operator, or the Hermitian conjugate operator, has a basis independent definition. It just needs that statement that we've written many times now, that T dagger u, v is defined via this relation. And it has nothing to do with a basis. It's true for arbitrary vectors. Nevertheless, how you construct T dagger, if you have a basis-- well, sometimes it's a Hermitian conjugate matrix, if your basis is orthonormal. But that statement, that the dagger is the Hermitian conjugate basis, is a little basis dependent, is not a universal fact about the adjoint. It's not always constructed that way. And there will be examples where you will see that. Questions? No questions?

Well, let's do brackets for a few minutes so that you see a few properties of them. With the same language, I'll write formulas that we've-- OK, I wrote a formula here, in fact. So for example, this formula-- if I want to write it with bras and kets, I would write u Tv. And I could also write it as u T v, because remember this means-- the bra and the ket-- just says a way to make clear that this object is a vector. But this vector is obtained by acting T on the vector v. So it's T on the vector v, because a vector v is just something, and when you put it like that that's still the vector v. The kit doesn't do much to it.

It's almost like putting an arrow, so that's why this thing is really this thing as well.

Now, on the other hand, this thing-- let's say that this is equal to v, T dagger u star. So then you would put here that this is v T dagger u star. So this formula is something that most people remember in physics, written perhaps a little differently. Change v and u so that this left hand side now reads u T dagger v. And it has a star, and the right hand side would become v T u. And just complex conjugated it. So u T dagger v is equal to v T u star-- a nice formula that says how do you get to understand what T dagger is.

Well, if you know T dagger's value in between any set of states, then you know-- well, if you know T between any set of states u and v, then you can figure out what T dagger is between any same two states by using this formula. What you have to do is that this thing is equal to the reverse thing. So you go from right to left and reverse it here. So you go v, then T, then u, and you put a star, and that gives you that object.

Another thing that we've been doing all the time when we calculate, for example, ei, T on ej. What is this? Well, you know what this is. Let's write it like that-- ei. Now T on ej is the matrix T kjek. If this is an orthonormal basis, here is a delta iek. So this is nothing else but Tij. So another way of writing that matrix element, ij, of a matrix is to put an ei, an ej here, and a T here. So people write it like that-- Tij is ei comma Tej. Or, in bracket language, they put ei T ej.

So I need it to be flexible and just be able to pass from one notation to the other, because it helps you. One of the most helpful things in this object is to understand, for example, in bra and ket notation, what is the following object? What is ei ei? This seems like the wrong kind of thing, because you were supposed to have bras acting on vectors. So this would be on the left of that, but otherwise it would be too trivial. If it would be on the left of it, it would give you a number. But think of this thing as a object that stands there.

And it's repeated endlessly, so it's summed. So what is this object? Well, this object is a sum of things like that, so this is really e1 e1 plus e2 e2, and it goes on like that.

Well, let it act on a vector. This kind of object is an operator. Whenever you have the bra and the ket sort of in this wrong position-- the ket first, and the bra afterwards-- this is, in Dirac's notation, an operator, a particular operator. And you will see in general how it is the general operator very soon. So look at this.

You have something like that, and why do we call it an operator? We call it an operator business if it acts on a vector-- you put a vector here, a bra-- this becomes a number, and there's still a vector left. So this kind of structure, acting on something like that, gives a vector, because this thing goes in here, produces a number, and the vector is left there. So for example, if you act with this thing on the vector a-- an arbitrary vector a-- what do you get?

Whatever this operator is is acted on a. Well, you remember that these thing are the components of a, and these are the basis vectors. So this is nothing else but the vector a again.

You see, you can start with a equals some alpha i's with ei's, and then you calculate what are the alpha i's. You put an ej a, and this ej on that gives you alpha j. So alpha j-- these numbers are nothing else but these things, these numbers.

So here you have the number times the vector. The only difference is that this is like ei alpha i. The number has been to the right. So this thing acting on any vector is the vector itself. So this is perhaps the most fundamental relation in bracket notation, is that the identity operator is this. Yes.

AUDIENCE: Is that just 1 e of i, or sum over all e of i?

PROFESSOR: It's sum of over all. So here implicit sum is the sum of all up to en en. You will see, if you take just one of them, you will get what is an orthogonal projector. Now this allows you to do another piece of very nice Dirac notation. So let's do that. Suppose you have an operator T. You put a 1 in front of it-- a T and a 1 in front of it. And then you say, OK, this 1, I'll put ei ei. Then comes the T, and then comes the ej ej-- another 1.

And then you look at that and you suddenly see a number lying there. Why? Because this thing is some number. So this is the magic of the Dirac notation. You write all this thing, and suddenly you see numbers have been created in between.

This number is nothing else but this matrix representation of the operator. T, between this, is Tij. So this is ei Tij ej. So this formula is very fundamental. It shows that the most general operator that you can ever invent is some sort of ket before a bra, and then you superimpose them with these numbers which actually happen to be the matrix representation of the operator.

So the operator can be written as a sum of, if this is an n by n matrix n squared thinks of this form-- 1 with 1, 1 with 2, 1 with 3, and all of them. Bu then, you know this formula is so important that people make sure that you realize that you're summing over i and j. So just put it there. Given an operator, these are its matrix elements. And this is the operator written back in abstract notation. The whole operator is back there for you.

I want to use the last part of the lecture to discuss a theorem that is pretty interesting, that allows you to understand things about all these Hermitian operators and unitary operators much more clearly. And it's a little mysterious, this theorem, and let's see how it goes. So any questions about this Dirac notation at this moment, anything that I wrote there? It takes a while to get accustomed to the Dirac notation. But once you get the hang of it, it's sort of fun and easy to manipulate.

No questions? Can't be. You can prove all kinds of things with this matrix representation of the identity. For example, you can prove easily something you proved already, that when you multiply two operators the matrices multiply. You can prove all kinds of things. Pretty much everything we've done can also be proven this way.

OK, so here comes the theorem I want to ask you about. Suppose somebody comes along, and they tell you, well, you know, here's a vector v, and I'm going to have a linear operator acting on this space. So the operator's going to be T, and I'm going act with the vector v.

And moreover, I find that this is 0 for all vectors v belonging to the vector space.

And the question is-- what can we say about this operator? From all vectors it's just 0. So is this operator 0, maybe? Does it have to be 0? Can it be something else? OK, we've been talking about real and complex vector spaces.

And we've seen that it's different. The inner product is a little different. But let's think about this. Take two dimensions, real vector space. The operator that takes any vector and rotates it by 90 degrees, that's a linear operator. And that is a non-trivial linear operator, and it gives you 0. So case settled-- there's no theorem here, nothing you can say about this operator. It may be non-zero.

But here comes the catch. If you're talking complex vector spaces, T is 0. It just is 0, can't be anything else. Complex vector spaces are different. You can't quite do that thing-- rotate all vectors by something and do things. So that's a theorem we want to understand. Theorem-- let v be a complex inner product space.

By that is a complex vector space with an inner product. Then v, Tv equals 0 for all v implies that the operator is just 0. I traced a lot of my confusions in quantum mechanics to not knowing about this theorem, that somehow it must be true. I don't know why it should be true, but somehow it's not, because it really has exceptions.

So here it is. We tried to prove that. It's so important, I think, that it should be proven. And how could you prove that? And at first sight it seems it's going to be difficult, because, if I do just a formal proof, how is it going to know that I'm not talking real or complex vector spaces. So it must make a crucial difference in the proof whether it's real or complex. So this property really sets the complex vector spaces quite apart from the real ones. So let's see what you would need to do.

Well, here's a strategy-- if I could prove that u, Tv is equal to 0 for all u and all v. You see, the problem here is that these two are the same vector. They're all vectors, but they're the same vector. If I could prove that this is 0 for all u and v, then what would I say? I would say, oh, if this is 0 for all u and v, then pick u equal to Tv. And then you find that Tv, Tv is 0, therefore Tv is the 0 vector. By the axiom of the inner product, for all v is a 0 vector, so T kills all vectors, therefore T is 0. So if I could prove this is true, I would be done. Now, of course, that's the difficulty. Well, I wouldn't say of course. This takes a leap of faith to believe that this is the way you're going to prove that. You could try to prove this, and then it would follow. But maybe that's difficult to prove. But actually that's possible to prove. But how could you ever prove that this is true? You could prove it if you could somehow rewrite u and Tv as some sort of something with a T and something plus some other thing with a T, and that other thing plus some-- all kinds of things like that. Because the things in which this is the same as that are 0. So if you can do that-- if you could re-express this left hand side as a sum of things of that kind-- that would be 0.

So let's try. So what can you try? You can put u plus v here, and T of u plus v. That would be 0, because that's a vector, same vector here. But that's not equal to this, because it has the u, Tu, and it has the v Tv. And it has this in a different order. So maybe we can subtract u minus v, T of u minus v. Well, we're getting there, but all this is question marks-- u, Tu, v, Tv-- these cancel-- u, Tu, v, Tv. But, the cross-products, what are they? Well here you have a u, Tv. And here you have a v, Tu. And do they cancel? No. Let's see. u, Tv, and up here is u minus Tv about. But there's another minus, so there's another one there. And v, Tu has a minus, minus is a plus. So actually this gives me two of this plus two of that.

OK, it shouldn't have been so easy anyway. So here is where you have to have the small inspiration. Somehow it shouldn't have worked, you know. If this had worked, the theorem would read different. You could use a real vector space. Nothing is imaginary there. So the fact that you have a complex vector space might help. So somehow you have to put i's there. So let's try i's here. So you put u plus iv and T of u plus iv.

Well, then you probably have to subtract things as well, so u minus iv, T of u minus iv. These things will be 0 because of the general structure-- the same operator here as here. And let's see what they are. Well, there's u, Tu, and here's minus u, Tu, so the diagonal things go away-- the minus iv, minus iv, iv, and a T. You have minus iv, minus iv subtracted, so that also cancels. So there's the cross-products.

Now you will say, well, just like the minus signs, you're not going to get anything because you're going to get 2 and 2. Let's see. Let's see what we get with this one. You get u with Tiv, so you get i u, Tv. But look, this i on the left, however, when you take it out, becomes a minus i, so you get minus i v, Tu.

And the other products [INAUDIBLE]. So let's look what you get here-- a u with a minus iv and a minus here gives you a 2 here. And the other term, v, Tu-- well, this goes out as a plus i. But with a minus, it becomes a minus i, so v, Tu is this. So there's a 2 here.

So that's what these terms give you. And now you've succeeded. Why? Because the relative sign is negative. So who cares? You can divide by i, and divide this by i. You are constructing something.

So let me put here what you get. I can erase this blackboard. So what do we get? I claim that if you put one quarter of u plus v, T u plus v minus u minus v, T of u minus v, then, let's see, what do we need to keep? We need to keep u and Tv. So divide this by i plus 1 over i u plus iv, T of u plus iv minus 1 over i, u minus iv, T of u minus iv. And close it. You've divided by i. You get here four of these ones, zero of these ones, and you got the answer you wanted. So this whole thing is written like that, and now, since this is equal to u with Tv, by the conditions of the theorem, any vector-- any vector here-- these are all 0. You've shown that this is 0, and therefore the operator is 0. And you should be very satisfied, because the proof made use of the fact that it was a complex vector space. Otherwise you could not add vectors with an imaginary number. And the imaginary number made it all work.

So the theorem is there. It's a pretty useful theorem, so let's use it for the most obvious application. People say that, whenever you find that v, Tv is real for all v, then this operator is Hermitian, or self-adjoint. That is, then, it implies T dagger equals T. So let's show that.

So let's take v, Tv. Proof. You take v, Tv, and now this thing is real. So since this is real, you can say it's equal to v, Tv star. Now, because it's real-- that's the assumption. The number is real. Now, the star off an inner product is Tv, v.

But on the other hand, this operator, by the definition of adjoint, can be moved here. And this is equal to T dagger v, v. So now you have done this is equal to this. So if you put it to one side, you get that T dagger minus T on v times v is equal to 0. Or, since any inner product that is 00-- it's complex conjugate is 0-- you can write it as v, T dagger minus v is 0 for all v.

And so this is an actually well known statement, that any operator that gives you real things must be Hermitian. But it's not obvious, because that theorem is not obvious. And now you can use a theorem and say, well, since this is true for all v, T dagger minus T is 0, and T dagger is equal to T. Then you can also show, of course, if T dagger is equal to T, this thing is real. So in fact, this arrow is both ways. And this way is very easy, but this way uses this theorem.

There's another kind of operators that are called unitary operators. We'll talk a little more about them next time. And they preserve the norm of vectors. People define them from you, and you see that they preserve the norm of vectors. On the other hand, you sometimes find an operator that preserves every norm. Is it unitary? You will say, yes, must be. How do you prove it? You need again that theorem. So this theorem is really quite fundamental to understand the properties of operators. And we'll continue that next time. All right.