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PROFESSOR: What we have to do today is study a very important example of a two- state system. That will be the ammonia molecule, and will lead us to understand how masers work. Masers are what for microwaves, the same thing as lasers are for light-- it's just a different frequency. Masers is microwaves and lasers is for light. It's the same thing.

So it's a very nice application of two-state systems. And then we'll discuss over the last part of the lecture some aspects of nuclear magnetic resonance. I don't think I'll get to the end of it, because it's quite a bit of material. But we'll try to see what we can do.

So let me remind you of the last thing we were doing last time, that this is going to be in the backdrop of what we do today. We spoke about Hamiltonians for a two-state system that were the most general two by two Hermitian matrix specified by four real numbers-- g0 and the three components of the vector g multiplied by the Pauli matrices. This is Hermitian.

This can be written in this way, in which we've identified Hamiltonians for spins, in the sense that g dot sigma-- really, sigma is proportional to S, so this is equal to omega dot S, where omega-- Larmor-- is 2g over h bar. And we explained last time that if you have a term omega l dot S, spins will rotate with angular velocity omega l vector, which means they rotate around the axis defined by the vector omega l, with an angular velocity equal to the magnitude of the vector omega l. So that's Larmor precession.

This Larmor precession in the case of a magnetic field is given by minus lambda times the magnetic field, gamma times the magnetic field, where gamma is that

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constant that relates the magnetic moment of the particle to the spin angular momentum of the particle. Then we got, moreover, that the energy levels of this Hamiltonian-- this is a two-state systems, so it's a two-dimensional vector space that can be at most two energy eigenstates. That's the simple thing about two-state systems.

These two energy eigenstates have the energies equal to g0 plus/minus g. And the plus corresponds to the spin state n plus, and the minus corresponds to the spin state n minus. And you don't have to talk spin states when you write this spin states over here. The plus, you should think of spinning in the plus direction, but the thing that we call plus is the first basis vector. And the thing that we call minus is the second basis vector of this state space.

Therefore, if you've given a matrix, Hamiltonian has nothing to do with spins. You still have the notion that the first basis vector, whatever it is-- an iron moving in this direction-- is the mathematical analog of a spin up. And the second basis vector-- whatever else it may be-- is the analog of the spin down.

So this will be important for what we do now, as we begin the study of the ammonia molecule, and its states. So having reviewed the key ideas from the last part of last lecture, are there any questions?

So let me begin with this ammonia molecule-- double M, M-O-N-I-A, is NH3. It's used as a fertilizer. It's a gas, has strong odor, no color, fertilizers, cleaning products, pharmaceuticals, all kinds of things.

It has the shape of a flattened tetrahedron with a nitrogen atom at one corner, say, and the base the three hydrogen atoms. If it would not be a flattened tetrahedron, this angle over here-- if it would be an equilateral, regular tetrahedron, this angle over there would be 60 degrees, because this every phase would be an equilateral triangle. But if it's a flattened tetrahedron, if it will be totally flat-- then n would be at the base. The angle in between these two edges would be 120 degrees, because they have to add up to 360.

Well, this has 108 degrees, so it's pretty flat, this tetrahedron. And you can imagine, actually, this is a two-state system, because just as the nitrogen can be like this-- can be up with respect to the base of the hydrogens, it could also be down. And so you can imagine this molecule rotating, and suddenly the up nitrogen goes down, and this keeps rotating-- a possible transition of the system, in which, well, I don't know if I'm drawing it well. I don't think so.

But roughly, it could be like this-- the nitrogen is down. And this, in principle, would be like two possible configurations of this system. There's a barrier.

This is in equilibrium, so if you try to push it, it's not easy. But if you did manage to push it, it would be stable in the other direction. It is as if you had a potential for the nitrogen-- a V of z-- the direction is z-- in which it can be here, up or down. And it's stable in either one, but there's a big barrier in between.

So that's the story of this nitrogen atom. And we're going to try to describe this as a two-state system. So I need some notation. Well, I'm going to have the first basis state to be called up for N up, and the second basis state is going to be called down for nitrogen down.

And now, I'm going to try to write the Hamiltonian for this system. Well, you know what sort of happens here. Your intuition in quantum mechanics with wave functions should be similar.

Look-- this is not the two-state system, because there may be many energy eigenstates, but you know that the ground state looks like a wave function, just like this. And the first excited state could look like a wave function that is like-- oops-- this. Pretty much the same thing, but you flip one, and if the barrier is sufficiently high, these two energy levels are not that different.

So the question is, how do we model this? There may be an energy E0 for the system, a ground state energy maybe, and a little bit of a higher energy. So we're going to write the Hamiltonian.

And I'm going to put E0 for the moment. And my first basis state is 1 and up. This would be the 1 0, and here would be the 0 1, the second basis state. And this is saying that 1 0, the N up, is an energy eigenstate of energy E0. And down is an energy eigenstate of energy in up as well.

But that can't be the story. There cannot be two degenerate energy eigenstates. Your intuition tells you that this is impossible. One dimensional potential wouldn't say that. So there must be something else happening.

This cannot be the whole Hamiltonian that describes the physics of the problem. So what we're going to do is try to tinker with this Hamiltonian, a simple tinkering that is going to give us the physics that we expect.

So I'm going to put a constant delta here. This should be Hermitian, so I should put the delta as well, another constant there. For convenience, however, I'd rather put the minus sign there. I will define delta to be positive for definiteness, and for convenience, however, I will put here minus delta.

Now, you could say look, you say that, but maybe it's not for convenience. Maybe it changes the physics. Well, it cannot change the physics, because these things are the matrix elements of the Hamiltonian-- the 1 2, and the 2 1 matrix elements.

And I could decide to change what they call the first basis vector, to call it minus the first basis vector. This would change the sign of this, change the sign of that, without changing those signs. So this sign is a matter of a basis. So we certainly have not made any assumption by putting that to be a minus sign over there.

Now, once you have this Hamiltonian, this delta is going to be some energy. And that's going to be what mimics the physics here, because these states are not going to be any more energy eigenstates. The matrix is not diagonal anymore. So the 1 0 vector, and the 0 1 vector are not any more energy eigenstates.

Moreover, it's interesting to try to figure out what it has to do with our previous system. So this is E0 times 1 minus delta times sigma 1. And that's a good thing to know.

So in this case, comparing to this g is the vector in the x direction, because its g multiplying by sigma 1. And it has magnitude delta. So we notice-- and we're going to make a picture later-- is that g, in this case so far, is equal to delta times the unit vector in the x direction, minus delta times the unit vector in the x direction.

So g is equal to delta. So, OK, we've written those. Let's then figure out what are the ground states and the excited states.

And this is a two by two matrix, and a simple one, at that. So you could just do it, or better to figure out what we're doing. We'll use our formulas before. Yes, George.

AUDIENCE: So why is it that we mandate that delta has to be real? I mean, that's not the most general form.

PROFESSOR: That's right, it's not the most general form. So at this moment, we're trying to do what any reasonable physicist does. Without delta, it doesn't match the physics. So let's try the simplest thing that can work, and a delta real-- we'll see if it works. And if it works, we'll worry later about different things.

So we'll put the simplest thing at this moment, but indeed, we could put more complicated things. So given this, in fact, we know what it the energy eigenstates should be, or we more or less can guess what the energy eigenstates should be. Let me tell you the energies are g0 plus/minus g, so you're going to get E0 plus delta and e0 minus delta as the energies-- E excited, and E ground. And the gap between these two energies-- the gap between these two energy levels is 2 delta.

So there we go. We've already produced something good. We have two energy eigenstates. There should be a small energy difference, and that gap is 2 delta.

Now, what are those states? Well, it's not too hard to see that the eigenstate that has this energy is the excited state, is 1 over square root of 2, 1 minus 1. If you add with this matrix on this, that's the energy eigenstate. And the energy eigenstate for this one is 1 over square root of 2, 1 1. Let's write them. These are the eigenvectors. Let's write them as 1 over square root of 2, nitrogen up minus

nitrogen down, and 1 over square root of 2 nitrogen up plus nitrogen down.

So I want to, even though it's not complicated to do this, and we have called these states that way, so it's all clear. I want you to see how that comes from our spin way of thinking. So you know there's this molecule, and for this molecule, only one direction matters. We could have called it x, if we wanted. In fact, maybe x would have been a better name.

On the other hand, for spin states, there are three dimensions-- x, y, and z. So we have to think in an abstract way. So where is this vector G? We said G is minus delta Ex. So this is the x-axis. This is the y-axis. This is the z-axis. G is here. The vector G goes back over here, is minus delta x hat vector.

Now, what if you have g in that way? You know that the excited state is one of these states over here. Let's see-- this N plus is the excited state, and N minus is the lower state. So the excited state should point in the direction of g vector, because N corresponds to the direction of the g vector.

G is positive here, this little g is positive. So g is in there. N is in there as well, because g and n are parallel. And the excited state should correspond to N, a vector in the plus n direction.

So the excited state should be here. It's a spin state in that direction. That's what that formula says. And the ground state should be a spin state in the minus N direction, so this must be the ground state.

So this I call the excited state, and this the ground state. And indeed, remember now that what is your translation. 1 and 2-- the 1 and 2 states are like the plus and minus of spins. So in terms of spin language, this excited state is the plus minus the minus. And this is the plus plus the minus, because the up is plus, the down is minus.

So indeed, this state-- you probably remember it. This is a spin along the x direction. So the ground state must be like a spin along the x direction. That's here. The excited state is a spin, the orthogonal one in the minus x direction, so it must be

a state orthogonal to this one, as to this, and it points in the other direction.

So those are our spins. And we had that the gap delta-- the gap is 2 omega. It's an energy, so it's what we called h omega naught a photon-- the transition energy.

I could give this energy in eVs, but I actually don't have it. I have the wavelength of the frequency of the associated photon. So this corresponds to a frequency nu of 23.87 gigahertz, and a lambda of about 1.26 centimeters-- more or less half an inch.

So that is the transition difference between these two levels. So this is something people knew-- there's two levels above this molecule, and they correspond to the result of this perturbation that splits them. So in a sense, we've got a nice model-- perfectly reasonable model, without introducing much complexity-- of what this thing is doing. So let's do a little exercise. Let's see how does the N up state evolve in time?

So we have psi at time equals 0 be in the state up. What is it later? There are many ways to do it-- many, many ways to do it. The quickest, in principle, is to think about spins, even if just a little painful.

But let's think about spins. Omega I is going to be around the direction of g. So think of the state of the spin. The N up state, the up state, is here. And then it's going to precess with angular frequency vector in the direction of g. So it's going to precess in the direction of g. So you can imagine now this vector precessing.

And it's going to go-- since it's essentially the minus x direction-- precession in time is going to flip it to the y-axis, and then make it rotate in the z-y axis. That's all what it's going to do. So you have a picture of what it's going to do.

We might as well calculate a little, although the picture is complete, and the frequency is known, and everything. But what you do here, of course, is you try to write it in terms of energy eigenstates. And the up state is the 1 over square root times the sum of e plus g. And you know the energies of those two states, so you know how they evolve in time.

It will be in the notes. You can do this. After you now evolve, with e to the minus i ht over h bar, you then go back from e to g, to up and down, because that's sort of the intuition that we want to

Have. So it's not difficult. None of these steps are difficult. e and g are written in terms of up and down. So what does one get? One gets psi of t is equal e to the minus i, et over h bar times cosine of t delta over h bar, times the state up plus i sine of t delta over h bar, state down.

This is the time evolution, so the probabilities, for example, to be up is the square of this one-- cosine squared of t delta over h bar. And the probability to be down is sine squared of the same thing. So this poor nitrogen molecule, if it happens to have the nitrogen up, is going to start rotating like crazy, even if you don't do anything.

It's just sitting there, and it's rotating up and down, with a speed doing this thing 23 billion times a second. Molecule's up and down, because it's not in a stationary eigenstate. Now, here, actually, you may think that something is a little funny, because you would say, well, the frequency of rotation is like delta over h bar, but the Larmor frequency is supposed to be 2g over h bar, so it would correspond to a Larmor frequency of 2 delta over h bar, which is exactly the frequency of the photons.

But there's no contradiction here. This is, in fact, rotating at that speed, at twice that speed. Because if you remember, for a spin state, this was the cosine of theta over 2. Therefore, as it changes, that's the way theta over 2 is changing. But theta, which is the angle of this physical rotation, changes twice as fast.

So it's, again, those 1/2s of spin states that are very confusing sometimes. But there's no contradiction. The sort of Larmor frequency of the equivalent spin problem is exactly the same as the frequency of the original problem.

So now we want to make this into something more practical. And for that, what we

explore is the fact that this molecule has an electric dipole moment. So the molecule as we pictured it there, as it happens, the electrons of the hydrogen sort of cluster near the nitrogen. So this up region is kind of negative. The bottom region is kind of positive, and there is an electric dipole moment pointing down.

So this is a pretty important property of this molecule, this dipole moment. And electric dipoles we usually call p. But for some reason-- maybe I should change the notes, at some stage or maybe this is discussed very nicely in Feynman's lectures on physics. He uses mu for this, like for magnetic dipole. So I will actually use mu as well, now.

So this thing has an electric dipole, and therefore the energy is the electric dipole dotted with the electric field. And that electric field is an external electric field. You have this little dipole, which is this molecule, and you put it inside an electric field, and there's a contribution to the energy, just because the dipole is sitting on an electric field.

And that means our Hamiltonian is now changed. So I will consider the case in which we have an electric field in the z direction-- a positive electric field in the z direction-- so that E is equal to E times z. And mu would be equal to minus mu times z, because it points down. We've assumed that the dipole is down. And the dipole is down for the case of spin in the z direction.

So look what we get here-- this energy contribution is essentially mu E. And it's the energy that is acquired by the state in which the nitrogen is up. This is for nitrogen up.

So what we've discovered-- if we want to model this in the Hamiltonian is that we can take the Hamiltonian that we have-- E0 minus delta, e0 minus delta, and the energy of the state up with nitrogen up, is this one-- mu E. So we add mu E. And the one with the spin down will be the opposite, so it will be minus mu E. And this is our reasonable expectation for the Hamiltonian of this molecule inside an electric field.

So this is the NH3 in E field. So again, we can wonder what kind of thing happens

here. And the best thing is to first say this is E0 1 minus delta sigma 1. And then you see, oh, it's mu E sigma 3.

So getting to that, and I realize this is a little more painful than the other one. And we don't have to do it, because we've solved the general problem. And the energies, this time, are going to be E of the excited one, and E of the lower one-- ground state. It's going to be E0 plus g. And g was the magnitude of the vector g. So it's the magnitude of the vector g that now has components minus delta 0 and mu E.

So here we get plus square root of delta squared plus mu E squared. And here is 0 minus square root of delta squared plus mu E squared.

So there we go. If we know how the energies behave, even if we have some electric field-- and typically delta is such, and mu is such that, for most electric fields that you ever have in the lab-- this is very small compared to that. The dipole moment is sufficiently small that the energies that you get from here are pale compared to the difference of energies over there.

So you can approximate. This is E0 plus delta plus 1/2 mu E squared over delta. This is for mu E small. Here E0 minus delta minus 1/2 mu E over delta squared. And this is when mu E is much smaller than delta.

Now, the only reason I point this out is because it does provide a technological opportunity to separate a beam of particles into excited- and the ground-state level. Sort of like Stern-Gerlach experiment, you put, now, this beam that has this ammonia molecules. And you put them inside an electric field that has a gradient.

In a gradient, this state is going to try to go to minimize its energy. So it's going to go to the regions of the electric field where the electric field is small. This particle minimizes its energy when it goes to the regions of the electric field when the electric field is big.

So it's like putting it in a Stern-Gerlach experiment. You have your beam, and you separate them. You have your beam and you manage to separate the things that

can be in an excited state, and the things that are in the ground state.

And now what you do is insert these excited states into a resonant cavity. Have a little hole here, and a little hole here, and E comes in, and something comes out. So we're getting now to the design of the maser.

The idea that we're trying to do is that we try to make a cavity tuned to 23.7 gigahertz-- the frequency associated with a gap. And we just insert those Es over there, these excited states over there, and hope that by the time they go out, they become a g. Because if they go from E-- say there was an electric field here to separate them, and then E's over here.

This is the excited state. There's no more electric fields over here. It just comes into the cavity as an excited state. The excited state has energy E0 plus delta. And then, if it manages to go out of the cavity as the ground state, then it would have energy E0 minus delta. It must've lost energy to delta.

That can go nicely into the electromagnetic field and become one photon-- a one photon state in the cavity of the right frequency, because the cavity is tuned to that. The only difficulty with this assumption is that E is an energy eigenstate. So energy eigenstates are lazy. They're stationary states. They don't like to change.

So there's no reason why it should go out as g. It's excited state. It's perfectly happy to remain excited forever. So what must happen somehow is that there's an electric field here in the cavity, and that stimulates this thing to make the transition, because once there's an electric field, E is not anymore an energy eigenstate. The E of the original system is not anymore an energy eigenstate, and nor is this. So then it's going to change in time.

So the problem is a delicate one in which we want to somehow have an electric field here that is self-consistent with the idea that this excited state goes out as the ground state. And that's why it's microwave amplification by stimulated emission of radiation, because you're going to amplify a signal here. It's a microwave, 1 centimeter wavelength-- that's a microwave. And the stimulation is the fact that this

wouldn't do it unless there's some electric field already.

So you could say, well, so how does it get started? There's no electric field to begin with. Well, you know quantum mechanics, and you know that in general, there are little fluctuations, and there's energies-- small photons, one or two photons that suddenly appear because of anything. Any motion of charges in here produces an electromagnetic wave.

So at the beginning, yes-- there's no many photons here. But somehow, by having it resonate at that frequency, it's very easy to get those photons. And a few appear, and a few molecules start to turn in, and then very soon this is full with energy, in which there's a consistent configuration of some electric field oscillating and producing precisely the right transitions here.

So I want to use the next 50 minutes to describe that math. How do we do this? Because it just shows really the sort of hard part of the problem. How do you get consistently a field, and the radiation going on?

So maybe I should call this E prime and g prime. They shouldn't be confused. E and g are these states that we had before. And E prime and g prime, we never wrote those states, but they are deformed states due to an electric field.

OK, so what do we have to do? Well here is E and g. And we had a Hamiltonian. There's going to be an electric field here, so this Hamiltonian is the relevant one.

The only problem with this Hamiltonian is that this is going to be a time-dependent field, something that we're a little scared of-- Hamiltonians with time dependence-- for good reason, because they're hard. But anyway, let's try to see. Today's all about Hamiltonians with time dependence.

So there's going to be a time-dependent is going to be the wave here. So that's the relevant Hamiltonian. But it's the Hamiltonian in the 1 2 basis, in that up nitrogen, down nitrogen basis. I want that Hamiltonian in the Eg basis. It's better. It's more useful.

So let's try to see how that looks-- Hamiltonian in the Eg basis. H prime in E equal 1 g equal 2 primes, maybe put basis. So here is the Hamiltonian in this basis. In the 1 2 basis, I have to pass to the other basis, the Eg basis.

So it's not complicated. It takes a little work, but it's nothing all that difficult. For example, in the 1 prime h 1 prime, which would be the 1 1 element of this matrix, I'm supposed to put here 1 prime is EhE. And now I'm supposed to say, OK, what is this?

Well, E is 1 over square root of 2. E was 1 over square root of 2, 1 minus 1. H is the original H, so it's E0 plus mu E minus delta minus delta E0 minus mu E. And E is 1 1, 1 minus 1, again. And there's also the square root of 2. So at the end, this is a 1/2.

So this is the kind of thing you have to do to pass to this basis. So I think I'll do that in the notes. And this calculation is simple. In this case, it gives E0 plus delta. And in retrospect, that sort of is pretty reasonable.

This is E0 plus delta, and this is E0 minus delta. And if you didn't have an electric field-- indeed in this basis, the first state is the excited state who has this energy. The second state is the ground state, and has this energy. And that makes sense if there's no mu E.

Well, the mu E still shows up, and it shows up here. So that is the Hamiltonian in this basis, and the general state in this basis is the amplitude to be excited, and the amplitude to be in the ground state. This is the general psi of t.

So your Schrodinger equation in this mu basis-- not mu basis, in the Eg basis, you see it's-- Eg basis is the basis of energy eigenstates if you don't have electric field. But once you have an electric field, it's not anymore energy eigenstates, and much worse if you have a time-dependent electric field. So the Schrodinger equation is i h bar d dt of CE Cg is equal to this mu matrix.

And now, E0 is totally irrelevant for everything. It's a constant of the unit matrix. Let's put E0 to 0. There's no need to keep it. E0 equal zero. So we have delta mu E

minus delta times CECg. I d dt of psi, h psi-- the Schrodinger equation.

Now, the real difficulty that we have is that E is a function of time. So this is not all that trivial. So what you do to solve this is simplify the equation by saying what would the solution be if you didn't have a function of time? Then you would have-- if you didn't have E, CE Cg of time would be E to the minus i ht.

So this would be i delta t-- the energy of this, if there's no electric field, the Hamiltonian will be delta minus delta. So here, I have my psi ht over h bar. And for the lower state, you would have E to the plus i delta t over h bar as solutions, if you didn't have this. This would be the solutions.

But we want now better. So what we're going to say is well, that's a solution if I neglect this. So this cannot be the real solution. So I'll put here beta sub E of t, and a beta sub g of t. And sure-- if no electric field, betas are 1. They're not necessary. But if there is an electric field, the betas are going to be complicated, so we need them there.

So this is like an [INAUDIBLE]. Now you could plug this back, and calculate what you get. And it should not be too surprising that you're going to get in here something in which these deltas are going to disappear, because this thing takes care of that. So there's a little bit of algebra here, maybe two, three lines of algebra.

And let me give you what you get-- I h bar d dt of beta E beta g. Now the equation is really for this quantities-- and it's 0 E to the i omega 0t times mu E, E to the minus i omega 0t times mu E0, beta E, beta g, where omega 0 is the Larmor frequency, or 2 delta over h bar.

So some calculation involved, but I hope you don't lose track of the physics here. The physics is that the amplitude to be in E and the amplitudes to be in g have now been replaced by beta and beta g, which are expected to be simpler things to calculate. And in fact, since the probability to be in E is the norm of to this thing squared, beta is as good as C to know how likely is the particle to be in E, or how likely is the particle to be in the ground state. You could use beta, because they

differ by a phase.

So betas have still the physical significance of the amplitude to be in E or g. And we're still having an electric field that is time dependent here. So it's time to put the time dependence, and what are we going to do?

Well we're going to find a sort of self-consistent operation of this device, in which we will have an electromagnetic field, E. E of t will be 2 E-- letter E, that now is going to be a constant with an E0, cosine omega 0 t-- again the Larmor frequency or the photon frequency that is emitted by the possible transition. So we will consider the case when the cavity has already that electric field that is going precisely at the right speed to do things well. So this Et, with the 2 conveniently put here, is equal to E0 E to the i omega naught t plus e to the minus i omega naught t.

So when you multiply these things, what do you get? Let me do one of them for you to see. You get this i-- the top one-- h I'm going to put to the other side. Beta E is going to be beta E dot, is going to couple with this to beta g. So that thing is going to be this electric field times mu, so mu e0, the h bar from the left-hand side, and you have E to the i omega naught t multiplying this, so it's 1 plus E to the 2i omega naught t times beta g.

So that's the first equation. Not all that easy, for sure. Second equation-- i beta g dot is equal to mu E0 over h bar, 1 plus e to the minus 2i omega naught t beta E of t. And now you have to think maybe a little like engineers, or physicists, or mathematicians-- whatever you prefer. But you want to convince yourself that what you want to do is true.

And what do you want to do? I like to forget about these curves, basically. That's what we want to do. Why would that be true? Well, here's a reason why it's true. This is a number that sort of goes between 1 and minus 1, with some phase. Mu E0, however, over h bar, is a very small number. Mu E0 we're thinking of-- we're saying compared to the natural scales of the problem, this energy, mu E0, is much smaller than delta. And delta is the thing that is related to omega naught, which h bar omega naught is equal to 2 delta.

So essentially, mu E0 is an energy which is very, very slow compared to h omega naught. Now, being very small, whatever this is, this time derivative is going to be very small. So beta E and beta g are going to be very small-time derivatives-- going to move slowly. If they move slowly over the time that this oscillates, this hasn't changed a lot. And therefore, the average of this function over that time is 0, and it doesn't contribute to the differential equation.

It's actually a good argument. You can try to convince yourselves, or maybe I'll try better in some way, or something. But it is right. I've tried it, actually, out with computers and with Mathematica, and things like that. And it's really absolutely true-- that if you think of differential equations as integrals, that you integrate with this in the right-hand side, you can see that if this-- really, the time derivative is controlled by this, that corresponds to a frequency much smaller than omega naught. These ones don't matter.

So it's really interesting, and still not trivial. But this is a case where we end up ignoring part of the thing. So what do we get, then? i beta E dot is equal mu E0 over h bar beta g.

And the second equation-- i beta g dot equal mu E0 over h bar beta E. Therefore, if you multiply by another i here, and differentiate i-- i beta double dot of E is equal mu E0 over h bar. The i extra that we borrowed, the dot here. You can use the second equation.

So you get mu E0 over h-bar squared beta E. Therefore, beta E double dot is equal to minus mu E0 over h-bar beta E. And you, see your you're rotating with the a of-- put the h-bar-- with a frequency that i much smaller than omega 0. So this as a frequency, this is like omega 0.

So indeed, the rate of change of this thing goes with the frequency that's much smaller. And it's all right, actually. So, we've had that, and then we can write the solution, finally. So, what is it?

Beta E of T is cosine mu E0 t over h-bar. And the probability to be in the E state is the square of that amplitude, so the probability to be in the E state at time t is the square of that. So it's cos squared mu E 0 t over h-bar.

Again, I'm sorry, this beta is here, and the probability to be in the excited state is this square of ce, but the square of ce is the square of beta E, so I just square this. So there you go. We have this thing, and we now have an understanding of how this goes as time goes by. In fact, this mu e 0 T or h-bar under bar goes by. This starts to as a cosine squared, and then it goes like this. And this is the place when this is equal to pi over 2.

So, what do we need? We need the place where this happens-- we can call that time T-- be such that mu E 0 T over h-bar be equal to pi over 2, or 3 pi over 2. For those values of time, the probability to be excited is zero, and therefore you must be in the ground state. These two probabilities squared added go to 1. So you're either E or G. So if you have zero probability to be in E, you will be in the state G.

So, the whole issue is basically at this moment that you must give the right speed, or for a given the speed of the molecules, there will be a time that it takes to traverse. That time is related to the steady state value of this electric field by this relation. So you need the right velocity-- the molecules have to be at the right temperature, that's when they'll have a velocity-- so that they travel in such a way that is consistent with this.

As they do that, each of these particles that goes from E to G gives out one photon, one quantum of the electromagnetic field, and helps build the time dependent electric field that we started at the beginning.

Now if you, at some point for some speed of the molecules, you saturate this and you build some electric field, and then you have your cavity operating at the nominal way. And then, of course, you want to use this for something, so you shine, let it go out, and shine those microwaves or do something. And if you want to recharge it, you keep adding nitrogen molecules.

Now, this was a great discovery, actually. Charles Townes, Gordon, and Zeiger, built this ammonia maser in 1953. They got the Nobel Prize. Nobel Prize, Charles Townes got it in 1964. And he emphasized that this masers do the most perfect amplification consistent with the uncertainty principle. This is a coherent state of flight that is built here, and it's much better than any vacuum amplifier or anything like that, because the thing that this giving out those photons is a molecule that this uncharged. So it doesn't disturb the electromagnetic field in the cavity as it goes through.

Many times you use an electron, for example, to give out some energy. But the electron itself is charged, so it produces additional electromagnetic fields, shot noise, all kinds of noise. This is absolutely quiet device in which is this turns from one state to another smoothly, stimulated by the electric field, because if there was no electric field would be an energy eigenstate and then it gives out photon after photon.

So the uncertainties actually, in this noble lecture, which is fun to read-- in fact, you remember delta N delta phi supposed to be greater than 1/2. Well, for coherent states as we more or less discussed, this thing is saturated. And what do you have now? You have a coherent state of light.

You may remember that the expectation value of N in a coherent state alpha was alpha squared. And you call this the number n of photons. And the uncertainty in N was, in fact, alpha. So, it's square root of n.

So for this thing, we have the situation in which we are working with a coherent state that saturates this. So delta n times delta phi is about one half. And delta n is square root of the number of photons. So delta phi is about to 1 over 2 square root of the number of photons. And you can imagine the cavity easily can have 10 to the 12 photons, and to 15 photons. Something fairly big. And you get an uncertainty phase. The thing is coherent. All the pieces of that electromagnetic wave, the phases are very coherent up to an incredibly great accuracy.

So it's a great discovery and the beginning of many things that were done here, in

fact, by Professor Kleppner and others in the '60s with other type of lasers and masers and this whole thing.

So, that's pretty much what's I wanted to say about the ammonia and these things, and we're going to use the last 15 minutes to begin something else, NMR. And, so are there any questions so this Point? Yes?

AUDIENCE: [INAUDIBLE]. If it started changing between states, I sort of imagine a photon being omitted and absorbed, what exactly is happening then? [INAUDIBLE]? I just happened to catch it in the right moment.

PROFESSOR: Well, in this case, it's basically omitted all the time, because we've tuned the cavity in such a way that if it comes as E, goes out as G. So the whole process by the time that it entered and it went out, it has to have omitted one photon.

AUDIENCE: In the midst of the process, how does it get back somehow, because it's sort of oscillating between states.

PROFESSOR: It doesn't get it back. If the cavity would be badly designed in such a way that it may be twice as long, it would out still come as E. So it would make the one transition and then absorb another thing, it would just not generate anything.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: Interesting.

PROFESSOR: And you know, a more complete discussion of this, of course, if you really want to do everything, you would have to treat the photon states. Here, we treated this as a wave coupled to the quantum system of the molecule. You could treat the photons themselves as quanta and do quantum mechanics of the photon field on that. And that would be a wave that you could calculate things more completely . But this is not what we do now.

Any other questions? Yes?

AUDIENCE: So physically, are the nitrogen molecules all fixed in the same orientation going in to this device? How does that happen?

PROFESSOR: Are the what?

AUDIENCE: The molecules all fixed in the same orientation?

PROFESSOR: Yes. Essentially it's not quite like an orientation. It's an energy eigenstate. So basically what you have to do is have this beam of ammonia molecules and do this beam splitter that we talked about with the electric field. And you split the beam into some that are all excited, and some that are ground. And that's it. You need that everything that enters here is excited state.

OK. So another time dependent problem that we're going to discuss today and continue next time is the NMR problem, nuclear magnetic resonance. So this is a pretty interesting problem. Our nuclear magnetic resonance.

And it all begins by having a magnetic field that has a big component here, B0, I think. It's a good name for it. B0 Z, indeed. And then you're going to have some magnetic field that is going to be rotating here in time, in the xy plane. We'll assume that time equals 0, is here, and then it's rotating with some angular frequency omega.

So the total magnetic field is B0 Z hat plus some number, B 1-- I'll write it here-- plus B 1 cosine omega t times x hat minus sine omega t times y hat. So, indeed, in the xy plane, seen like this, you have it here and it's rotating with angular frequency omega this way, clockwise.

All right. So we have this magnetic field. And of course we're going to try to figure out what spins do in it. And the magnetic field is time dependent. So we're in risk of getting a time dependent Hamiltonian.

So what is the Hamiltonian? This possibly time dependent, it's supposed to be minus gamma B times the spin. So what is that? It's minus gamma B0 Sz-- The z component matches with the z component of the spin-- plus B1, Sx cosine omega t

minus Sy sine omega t.

And, well, it's as bad as you could imagine, pretty much. This Hamiltonian is time dependent. And there's some good news if even if it's time dependent, they commute at different times. The time evolution is EC. But no, they don't commute at different times. Time equal 0, for example, you have Sz and Sx, but at a later time, you will have Sz and Sy, and they just don't commute.

So we have no cookbook recipe to solve this problem. We have to figure it out. And there are several ways to figure it out. I had a way to figure it out that I explained in previous years, but today I suddenly thought I could explain it in a different way that is maybe a little harder conceptually, but explains more of what's going on.

So this is what I want to try to do now. And basically what we're going to try to do is get the main intuition going for this problem. I have the Schrodinger equation for this problem that is very complicated with a time dependent Hamiltonian.

So if you have a wave function-- and now I'm going to be a little rough for a little while. A h-bar D T H psi. And it-- oops. What is it? Let's see which lights-- Lights, general. OK.

So if you have a problem like this, you might say, well, I would like to do something non-trivial to this, so I want to maybe change the Hamiltonian and do the same physical problem. Now that's not easy, because if it's the same physical problem, how can you change the Hamiltonian all that much? So one thing you could do is you can try to change the states by doing a unitary transformation, and then hope that the unitary transformation acts on this Hamiltonian. So this will be the new states. And you would hope that this unitary transformation would somehow, working with these new states, would simplify the Hamiltonian.

But unitary transformations in general is just like a change of bases. It's not going to do all that much, unless the unitary transformation has time dependence, in which case it doesn't just rotate the Hamiltonian but messes up this term. But this is really your only possibility of fixing things, is to try to do a time independent unitary

transformation that somehow you started with a spin Hamiltonian, and you're going to do a unitary transformation, and perhaps it's going to be a time independent one.

But this U is going to depend on time. And this is not the whole Hamiltonian, because there's a problem with this part. So the idea is sort of vague, but you're getting a little of the gist of what we try to do.

So the first thing I tried to do is something maybe intuitive about though this system. I could say that suppose I have a system here, and the Hamiltonian is 0. Nothing. Nothing happens in that system.

Now the thing that this curious about this, that this magnetic field is rotating. So let's try to imagine how would physics look if you have the xy axis, and you have the xy axis rotating with angular velocity omega. So the xy in the plane is rotating with angular velocity omega. What would happen?

So there's an Hs that is originally 0. You say it's 0, and any spin state stays there. There's no time evolution. H is equal to 0.

Nevertheless, if you jump into this rotating frame, all the spin states that were static, for you they're rotating. And in fact, for you, they are processing around the z-axis if you're in the rotating frame. Therefore, in the rotating frame, you must have some Hamiltonian, even though there's no Hamiltonian in the static frame. Because in the rotating frame, you feel things spinning.

So in the rotating frame, if there's a static spin along the x direction, you now see it spinning around the z-axis with frequency omega, in fact, going the plus z direction. So in the rotating frame you can ask, what is the new Hamiltonian in the rotating frame?

I mean, the rotating frame, the Hamiltonian should be such that it rotates the spins with angular velocity omega around the z-axis. You may recall that this is done by all the unitary operator E to the minus i omega t Sz hat over h-bar. This is the operator that rotates, spins, with omega t around the z direction, which is precisely what you want this Hamiltonian to do. So the Hamiltonian must be e to the minus i

ht. So this rotating Hamiltonian must be omega Sz hat.

So the Hamiltonian in the new frame is that because that Hamiltonian produces the rotation of states that you see because your friend that is not rotating is telling you that no state is moving. H is 0. So the intuition is that you've passed from the original [INAUDIBLE] to this one, and you have to add 6 this. But what we'll see now is that you have to add the little-- you have to do other things a little if this is not equal to 0.

Here is what we want to do. Now there's also a couple of ways of doing this, but let me try a little calculation to do this second part. So, we're going to take, therefore, to have, for example of psi R a rotating wave function that is going to be given by a unitary operator times the physical wave function you want to solve. This is what you want, and this is what you hope has a simpler dynamics, psi R.

So, let's try to figure out what is the Schrodinger equation for psi R if you know the Schrodinger equation for psi. So, I dt I h-bar dt psi is equal to Hs psi, let's see what is the Schrodinger equation satisfied by this one. So I h-bar dt of psi R is I h-bar dt of U psi. So let's differentiate this. This is I h-bar dt of U, and I will have the psi. But then I will put them a U dagger, there and another U acting on psi so that it gives me psi R.

So the first derivative is acting just on the U. I acted, and then I put U dagger U and recreated the psi R. The other terms is in which it acts on psi, so I get plus I h-bar. The U goes away, and now we get dt of psi, which is I h-bar dt of psi is Hs I psi, so I must delete this.

So the second term, when I acted here, I act with the whole thing. I ddt, I h-bar ddt on psi. The U is out, and I put Hs psi. And here, I can, of course, put a U dagger U and put back the psi R, so I will do that as well. U dagger psi R.

So actually you have now a the Schrodinger equation which is of the form I h-bar dt of psi R is equal to U Hs U dagger plus I h-bar dt of U U dagger of psi R. Now that's it. This is the new Hamiltonian. This is the rotating Hamiltonian, essentially, that

we're trying to figure out that is going to be simpler, we hope. And there we go here. We have this similarity transformation of the Hamiltonian and we have this extra term.

Now suppose the original Hamiltonian had been 0. We want the new Hamiltonian, given our argument, to be a this for the rotating systems. So I will say that this U is such that you get I h-bar dt U U dagger being precisely omega Sz. And for that the U is nothing else but e to the minus I omega tSz over h-bar, which is in fact that thing that we had there.

So that's what U is. And in that way, this whole term becomes just omega Sz hat. So we're almost, in a sense, done with this thing, because we have made some good progress. Except that we still don't know if everything has worked out or not. I'll continue here. And just one minute to close up the discussion by saying what has happened.

So, what has happened is that we have found that there's a new Hamiltonian, HR equal U Hs U plus I h-bar dtU U dagger. And then U is given by e to the minus I omega tSz hat over h-bar. And psi-- as we said, psi of t is equal to e to the I omega tSz hat over h-bar psi R of t.

So this came because we said that psi R was going to be U psi, and therefore I took U and the inverse and you get this. So look what the problem has turned into. It has turned into a problem for psi R with a Schrodinger equation that has a Hamilton HR in which in this piece is very simple. It's omega Sz hat.

And now the crucial point is whether-- here I have U dagger-- whether this thing is time independent. And this should be time independent if we got our physics right, and that's exactly where we'll take on the next time, and prove that it's time independent and then we can solve the complete problem.

All right, there will be notes posted on this soon. Good luck finishing your homework, and I think Monday is a holiday. Is that right? Well, no class Monday. We'll see you on Wednesday.