PROFESSOR: So it's a beautiful recording. OK, so to get started, questions? From last time? Barton covered for me last time. I fled. I was out of town. I was at a math conference. It was pretty surreal. Questions, yes.

AUDIENCE: [INAUDIBLE] about the exam?

PROFESSOR: About--?

AUDIENCE: The exam.

PROFESSOR: The exam. Yes, absolutely. So the exam is, as you all know, on Thursday, a week hence. So on Tuesday we will have a lecture. The material Tuesday will not be covered on the exam. The exam will be a review of everything through today's lecture, including the problems that, which for some technical reason I don't know why didn't get posted. But it should be up after lecture today. The exam will be a combination of short questions and computations. It will not focus on an enormous number of computations. It will focus more on conceptual things. But there will be a few calculations on the exam. And I will post some practice problems over the next couple of days.

AUDIENCE: Do we have a problem set [INAUDIBLE]?

PROFESSOR: You do have a problem set due Tuesday. And that is part of your preparation for the exam. Here's a basic strategy for exams for this class. Anything that's on a problem set is fair game. Anything that's not covered on a problem set is not going to be fair game. If you haven't seen a new problem on it, broadly construed, then you won't-- I won't test you on a topic you haven't done problems on before. But I will take problems and ideas that you've studied before and spin them slightly differently to

make you think through them in real time on the exam. OK?

From my point of view, the purpose of these exams is not to give you a grade. I don't care about the grade. The purpose of these exams is to give you feedback on your understanding. It's very easy to slip through quantum mechanics and think, oh yeah, I totally-- I got this. This is fine. But it's not always an accurate read. So that's the point.

Did that answer your question? Other questions? Exam or-- yeah.

AUDIENCE: About the harmonic oscillator actually.

PROFESSOR: Excellent.

AUDIENCE: So when we solved it Tuesday using the series method, so there are two solutions technically, the even solution and the odd term solution. So did boundary conditions force the other one to be completely zero, like the coefficient in front of it? So there's like an A0 term which determines all the other ones. But there's an A0 term and an A1 term for the evens and odds. So did the other ones just have to be 0?

PROFESSOR: This is a really good question. This is an excellent question. Let me ask the question slightly differently. And tell me if this is the same question. When we wrote down our differential equation-- so last time we did the harmonic oscillator. And Barton did give you the brute force strategy for the harmonic oscillator. We want to find the energy eigenstates, because that's what we do to solve the Schrodinger equation. And we turn that into a differential equation. And we solve this differential equation by doing an asymptotic analysis and then a series expansion.

Now, this is a second order differential equation. Everyone agree with that? It's a second order differential equation. However, in our series expansion we ended up with one integration constant, not two. How does that work? How can it be that there was only one integration constant and not two? It's a second order differential equation. Is this he question?

AUDIENCE: Yeah.

PROFESSOR: OK, and this is an excellent question. Because it must be true, that there are two solutions. It cannot be that there is just one solution. It's a second order differential equation. Their existence in uniqueness theorems, which tell us there are two integration constants. So how can it possibly be that there was only one?

Well, we did something rather subtle in that series expansion. For that series expansion there was a critical moment, which I'm not going to go through but you can come to my office hours again, but just look through the notes. There's an important moment in the notes when we say, aha, these terms matter. But what we did is we suppressed a singular solution. There's a solution of that differential equation which is not well-behaved, which is not smooth, and in particular which diverges.

And we already did, from the asymptotic analysis, we already fixed that the asymptotic behavior was exponentially falling. But there's a second solution which is exponentially growing. So what we did, remember how we did this story? We took our wave function and we said, OK, look, we're going to pull off-- we're going to first asymptotic analysis. And asymptotic analysis tells us that either we have exponentially growing or exponentially shrinking solutions.

Let's pick the exponentially shrinking solutions. So phi e is equal to e to the minus x over 2a squared squared, times some-- I don't remember what Barton called it. I'll call it u of x. So we've extracted, because we know that asymptotically it takes this form. Well, it could also take the other form. It could be e to the plus, which would be bad and not normalizable. We've extracted that, and then we write down the differential equation for u. And then we solve that differential equation by series analysis, yeah?

However, if I have a secondary differential equation for phi, this change of variables doesn't change the fact that it's a secondary differential equation for u, right? There's still two solutions for u. One of those solutions will be the solution of the equation that has this asymptotic form. But the other solution will be one that has an e to the plus x squared over a squared so that it cancels off this leading factor and

gives me the exponentially growing solution. Everyone cool with that?

So in that series analysis there's sort of a subtle moment where you impose that you have the convergent solution. So the answer of, why did we get a first order relation, is that we very carefully, although it may not have been totally obvious, when doing this calculation one carefully chooses the convergent solution that doesn't have this function blowing up so as to overwhelm the envelope. That answer your question?

AUDIENCE: Yep.

PROFESSOR: Great. It's a very good question. This is an important subtlety that comes up all over the place when you do asymptotic analysis. I speak from my heart. It's an important thing in the research that I'm doing right now, getting these sorts of subtleties right. It can be very confusing. It's important to think carefully through them. So it's a very good question. Other questions before we move on? OK.

So I'm going to erase this, because it's not directly germane, but it is great. OK, so one of the lessons of this brute force analysis was that we constructed the spectrum, i.e., the set of energy eigenvalues allowed for the quantum harmonic oscillator, and we constructed the wave functions. We constructed the wave functions by solving the differential equation through asymptotic analysis, which give us the Gaussian envelope, and series expansion, which give us the Hermite polynomials. And then there's some normalization coefficient.

And then we got the energy eigenvalues by asking, when does this series expansion converge? When does it, in fact truncate, terminate, so that we can write down an answer? And that was what gave us these discrete values. But fine, we can see that it would be discrete values. We're cool with that. In fact, Barton went through the discussion of the node theorem and the lack of degeneracy in one dimensional quantum mechanics. So it's reasonable that we get a bunch of discrete energy eigenvalues, as we've talked about now for two lectures.

However, there's a surprise here, which is that these aren't just discrete, they're

evenly spaced. We get a tower, starting with the lowest possible energy corresponding to a-- sorry, E0-- starting with the lowest possible energy, which is greater than 0, and a corresponding ground state wave function. And then we have a whole bunch of other states, phi 1, phi 2, phi 3, phi 4, labeled by their energies where the energies are evenly spaced.

They needed to be discrete, because these are bound states. But evenly spaced is a surprise. So why are they evenly spaced? Anyone, based on the last lecture's analysis? Yeah, you don't have a good answer to that from last lecture's analysis. It's one of the mysteries that comes out of the first analysis. When you take a differential equation, you just beat the crap out of it with a stick by solving it. With differential equations strategies like this you don't necessarily get some of the more subtle structure.

One of the goals of today's lecture is going to be to explain why we get this structure. Why just from the physics of the problem, the underlying physics, should you know that the system is going to have evenly spaced eigenvalues? What's the structure?

And secondly, I want to show you a way of repeating this calculation without doing the brute force analysis that reveals some of that more fine grain structure of the problem. And this is going to turn out to be one of the canonical moves in the analysis of quantum mechanical systems. So from quantum mechanics to quantum field theory this is a basic series of logic moves.

What I'm going to do today also has an independent life in mathematics, in algebra. And that will be something you'll studying in more detail in 8.05, but I would encourage you to ask your recitation instructors about it, or me in office hours.

So our goal is to understand that even spacing and also to re-derive these results without the sort of brutal direct assault methods we used last time. So what I'm going to tell you about today is something called the operator method. It usually goes under the name of the operator method.

To get us started let's go back to look at the energy operator for the harmonic oscillator, which is what, at the end of the day, we want to solve. p squared over 2m plus m omega squared upon 2, x squared. And this is the operator that we want to-- whose eigenvalues we want to construct, whose eigenfunctions we want to construct. Before we do anything else, we should do dimensional analysis.

First thing you when you look at a problem is do some dimensional analysis. Identify the salient scales and make things, to the degree possible, dimensionless. Your life will be better. So what are the parameters we have available to us? We have h bar, because it's quantum mechanics. We have m, because we have a particle of mass, m. We have omega, because this potential has a characteristic frequency of omega. What other parameters do we have available to us? Well, we have c. That's available to us. But is it relevant? No. If you get an answer that depends on the speed of light, you made some horrible mistake. So not there.

What about the number of students in 8.04? No. There are an infinite number of parameters that don't matter to this problem. What you want to know is, when you do dimensional analysis, what parameters matter for the problem. What parameters could possibly appear during the answer? And that's it. There are no other parameters in this problem.

So that's a full set of parameters available to us. This has dimensions of momentum times length. This has dimensions of mass, and this has dimensions of one upon the time. And so what characteristic scales can we build using these three parameters? Well, this is a moment times a length. If we multiply by a mass, that's momentum times mass times x, which is almost momentum again. We need a velocity and not a position, but we have 1 over time.

So if we take h bar times and omega, so that's px times m over t, that has units of momentum squared. And similarly, this is momentum which is x, which is length mass over time. I can divide by mass and divide by frequency or multiply by time, so h bar upon m omega. And this is going to have units of length squared. And with a little bit of foresight from factors of two I'm going to use these to define two link

scales. x0 is equal to h bar-- I want to be careful to get my coefficients. I always put the two in the wrong place. So 2 h bar upon m omega. Square root. And I'm going to define p0 as equal to square root of 2 h bar times m omega.

So here's my claim. My claim is at the end of the day the salient link scales for this problem should be integers or dimensionless numbers times this link scale. And salient momentum scales should be this scale. Just from dimensional analysis. So if someone at this point says, what do you think is the typical scale, what is the typical size of the ground state wave function, the typical link scale over which the wave function is not 0? Well, that can't possibly be the size of Manhattan. It's not the size of a proton. There's only one link scale associated with the system. It should be of order x0. Always start with dimensional analysis. Always.

OK, so with that we can rewrite this energy. Sorry, and there's one last energy. We can write an energy, the thing with energy, which is equal to h bar omega. And this times a frequency gives us an energy.

So we can rewrite this energy operator as h bar omega times p squared over p0. So this has units of energy. So everything here must be dimensionless. And it turns out to be p squared over p0 squared plus x operator squared over x0 squared. So that's convenient. So this has nothing to do with the operator method. This is just being reasonable.

Quick thing to note, x0 times p0 is just-- the m omegas cancel, so we get root 2h bar squared, 2 h bar. Little tricks like that are useful to keep track of as you go.

So we're interested in this energy operator. And it has a nice form. It's a sum of squares. And we see the sum of squares, a very tempting thing to do is to factor it. So for example, if I have two classical numbers, c squared plus d squared, the mathematician in me screams out to write c minus id times c plus id. I have factored this. And that's usually a step in the right direction. And is this true? Well yes, it's true. c squared plus d squared and the cross terms cancel. OK, that's great. Four complex numbers, or four C numbers.

Now is this true for operators? Can I do this for operators? Here we have the energy operator as a sum of squares. Well, let's try it. I'd like to write that in terms of x and p. So what about writing the quantity x minus ip over x0 over p0, operator, times x over x0 plus ip over p0.

We can compute this. This is easy. So the first term gives us the x squared over x0 squared. That last term gives us-- the i's cancel, so we get p squared over p0, squared. But then there are cross terms. We have an xp and a minus px, with an overall i. So plus i times xp over x0 times p0. x0 times p0, however, is 2 h bar. So that's over 2 h bar. And then we have the other term, minus px. Same thing. So I could write that as a commutator, xp minus px. Everyone cool with what?

Unfortunately this is not what we wanted. We wanted just p squared plus x squared. And what we got instead was p squared plus x squared close plus a commutator. Happily this commutator is simple. What's the commutator of x with p?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, exactly. Commit this to memory. This is your friend. So this is just i h bar, so this is equal to ditto plus ditto plus i h bar. Somewhere I got a minus sign. Where did I get my minus sign wrong? x with ip. Oh now, good. This is good. So x with p is i-- so we get an i h bar. No, I really did screw up the sign. How did I screw up the sign? No I didn't. Wait. Oh! Of course. No, good. Sorry, sorry. Trust your calculation, not your memory.

So the calculation gave us this. So what does this give us? It gives us i h bar. So plus. But the i h bar times i is going to give me a minus. And the h bar is going to cancel, because I've got an h bar from here and an h bar at the denominator minus 1/2. So this quantity is equal to the quantity we wanted minus 1/2. And what is the quantity we wanted, x0 squared plus p0 squared? This guy.

So putting that all together we can write that the energy operator, which was equal to h bar omega times the quantity we wanted, is equal to-- well, the quantity we wanted is this quantity plus 1/2. h bar omega times-- I'll write this as x over x0 plus

ip over p0, x over x0 plus ip over p0, hat, hat, hat, plus 1/2. Everyone cool with that? So it almost worked. We can almost factor.

So at this point it's tempting to say, well that isn't really much an improvement. You've just made it uglier. But consider the following. And just trust me on this one, that this is not a stupid thing to do. That's a stupid symbol to write, though.

So let's define an operator called a, which is equal to x over x0 plus ip over p0, and an operator, which I will call a dagger. "Is that a dagger I see before me?" Sorry. x over x0 minus ip over p0. *Hamlet* quotes are harder. So this is a dagger.

And we can now write the energy operator for the harmonic oscillator is equal to h bar omega times a dagger a plus 1/2. Everyone cool with that? Now, this should look suggestive. You should say, aha, this looks like h bar omega something plus 1/2. That sure looks familiar from our brute force calculation. But, OK, that familiarity is not an answer to the question.

Meanwhile you should say something like this. Look, this looks kind of like the complex conjugate of this guy. Because there's an i and you change the sign of the i. But what is the complex conjugate of an operator? What does that mean? An operator is like take a vector and rotate. What is the complex conjugation of that? I don't know. So we have to define that.

So I'm now going to start with this quick math aside. And morally, this is about what is the complex conjugate of an operator. But before I move on, questions? OK.

So here's a mathematical series of a facts and claims. I claim the following. Given any linear operator we can build-- there's a natural way to build without making any additional assumptions or any additional ingredients. We can build another operator, o dagger, hat, hat, in the following way. Consider the inner product of f with g, or the bracket of f with g. So integral dx of f complex conjugate g.

Consider the function we're taking here is actually the operator we have on g. I'm going to define my-- so this is a perfectly good thing. What this expression says is, take your function g. Act on it with the operator o. Multiply by the complex

conjugate. Take the integral. This is what we would have done if we had taken the inner product of f with the function we get by taking o and acting on the function g.

So here's the thing. What we want-- just an aside-- what we want to do is define a new operator. And here's how I'm going to define it. We can define it by choosing how it acts. I'm going to tell you exactly how it acts, and then we'll define the operator.

So this operator, o with a dagger, called the adjoint, is defined in the following way. This is whatever operator you need, such that the integral gives you-- such that the following is true. Integral dx o on f, complex conjugate, g. So this is the definition of this dagger action, the adjoint action. OK so o dagger is the adjoint. And sometimes it's called the Hermetian adjoint. I'll occasionally say Hermetian and occasionally not, with no particular order to it.

So what does this mean? This means that whatever o dagger is, it's that operator that when acting on g and then taking the inner product with f gives me same answer as taking my original operator and acting on f and taking the inner product with g. Cool? So we know how-- if we know what our operator o is, the challenge now is going to be to figure out what must this o dagger operator be such that this expression is true. That's going to be my definition of the adjoint. Cool? So I'm going to do a bunch of examples. I'm going to walk through this.

So the mathematical definition is that an operator o defined in this fashion is the Hermetian adjoint of o. So that's the mathematical definition. Well, that's our version of the mathematical definition. I just came back from a math conference, so I'm particularly chastened at the moment to be careful.

So let's do some quick examples. Example one. Suppose c is a complex number. I claim a number is also an operator. It acts by multiplication. The number 7 is an operator because it takes a vector and it gives you 7 times that vector. So this number is a particularly simple kind of operator. And what's the adjoint? We can do that. That's easy.

So c adjoint is going to be defined in the following way. It's integral dx f star of c adjoint g is equal to the integral dx of c on f complex conjugate times g. But what is this? Well, c is just a number. So when we take its complex conjugate we can just pull it out. So this is equal to the integral dx c complex conjugate, f star g. But I'm now going to rewrite this, using the awesome power of reordering multiplication, as c star. And I'm going to put parentheses around this because it seems like fun.

So now we have this nice expression. The integral dx of f c adjoint g is equal to the integral dx of f c star g, c complex conjugate g. But notice that this must be true for all f. It's true for all. Because I made no assumption about what f and g are, true for all f and g. And therefore the adjoint of a complex number is its complex conjugate.

And this is the basic strategy for determining the adjoint of any operator. We're going to play exactly this sort of game. We'll put the adjoint in here. We'll use the definition of the adjoint. And then we'll do whatever machinations are necessary to rewrite this as some operator acting on the first factor. Cool? Questions?

OK, let's do a more interesting operator. By the way, to check at home, and I think this might be on your problem set-- but I don't remember if it's on or not. So if it's not, check this for yourself. Check that the adjoin of the adjoint is equal to the operator itself. It's an easy thing to check.

So next example. What is the adjoint of the operator derivative with respect to x? Consider the operator, which is just derivative with respect to x. And I want to know what is the adjoint of this beast. So how do we do this? Same logic as before. Whatever the operator is, it's defined in the following way. Integral dx, f complex conjugate on dx dagger on g. This is equal to the integral dx of-- how we doing on time? Good-- integral dx of dx, f complex conjugate on g.

Now, what we want is we want to turn this into an expression where the operator is acting on g, just as our familiar operator ddx. So how do I get the ddx over here? I need to do two things. First, what's the complex conjugate of the derivative with respect to x of a complex function? MIT has indigestion. So this is integral dx, derivative with respect to x of f complex conjugate, g.

And now I want this operator. I want derivative acting on g. That's the definition. Because I want to know what is this operator. And so I'm going to do integration by parts. So this is equal to the integral, dx. When I integrate by parts I get an F complex conjugate, and then an overall minus sign from the integration by parts minus f complex conjugate dx g. Was I telling you the truth earlier? Or did I lie to you? OK, keep thinking about that.

And this is equal to, well the integral of dx, f complex conjugate if minus the derivative with respect to x acting on g. Everyone cool with that? But if you look at these equalities, dx adjoint acting on g is the same as minus dx acting on g. So this tells me that the adjoint of dx is equal to minus dx. Yeah?

AUDIENCE: Are you assuming that your surface terms vanish?

PROFESSOR: Thank you! I lied to you. So I assumed in this that my surface terms vanished. I did a variation by parts. And that leaves me with a total derivative. And that total derivative gives me a boundary term. Remember how integration by parts works. Integration by parts says the integral of AdxB is equal to the integral of-- well, AdxB can be written as derivative with respect to x of AB minus B derivative with respect to x of A. Because this is A prime B plus B prime A. Here we have AB prime. So we just subtract off the appropriate term.

But this is a total derivative. So it only gives us a boundary term. So this integral is equal to-- can move the integral over here-- the integral and the derivative, because an integral is nothing but an antiderivative. The integral and the derivative cancel, leaving us with the boundary terms. And in this case, it's from our boundaries which are minus infinity plus infinity, minus infinity and plus infinity.

Now, this tells us something very important. And I'm not going to speak about this in detail, but I encourage the recitation instructors who might happen to be here to think to mention this in recitation. And I encourage you all to think about it. If I ask you, what is the adjoint of the derivative operator acting on the space of functions which are normalizable, so that they vanish at infinity, what is the adjoint of the

derivative operator acting on the space of functions which is normalizable at infinity? We just derive the answer. Because we assume that these surface terms vanish. Because our wave functions, f and g, vanish at infinity. They're normalizable.

However, if I had asked you a slightly different question, if I had asked you, what's the adjoint of the derivative operator acting on a different set of functions, the set of functions that don't necessarily vanish at infinity, including sinusoids that go off to infinity and don't vanish. Is this the correct answer? No. This would not be the correct answer, because there are boundary terms.

So the point I'm making here, first off, in physics we're always going to be talking about normalizable beasts. At the end of the day, the physical objects we care about are in a room. They're not off infinity. So everything is going to be normalizable. That is just how the world works. However, you've got to be careful in making these sorts of arguments and realize that when I ask you, what is the adjoint of this operator, I need to tell you something more precise. I need to say, what's the adjoint of the derivative acting when this operator's understood as acting on some particular set of functions, acting on normalizable functions? Good. So anyway, I'll leave that aside as something to ponder.

But with that technical detail aside, as long as we're talking about normalizable functions so these boundary terms from the integration by parts cancel, the adjoint of the derivative operator is minus the derivative operator. Cool? OK, let's do another example.

And where do I want to do this? I'll do it here. So another example. Actually, no. I will do it here.

So we have another example, which is three. What's the adjoint of the position operator? OK, take two minutes. Do this on a piece of paper in front of you. I'm not going to call on you. So you can raise you hand if you-- OK, chat with the person next to you. I mean chat about physics, right? Just not-- [LAUGHS].

AUDIENCE: [CHATTING]

PROFESSOR: OK, so how do we go about doing this? We go about solving this problem by using the definition of the adjoint. So what is x adjoint? It's that operator such that the following is true, such that the integral dx of f complex conjugate with x dagger acting on g is equal to the integral dx of what I get by taking the complex conjugate of taking x and acting on f and then integrating this against g. But now we can use the action of x and say that this is equal to the integral dx of x f complex conjugate g. But here's the nice thing. What is the complex conjugate of f times the complex function of f? x is real. Positions are real. So that's just x times the complex conjugate of f. So that was essential move there.

And now we can rewrite this as equal the integral of f complex conjugate xg. And now, eyeballing this, x dagger is that operator which acts by acting by multiplying with little x. Therefore, the adjoint of the operator x is equal to the same operator. x is equal to its own adjoint. OK? Cool?

So we've just learned a couple of really nice things. So the first is-- where we I want to do this? Yeah, good. So we've learned a couple of nice things. And I want to encode them in the following definition. Definition-- an operator, which I will call o, whose adjoint is equal to o, so an operator whose adjoint is equal to itself is called Hermetian. So an operator which is equal to its own adjoint is called Hermetian.

And so I want to note a couple of nice examples of that. So note a number which is Hermetian is what? Real. An operator-- we found an operator which is equal to its own adjoint. x dagger is equal to x. And what can you say about the eigenvalues of this operator? They're real. We use that in the proof, actually. So this is real.

I will call an operator real if it's Hermetian. And here's a mathematical fact, which is that any operator which is Hermetian has all real eigenvalues. So this is really-- I'll state it as a theorem, but it's just a fact for us. o has all real eigenvalues.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah?

AUDIENCE: Is it if and only [INAUDIBLE]?

PROFESSOR: No. Let's see. If you have all real eigenvalues, it does not imply that you're Hermetian. However, if you have all real eigenvalues and you can be diagonalized, it does imply. So let me give you an example.

So consider the following operator. We've done this many times, rotation in real three-dimensional space of a vector around the vertical axis. It has one eigenvector, which is the vertical vector. And the eigenvalue is 1, so it's real. But that's not enough to make it Hermetian. Because there's another fact that we haven't got to yet with Hermetian operators, which is going to tell us that a Hermetian operator has as many eigenvectors as there are dimensions in the space, i.e., that the eigenvectors form a basis.

But there's only one eigenvector for this guy, even though we're in a three-dimensional vector space. So this operator, rotation by an angle theta, is not Hermetian, even though its only eigenvalue isn't in fact real. So it's not an only if. If you are Hermetian, your eigenvalues are all real. And you'll prove this on a problem set. Yeah?

AUDIENCE: If you're Hermetian, are your eigenfunctions normal?

PROFESSOR: Not necessarily. But they can be made normal. We'll talk about this in more detail later. OK. Let's do a quick check, last example. And I'm not actually going to go through this in detail, but what about p? What about the momentum operator? First off, do you think the momentum is real? It sure would be nice. Because its eigenvalues are the observable values of momentum. And so its eigenvalues should all be real. Does that make it Hermetian? Not necessarily, but let's check.

So what is the adjoint of p? Well, this actually we can do very easily. And I'm not going to go through an elaborate argument. I'm just going to know the following. p is equal to h bar upon i ddx. And this is an operator. This is an operator. So what's the adjoint of this operator? Well, this under an adjoint gets a minus sign, right? It's itself up to a minus sign. So is the derivative Hermetian? No, it's in fact what we anti-

Hermetian. Its adjoint is minus itself.

What about i? What's its adjoint? Minus i. Sweet. So this has an adjoint, picks up a minus. This has an adjoint, picks up a minus. The minuses cancel. p adjoint is p. So p is in fact Hermetian. And here's a stronger physical fact. So now we've seen that each of the operators we built is x and p, true of the operators we've looked at so far is Hermetian, those that correspond to physical observables.

Here's a physical fact. All the observables you measure with sticks are real. And the corresponding statement is that all operators corresponding to observables, all operators must be Hermetian. To the postulate that says, "Observables are represented by operators," should be adjoined the word "Hermetian." Observables are represented in quantum mechanics by Hermetian operators, which are operators that have a number of nice properties, including they have all real eigenvalues. Cool? OK. Questions? Yeah.

AUDIENCE: If it has to be Hermetian and not just have real eigenvalues, does that mean the eigenvalues always need to form some kind of basis?

PROFESSOR: Yeah, the eigenvectors will. This is connected to the fact we've already seen. If you take an arbitrary wave function you can expand it in states with definite momentum as a superposition. You can also expand it in a set of states of definite energy or of definite position. Anytime you have a Hermetian operator, its eigenvectors suffice to expand any function. They provide a basis for representing any function.

So that's the end of the mathematical side. Let's get back to this physical point. So we've defined this operator a and this other operator a dagger. And here's my question first. Is a Hermetian? No. That's Hermetian. That's Hermetian. But there's an i. That i will pick up the minus sign when we do the complex conjugation. Oh, look. Sure was fortuitous that I called this a dagger, since this is equal to a dagger.

So this is the adjoint of a. So this immediately tells you something interesting. x and p are both observables. Does a correspond to an observable? Is it Hermetian? Every intervals is associated to a Hermetian operator. This is not Hermetian. So a

does not represent an observable operator.

And I will post notes on the web page, which give us a somewhat lengthy discussion-- or it might be in one of the solutions-- a somewhat lengthy discussion of what it means for a and a dagger to not be observable. You'll get more discussion of that there.

Meanwhile, if a is not observable, it's not Hermetian, does it have real eigenvalues? Well, here's an important thing. I said if you're Hermetian, all the eigenvalues are real. If you're not Hermetian, that doesn't tell you you can't have any real eigenvalues. It just says that I haven't guaranteed for you that all the eigenvalues are real. So what we'll discover towards the end of the course when we talk about something called coherent states is that in fact, a does have a nice set of eigenvectors. They're very nice. They're great. We use them for lasers. They're very useful. And they're called coherent states. But their eigenvalues are not in general real. They're generically complex numbers. Are they things you can measure? Not directly. They're related to things you can measure, though, in some pretty nice ways.

So why are we bothering with these guys if they're not observable? Yeah.

AUDIENCE: E [INAUDIBLE].

PROFESSOR: Yeah, good. Excellent. That's really good. So two things about it. So one thing is this form for the energy operator is particularly simple. We see the 1/2. This looks suggestive from before. But it makes it obvious that E is Hermetian. And that may not be obvious to you guys. So let's just check.

Here's something that you'll show on the problem set. AB adjoint is equal to B adjoint A adjoint. The order matters. These are operators. And so if we take the adjective of this, what's this going to give us? Well we change the order. So it's going to be a dagger, and then we take the dagger of both of the a dagger. So this is self-adjoint, or Hermetian. So that's good. Of course, we already knew that, because we could have written it in terms of x and p. But this is somehow simpler.

And it in particular emphasizes the form, or recapitulates the form of the energy eigenvalues. Why else would we care about a and a dagger?

OK, now this is a good moment. Here's the second reason. So the first reason you care is this sort of structural similarity and the fact that it's nicely Hermetian in a different way. Here's the key thing. Key. a and a dagger satisfy the simplest commutation relation in the world. Well, the second simplest. The simplest is that it's 0 on the right-hand side. But the simplest not trivial commutation relationship. a with a dagger is equal to-- so what is a dagger equal to? We just take the definition. Let's put this in.

So this is x over x0 plus ip over p0, comma, x over x0 minus i, p over p0, hat, hat, hat, hat, bracket, bracket. Good. So here there are going to be four terms. There's x commutator x. What is that? What is the commutator of an operator with itself? 0. Because remember the definition of the commutator A, B is AB minus BA. So A with A is equal to AA minus AA. And you have no options there. That's 0.

So x with x is 0. p with p is 0. So the only terms that matter are the cross terms. We have an x with p. And notice that's going to be times a minus i with p0 and x0. And then we have another term which is p with x, which is i, p0 over x0. So you change the order and you change the sign. But if you change the order of a commutator, you change the side.

So we can put them both in the same order. Let me just write this out. So this is i over x0 p0. So this guy, minus i over x0p0. But x0p0 is equal to 2h bar, as we checked before. This was x with p. And then the second term was plus i, again over x0p0, which is 2h bar, p with x.

This x with p is equal to? i h bar. So the h bar cancels. The i gives me a plus 1. And p with x gives me minus i h bar. So the h bar and the minus i gives me plus 1. Well that's nice. This is equal to 1. So plus 1/2, therefore a with a dagger is equal to 1. As advertised, that is about as simple as it gets.

Notice a couple of other commutators that follow from this. a dagger with a is equal

to minus 1. We just changed the order. And that's just an overall minus sign. And a with a is what? 0. a dagger with a dagger? Good. OK.

So we are now going to use this commutation relation to totally crush the problem into submission. It's going to be weeping before us like the Romans in front of the Visigoths. It's going to be dramatic. OK, so let's check.

So let's combine the two things. So we had the first thing is that this form is simple. The second is that the commutator is simple. Let's combine these together and really milk the system for what it's got. And to do that, I need two more commutators. And the lesson of this series of machinations, it's very tempting to look at this and be like, why are you doing this? And the reason is, I want to encourage you to see the power of these commutation relations. They're telling you a tremendous amount about the system. So we're going through and doing some relatively simple calculations. We're just computing commutators. We're following our nose. And we're going to derive something awesome. So don't just bear with it. Learn from this, that there's something very useful and powerful about commutation relations. You'll see that at the end. But I want you to on to the slight awkwardness right now, that it's not totally obvious beforehand where this is going.

So what is E with a? That's easy. It's the h bar omega a dagger a plus 1/2. So the 1/2, what's 1/2 commutator with an operator? 0. Because any number commutes with an operator. 1/2 operator is operator 1/2. It's just a constant.

That term is gone. So the only thing that's left over is h bar omega, a dagger a with a. The h bar omega's just a constant. It's going to pull out no matter which term we're looking at. So I could just pull that factor out.

So this is equal to h bar omega times a dagger a minus a a dagger a. But this is equal to h bar omega-- well, that's a dagger a a, a a dagger a. You can just pull out the a on the right. a dagger a minus a a dagger a. That's equal to h bar omega. Well, a dagger with a is equal to a dagger with a minus 1 is equal to minus h bar omega. And we have this a leftover, a.

So E with a is equal to minus a. Well, that's interesting. Now, the second commutator-- I'm not going to do it-- E with a dagger is going to be equal to-- let's just eyeball what's going to happen. They can be a dagger. So we're going to have a dagger a dagger minus a dagger a dagger a. So we're going to have an a dagger in front and then a dagger. So all we're going to get is a sign. And it's going to be a dagger plus a dagger. I shouldn't written that in the center. Everyone cool with that? Yeah.

AUDIENCE: The h bar where?

PROFESSOR: Oh shoot, thank you! h bar here. Thank you. We would have misruled the galaxy. OK, good. Other questions?

You don't notice-- you haven't noticed yet, but we just won. We just totally solved the problem. And here's why. Once you see this, any time you see this, anytime you see this commutator, an operator with an a is equal to plus a times some constant, anytime you see this, cheer. And here's why. Yeah, right. Exactly. Now. Whoo!

Here's why. Here's why you should cheer. Because you no longer have to solve any problems. You no longer have to solve any differential equations. You can simply write down the problem. And let's see that you can just write down the answer.

Suppose that we already happened to have access-- here in my sleeve I have access to an eigenfunction of the energy operator. E on phi E is equal to E phi E. Suppose I have this guy. Cool? Check this out. Consider a new state, psi, which is equal to a-- which do I want to do first? Doesn't really matter, but let's do a.

Consider psi is equal to a on phi E. What can you say about this state? Well, it's the state you get by taking this wave function and acting with a. Not terribly illuminating. However, E on psi is equal to what? Maybe this has some nice property under acting with E. This is equal to E on a with pfi E.

Now, this is tantalizing. Because at this point it's very-- look, that E, it really wants to hit this phi. It just really wants to. There's an E it wants to pull out. It'll be great. The problem is it's not there. There's an a in the way. And so at this point we add 0. And

this is a very powerful technique. This is equal to Ea minus aE plus aE, phi E.

But that has a nice expression. This is equal to Ea minus aE. That's the commutator of E with a. Plus a. What's E acting on phi E? Actually, let me just leave this as aE. So what have we done here before we actually act? What we've done is something called commuting an operator through. So what do I mean by commuting an operator through? If we have an operator A and an operator B and a state f, and I want A to act on f, I can always write this as-- this is equal to the commutator of A would be plus BA acting on f.

So this lets me act A on f directly without B. But I have to know what the commutator of these two operators is. So if I know what the commutator is, I can do this. I can simplify. When one does this, when one takes AB and replaces it by the commutator of A with B, plus BA, changing the order, the phrase that one uses is I have commuted A through B. And commuting operators through other is an extraordinarily useful tool, useful technique.

Now let's do y. So here what's the commutator of E with a? We just did that. It's minus h bar omega a. And what's aE on phi E? What's E on phi E? E. Exactly. Plus Ea on phi. And now we're cooking with gas. Because this is equal to minus h bar omega a plus Ea, hat. I'm going to pull out this common factor of a.

So if I pull out that common factor of a, plus E, a phi E, and now I'm going to just slightly write this instead of minus h bar omega plus E, I'm going to write this as E minus h bar omega. I'm just literally changing the order of the algebra. E minus h bar omega. And what is aE? Psi. That was the original state we started with, psi.

Well, that's cool. If I have a state with energy E and I act on it with the operator a, I get a new state, psi, which is also an eigenstate of the energy operator, but with a slightly different energy eigenvalue. The eigenvalue is now decreased by h bar omega. Cool?

And that is what we wanted. Let's explore the consequences of this. So if we have a state with eigenvalue E, we have phi E such that E on phi E is equal to E phi E.

Then the state a phi E has eigenvalue as energy, eigenvalue E minus h bar omega.

So I could call this phi sub E minus h bar omega. It's an eigenfunction of the energy operator, the eigenvalue, E minus h bar omega. Agreed? Do I know that this is in fact properly normalized? No, because 12 times it would also be a perfectly good eigenfunction of the energy operator. So this is proportional to the properly normalized guy, with some, at the moment, unknown constant coefficient normalization. Everyone cool with that?

So now let's think about what this tells us. This tells us if we have a state phi E, which I will denote its energy by this level, then if I act on it with a phi E I get another state where the energy, instead of being E, is equal E minus h bar omega. So this distance in energy is h bar omega. Cool? Let me do it again.

We'll tack a on phi E. By exactly the same argument, if I make psi as equal to a on a phi E, a squared phi E, I get another state, again separated by h bar omega, E minus 2h bar omega. Turtles all the way down. Everyone cool with that?

Let's do a slightly different calculation. But before we do that, I want to give a a name. a does something really cool. When you take the state phi E that has definite energy E, it's an energy eigenfunction, and you act on it with a, what happens? It lowers the energy by h bar omega. So I'm going to call a the lowering operator. Because what it does is it takes a state with phi E, with energy eigenvalue E to state with energy E minus h bar omega. And I can just keep doing this as many times as I like and I build a tower. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Very good question. Hold on to that for a second. We'll come back to that in just a second. So this seems to build for me a ladder downwards. Everyone cool with that? But we could have done the same thing with a dagger. And how does this story change? What happens if we take a dagger instead of a? Well, let's go through every step here. So this is going to be E on a dagger. And now we have E a dagger, a dagger, E, a dagger.

What's E with a dagger? E with a dagger is equal to same thing but with a plus. And again, psi. Same thing, because the a dagger factors out. Yeah? So we go down by acting with a. We go up by acting with a dagger. And again, the spacing is h bar omega. And we go up by acting with a dagger again.

So a and a dagger are called the raising and lowering operators. a dagger, the raising operator. a dagger phi E plus h bar omega. So what that lets us do is build a tower of states, an infinite number of states where, given a state, we can walk up this ladder with the raising operator, and we can walk down it by the lowering operator.

So now I ask you the question, why is this ladder evenly spaced? There's one equation on the board that you can point to-- I guess two, technically-- there are two equations on the board that you could point to that suffice to immediately answer the question, why is the tower of energy eigenstates evenly spaced. What is that equation?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, those commutators. These commutators are all we needed. We didn't need to know anything else. We didn't even need to know what the potential was. If I just told you there's an energy operator E and there's an operator a that you can build out of the observables of the system, such that you have this commutation relation, what do you immediately know? You immediately know that you get a tower of operators. Because you can act with a and raise the energy by a finite amount, which is the coefficient of that a in the commutator.

This didn't have to be the quantum mechanics of the harmonic oscillator at this point. We just needed this commutator relation, E with a, E with a dagger. And one of the totally awesome things is how often it shows up. If you take a bunch of electrons and you put them in a magnetic field, bunch of electrons, very strong magnetic field, what you discover is the quantum mechanics of those guys has nothing to do with the harmonic oscillator on the face if it's magnetic fields, Lorentz force law, the whole thing.

What you discover is there's an operator, which isn't usually called a, but it depends on which book you use-- it's n or m or l-- there's an operator that commutes with the energy operator in precisely this fashion, which tells you that the energy eigenstates live in a ladder. They're called Landau levels. This turns out to be very useful. Any of you who are doing a [INAUDIBLE] in the lab that has graphene or any material, really, with a magnetic field, then this matters.

So this commutator encodes an enormous amount of the structure of the energy eigenvalues. And the trick for us was showing that we could write the harmonic oscillator energy operator in terms of operators that commute in this fashion. So we're going to run into this structure over and over again. This operator commutes with this one to the same operator times a constant that tells you have a ladder. We're going to run into that over and over again when we talk about Landau levels, if we get there. When we talk about angular momentum we'll get the same thing. When we talk about the harmonic oscillator we'll get the same thing. Sorry, the hydrogen system. We'll get the same thing.

So second question, does this ladder extend infinitely up? Yeah, why not? Can it extend infinitely down?

AUDIENCE: Nope.

PROFESSOR: Why?

AUDIENCE: Ground state.

PROFESSOR: Well, people are saying ground state. Well, we know that from the brute force calculation. But without the brute force calculation, can this ladder extend infinitely down?

AUDIENCE: [INAUDIBLE] you can't go [INAUDIBLE].

PROFESSOR: Brilliant. OK, good. And as you'll prove on the problems, that you can't make the energy arbitrarily negative. But let me make that sharp. I don't want to appeal to

something we haven't proven. Let me show you that concretely. In some state, in any state, the energy expectation value can be written as the integral of phi complex conjugate-- we'll say in this state phi-- phi complex conjugate E phi.

But I can write this as the integral, and let's say dx, integral dx. Let's just put in what the energy operator looks like. So psi tilda, we can take the 4a transfer and write the psi tilda p, p squared upon 2m-- whoops, dp-- for the kinetic energy term, plus the integral-- and now I'm using the harmonic oscillator-- plus the integral dx of psi of x norm squared, norm squared, m omega squared upon 2x squared. Little bit of a quick move there, doing the 4a transfer for the momentum term and not doing the 4a [INAUDIBLE] but it's OK. They're separate integrals. I can do this.

And the crucial thing here is, this is positive definite. This is positive definite, positive definite, positive definite. All these terms are strictly positive. This must be greater than or equal to 0. It can never be negative. Yeah? So what that tells us is there must be a minimum E. There must be a minimum energy. And I will call it minimum E0. We can't lower the tower forever.

So how is this possible? How is it possible that, look, on the one hand, if we want, if we have a state, we can always build a lower energy state by acting with lowering operator a. And yet this is telling me that I can't. There must be a last one where I can't lower it anymore. So what reaches out of the chalkboard and stops me from acting with a again? How can it possibly be true that a always lowers the eigenfunction but there's at least one that can't be lowered any further. Normalizable's a good guess. Very good guess. Not the case. Because from this argument we don't even use wave functions.

AUDIENCE: [INAUDIBLE]

PROFESSOR: That would be bad. Yes, exactly. So that would be bad, but that's just saying that there's an inconsistency here. So I'm going to come back to your answer, a non-normalizable. It's correct, but in a sneaky way. Here's the way it's sneaky.

Consider a state a on phi-- let's say this is the lowest state, the lowest possible

state. It must be true that the resulting statement is not phi minus h bar omega. There can't be any such state. And how can that be? That can be true if it's 0. So if the lowering operator acts on some state and gives me 0, well, OK, that's an eigenstate. But it's a stupid eigenstate. It's not normalizable. It can't be used to describe any real physical object. Because where is it? Well, it's nowhere. The probability density, you'd find it anywhere. It's nowhere, nothing, zero.

So the way that this tower terminates is by having a last state, which we'll call phi 0, such that lowering it gives me 0. Not the state called 0, which I would call this, but actually the function called 0, which is not normalizable, which is not a good state. So there's a minimum E0. Associated with that is a lowest energy eigenstate called the ground state.

Now, can the energy get arbitrarily large? Sure. That's a positive definite thing, and this could get as large as you like. There's no problem with the energy eigenvalues getting arbitrarily large. We can just keep raising and raising and raising. I mention that because later on in the semester we will find a system with exactly that commutation relation, precisely that commutation relation, where there will be a minimum and a maximum.

So the communication relation is a good start, but it doesn't tell you anything. We have to add in some physics like the energy operators bounded below for the harmonic oscillator. Questions at this point? Yeah?

AUDIENCE: So you basically [INAUDIBLE] this ladder has to [INAUDIBLE] my particular energy eigenstate and I can kind of construct a ladder. How do I know that I can't construct other, intersecting ladders?

PROFESSOR: Yeah, that's an excellent question. I remember vividly when I saw this lecture in 143A, and that question plagued me. And foolishly I didn't ask it. So here's the question. The question is, look, you found a bunch of states. How do you know that's all of them? How do you know that's all of them? So let's think through that. That's a very good question. I'm not going to worry about normalization. There's a discussion of normalization in the notes.

How do we know that's all of them? That's a little bit tricky. So let's think through it. Imagine it's not all of them. In particular, what would that mean? In order for there to be more states than the ones that we've written down, there must be states that are not on that tower. And how can we possi-- wow, this thing is totally falling apart. How do we do that? How is that possible?

There are two ways to do it. Here's my tower of states. I'll call this one phi 0. so I raise with a dagger and I lower with a. So how could it be that I missed some states? Well, there are ways to do it. One is there could be extra states that are in between. So let's say that there's one extra state that's in between these two. Just imagine that's true.

If there is such a state, by that commutation relation there must be another tower. So there must be this state, and there must be this state, and there must be this state, and there must be this state. Yeah? OK, so that's good so far.

But what happens? Well, A on this guy gave me 0. And this is going to be some phi tilde 0. Suppose that this tower ends. And now you have to ask the question, can there be two different states with two different energies with a0? Can there be two different states that are annihilated by a0? Well, let's check.

What must be true of any state annihilated by a0? Well, let's write the energy operator acting on that state. What's the energy of that state? Energy on phi 0 is equal to h bar omega. This is a very good question, so let's go through it. So it's equal to h bar omega times a dagger a plus 1/2 on phi 0. But what can you say about this? Well, a annihilates phi 0. It gives us 0. So in addition to a being called the lowering operator it's also called the annihilation operator, because, I don't know, we're a brutal and warlike species.

So this is equal to h bar omega-- this term kills phi 0-- again with the kills-- and gives me a 1/2 half leftover. 1/2 h bar omega phi 0. So the ground state, any state-- any state annihilated by a must have the same energy. The only way you can be annihilated by a is if your energy is this. Cool?

So what does that tell you about the second ladder of hidden seats that we missed? It's got to be degenerate. It's got to have the same energies. I drew that really badly, didn't I? Those are evenly spaced.

So it's got to be degenerate. However, Barton proved for you the node theorem last time, right? He gave you my spread argument for the node theorem? In particular, one of the consequences of that is it in a system with bound states, in a system with potential that goes up, you can never have degeneracies in one dimension. We're not going to prove that carefully in here. But it's relatively easy to prove. In fact, if you come to my office hours I'll prove it for you. It takes three minutes. But I don't want to set up the math right now.

So how many people know about the Wronskian? That's awesome. OK, so I leave it to you as an exercise to use the Wronskian to show that there cannot be degeneracies in one dimension, which is cool. Anyway, so the Wronskian for the differential equation, which is the energy eigenvalue equation.

There can be no degeneracies in one dimensional potentials with bound states. So what we've just shown is that the only way that there can be extra states that we missed is if there's a tower with exactly identical energies all the way up. But if they have exactly identical energies, that means there's a degenerate. But we can prove that there can't be degeneracies in 1D. So can there be an extra tower of states we missed? No. Can we have missed any states? No. Those are all the states there are. And we've done it without ever solving a differential equation, just by using that commutation relation.

Now at this point it's very tempting to say, that was just sort of magical mystery stuff. But what we really did last time was very honest. We wrote down a differential equation. We found the solution. And we got the wave functions. So, Professor Adams, you just monkeyed around at the chalkboard with commutators for a while, but what are the damn wave functions? Right? We already have the answer. This is really quite nice.

Last time we solved that differential equation. And we had to solve that differential

equation many, many times, different levels. But now we have a very nice thing we can do. What's true of the ground state? Well, the ground state is annihilated by the lowering operator. So that means that a acting on phi 0 of x is equal to 0.

But a has a nice expression, which unfortunately I erased. Sorry about that. So a has a nice expression. a is equal to x over x0 plus ip over p0. And so if you write that out and multiply from appropriate constants, this becomes the following differential equation. The x is just multiplied by x. And the p is take a derivative with respect to x, multiply by h bar upon i. And multiplying by i over h bar to get that equation, this gives us dx plus p over h bar x0-- sorry, that shouldn't be an i. That should be p0. x on phi 0 is equal to 0.

And you solved this last time when you did the asymptotic analysis. This is actually a ridiculously easy equation. It's a first order differential equation. There's one integration constant. That's going to be the overall normalization. And so the form is completely fixed. First order differential equation.

So what's the solution of this guy? It's a Gaussian. And what's the width of that Gaussian? Well, look at p0 over h bar x0. We know that p0 times x0 is twice h bar. So if I multiply by x amount on the top and bottom, you get 2 h bar. The h bars cancel. So this gives me two upon x0 squared. Remember I said it would be useful to remember that p0 times x0 is 2h bar? It's useful.

So it gives us this. And so the result is that phi 0 is equal to, up to an overall normalization coefficient, e to the minus x squared over x0 squared. Solid. So there. We've solved that differential equation. Is the easiest, second easiest differential equation. It's our first order differential equation with a linear term rather than a constant. We get a Gaussian.

And now that we've got this guy-- look, do you remember the third Hermite polynomial? Because we know the third excited state is given by h3 times this Gaussian. Do you remember it off the top of your head? How do you solve what it is? How do we get phi 3? First off, how do we get phi 1? How do we get the next state in the ladder? How do we get the wave function? Raising operator. But what is

the raising operator? Oh, it's the differential operator I take with-- OK, but if I had a dagger, it's just going to change the sign here.

So how do I get phi 1? Phi 1 is equal to up to some normalization. dx minus 2 over x0 squared, x0, phi 0. So now do I have to solve the differential equation to get the higher states? No. I take derivatives and multiply by constants.

So to get the third Hermite polynomial what do you do? You do this three times. This is actually an extremely efficient way-- it's related to something called the generating function, and an extremely efficient way to write down the Hermite polynomials. They're the things that you get by acting on this with this operator as many times as you want. That is a nice formal definition of the Hermite polynomials.

The upshot of all of this is the following. The upshot of all this is that we've derived that without ever solving the differential equation the spectrum just from that commutation relation, just from that commutation relation-- I cannot emphasize this strongly enough-- just from the commutation relation, Ea is minus a times the constant, and Ea dagger is a dagger times the constant. We derive that the energy eigenstates come in a tower. You can move along this tower by raising with the raising operator, lowering with the lowering operator. You can construct the ground state by building that simple wave function, which is annihilated by the lowering operator. You can build all the other states by raising them, which is just taking derivatives instead of solving differential equations, which is hard.

And all of this came from this commutation relation. And since we are going to see this over and over again-- and depending on how far you take physics, you will see this in 8.05. You will see this in 8.06. You will see this in quantum field theory. This shows up everywhere. It's absolutely at the core of how we organize the degrees of freedom. This structure is something you should see and declare victory upon seeing. Should see this and immediately say, I know the answer, and I can write it down. OK?

In the next lecture we're going to do a review which is going to introduce a slightly more formal presentation of all these ideas. That's not going to be material covered

on the exam, but it's going to help you with the exam, which will be on Thursday. See you Tuesday.