8.04: Quantum Mechanics
Massachusetts Institute of Technology

Professor Allan Adams
Wednesday April 18 2012

Exam 2

Last Name: ______________________

First Name: ______________________

Check	Recitation	Instructor	Time
	R01	Barton Zwiebach	10:00
	R02	Barton Zwiebach	11:00
	R03	Matthew Evans	3:00
	R04	Matthew Evans	4:00

Instructions:
Show all work. All work must be done in this exam packet.
This is a closed book exam – books, notes, phones, calculators etc are **not allowed**.
You have 1.5 hours to solve the problems. Exams will be collected at 12:30pm sharp.

Problem	Max Points	Score	Grader
1	40		
2	30		
3	30		
Total	100		

Formula Sheet 1

Fourier Transform Conventions:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{ikx} \tilde{f}(k) \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; e^{-ikx} f(x)$$

Delta Functions:

$$\int_{-\infty}^{\infty} dx \, f(x) \, \delta(x-a) = f(a) \qquad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx}$$

$$\delta(x) = \left\{ \begin{array}{cc} 0 & x \neq 0 \\ \infty & x = 0 \end{array} \right\} \qquad \delta_{mn} = \left\{ \begin{array}{cc} 0 & m \neq n \\ 1 & m = n \end{array} \right\}$$

Operators and the Schrödinger Equation:

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \qquad [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{E} \, \psi(x,t)$$

$$\hat{E} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \qquad E \, \phi_E(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_E(x) \; + \; V(x) \, \phi_E(x)$$

Common Integrals:

$$\int_{-\infty}^{\infty} dx \, e^{-x^2} = \sqrt{\pi} \qquad (f|g) = \int_{-\infty}^{\infty} dx \, f(x)^* \, g(x)$$

For an infinite square well with $0 \leq x \leq L$:

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin(k_n x) \qquad (\phi_n|\phi_m) = \delta_{mn}$$

$$k_n = \frac{(n+1)\pi}{L} \qquad E_n = \frac{\hbar^2 k_n^2}{2m}$$

Continuity Condition for $V(x) = W_o \delta(x-a)$:

$$\phi_E(a^+) = \phi_E(a^-) \qquad \phi'_E(a^+) - \phi'_E(a^-) = \frac{2mW_o}{\hbar^2} \phi_E(a)$$

Planck's Constant:

$$\hbar \simeq 6.6 \cdot 10^{-16} eV \cdot s$$

Formula Sheet 2

The Probability Current:

$$\mathcal{J}(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

Definition of the S-matrix and the scattering phase:

$$\begin{pmatrix} B \\ C \end{pmatrix} = S\begin{pmatrix} A \\ D \end{pmatrix}, \qquad t = |t|\, e^{-i\varphi}, \qquad T = |t|^2$$

Raising and Lowering Operators for the 1d Harmonic Oscillator ($\beta^2 = \hbar/m\omega$):

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{\beta}\hat{x} + i\frac{\beta}{\hbar}\hat{p}\right) , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\beta}\hat{x} - i\frac{\beta}{\hbar}\hat{p}\right) , \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

Normalization and Orthonormality of 1d HO wavefunctions :

$$\phi_n(x) = A_n e^{-x^2/2\beta^2} H_n\left(\frac{x}{\beta}\right) \qquad A_n = (2^n n!\, \beta\sqrt{\pi})^{-1/2} \qquad (\phi_n|\phi_m) = \delta_{nm}$$

Laplacian in Spherical Coordinates.

$$\vec{p} = -i\hbar\vec{\nabla} \qquad \vec{\nabla}^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Angular Momentum Operators in Spherical Coordinates:

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} , \qquad \hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Angular Momentum Commutators

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x , \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y , \qquad [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z , \qquad [\hat{L}_i, \hat{L}^2] = 0$$

Angular Momentum Raising and Lowering Operators

$$\begin{aligned} \hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \hbar e^{+i\phi}\left(i\cot\theta\,\partial_\phi + \partial_\theta\right) \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar\,\hat{L}_\pm \ . \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \hbar e^{-i\phi}\left(i\cot\theta\,\partial_\phi - \partial_\theta\right) \qquad [\hat{L}^2, \hat{L}_\pm] = 0 \ . \end{aligned}$$

First Few Spherical Harmonics

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}} , \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta , \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} , \qquad (Y_{lm}|Y_{l'm'}) = \delta_{ll'}\delta_{mm'} \ .$$

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1. **(35 points) Short Answer**

(a) Fill in the blanks for the following four commutators:

$[\hat{z}, \hat{p}_z] = \boxed{}$ $[\hat{z}, \hat{L}_z] = \boxed{}$

$[\hat{x}, \hat{p}_z] = \boxed{}$ $[\hat{x}, \hat{L}_z] = \boxed{}$

(b) Let $\hat{A}$ and $\hat{B}$ be Hermitian, *i.e.* $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$. Beneath each of the following operators, write its Hermitian adjoint.

$i\hat{A}$	$\hat{A}\hat{B}$	$\hat{A}\hat{B} + \hat{B}\hat{A}$	$\hat{A}\hat{B} - \hat{B}\hat{A}$
$\boxed{}$	$\boxed{}$	$\boxed{}$	$\boxed{}$

(c) Let $\hat{A}$ and $\hat{B}$ be Hermitian, *i.e.* $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$. Which of the following is a Unitary operator? Circle each that is Unitary:

$e^{i\hat{A}}$ $e^{\hat{A}\hat{B}}$ $e^{\hat{A}\hat{B}+\hat{B}\hat{A}}$ $e^{\hat{A}\hat{B}-\hat{B}\hat{A}}$ *None of these*

(d) It is commonly believed that Unicorns can be described by 3 observables, Color, Happiness and Location. Quantum Unicorns are thus described by the operators, $\hat{C}$, $\hat{H}$ and $\hat{L}$. These operators have been conjectured to satisfy the commutator

$$[\hat{C}, \hat{H}] = \hbar \hat{L}$$

In a landmark measurement, Prof. Evans has directly observed a Unicorn and determined its Color to be Purple with absolute certainty (he's an excellent experimentalist). Can he simultaneously determine its Happiness with absolute certainly? Circle and explain your answer:

Always, because:

Sometimes, if:

Never, because:

(e) Shortly after Prof. Evans's announcement, the OPERA collaboration announced its own measurement of a perfectly Happy Unicorn. Given these measurements of C and H, the corresponding quantum operators $\hat{C}$ and $\hat{H}$ must be Hermitian. Is $\hat{L}$ also Hermitian? Explain why or why not.

Yes No

What does this imply about the eigenvalues of the Unicorn Location operator $\hat{L}$?

The eigenvalues of $\hat{L}$ must be $\boxed{}$.

(f) Given that $Y_{7,-7} = C\,(\sin\theta)^7 e^{-7i\phi}$, where C is a constant, what is $Y_{7,-6}$? (Don't worry about the overall normalization.)

$$Y_{7,-6} \propto \boxed{}$$

(g) Suppose a particle on a sphere is in the state,

$$\psi(\theta,\phi) = i\sqrt{\frac{3}{4\pi}}\,\sin\theta\,\sin 3\phi\ .$$

What possible values of L_z can measurement find and with what probabilities will these be observed?

$$L_z = \boxed{}$$

(h) An engineer has to design a $1D$ potential for a particle of mass m in which the three lowest eigenstates have energies 1meV, 3meV and 5meV. Propose a simple solution using the potentials you have learned in 8.04. Specify the value of any constants in your potential.

$V(x) = \boxed{}$ $\qquad$ $Constants: \boxed{}$

(i) Suppose you discover a system which includes an operator $\hat{J}$ such that

$$[\hat{E}, \hat{J}] = \epsilon \hat{J} \ ,$$

where $\hat{E}$ is the energy operator and ϵ is a constant with dimensions of energy. What can you say about the set of eigenvalues of $\hat{E}$?

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2. **(35 points) Decoding the S-Matrix**

A beam of particles of mass m, wavenumber k and energy $E = \frac{\hbar^2 k^2}{2m} > 0$, is scattered off an unknown $1d$ potential, $V(x)$, which asymptotes to zero in both directions. After extensive measurement, it is determined that the S-matrix for the system is extremely well approximated by,

$$S = \frac{1}{\eta^2 + g^2(1 - e^{2i\eta})} \begin{pmatrix} 2g(g + i\eta)i\sin(\eta) & \eta^2 \\ \eta^2 & 2g(g - i\eta)i\sin(\eta) \end{pmatrix},$$

where $\eta = 2kL$, with L and g parameters that have been tuned to match the data. Your job is to deduce as much as possible about the unknown potential $V(x)$.

(a) Is $V(x)$ invariant under Parity, $x \to -x$? Briefly explain why or why not.

Yes No

(b) What is the probability for a particle incident from the left to transmit to the right, T_{LR}? What about T_{RL} from right to left?

You may find the following useful: $|\eta^2 + g^2(1-e^{2i\eta})|^2 = \eta^4 + 4g^2(g^2+\eta^2)(\sin\eta)^2$

$T_{LR} =$

$T_{RL} =$

(c) Are there any resonances in this system? In not, explain why not. If so, list the values of η for which there are resonances.

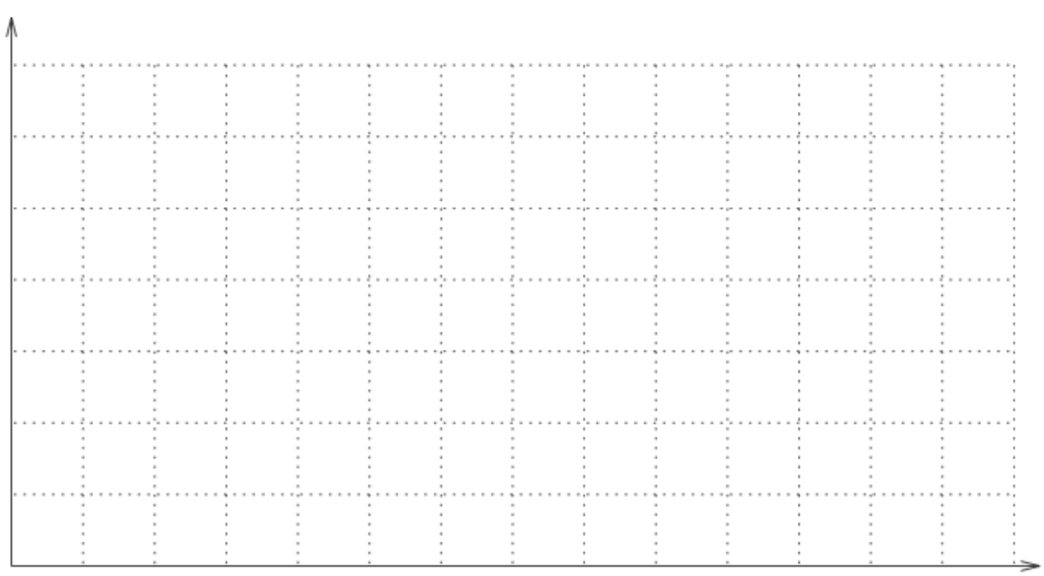
Yes No

(d) How does the transmission probability T_{LR} behave at low energy? At high energy? If either limiting value is zero, give the leading term.

$T_{LR} \xrightarrow{E\to 0}$

$T_{LR} \xrightarrow{E\to \infty}$

(e) Sketch the transmission probability T_{LR} as a function of η for a typical value of g, being careful to explicitly indicate the features discussed in parts (c) and (d).

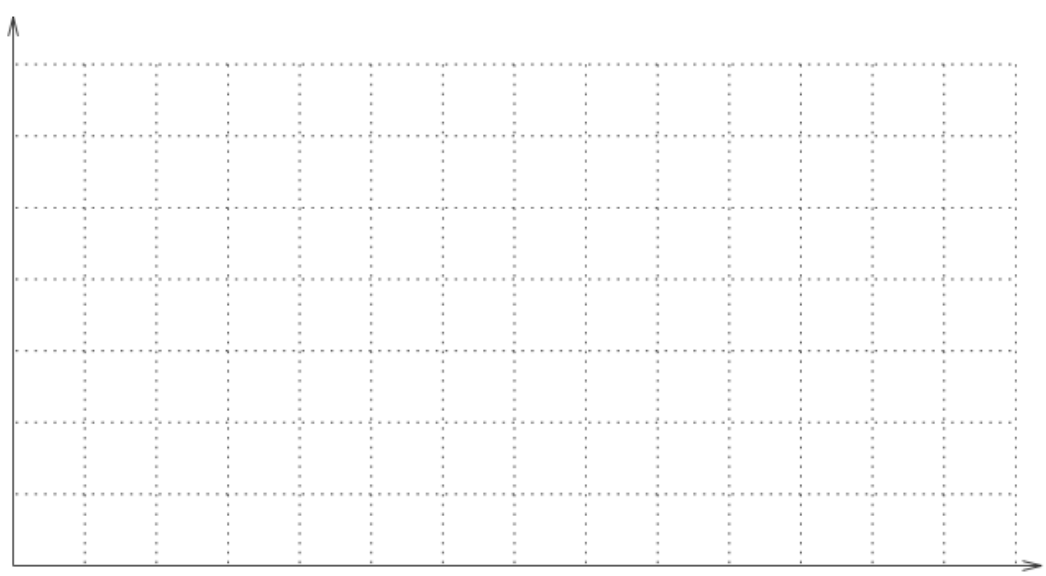


(f) Suppose there exist bound states with energies $E < 0$. Use the S-matrix to derive an equation which must be satisfied by the energy of any bound state. Put your equation in the form, $f(\beta) = g^2$, where $\beta = \alpha L = \sqrt{\frac{-2mE}{\hbar^2}}\, L$ is dimensionless.

NOTE: DO NOT ATTEMPT TO SOLVE THIS EQUATION ANALYTICALLY!

$$\boxed{\qquad\qquad\qquad = \; g^2}$$

(g) Solve your equation graphically in the space below. How many bound states does this potential support? Does this number depend on the value of g?



(h) Based on everything you have deduced above, list the properties the potential must have, then conjecture a potential $V(x)$ which could produce the S-matrix we measured above.

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3. **(30 points) Quantum Epicycles**

   Imagine a planet which is constrained to move freely along a sphere, forming a symmetric rigid rotor with moment of inertia I. Let $\vec{L}$ denote the angular momentum of the planet about the center of the sphere. Recall that the classical kinetic energy for a symmetric rotor with moment of inertia I is given by, $E = \frac{1}{2I}\vec{L}^2$.

   (a) What are the energy eigenvalues and eigenfunctions for this system? *You do not need to give the functional forms of the eigenfunctions, just identify them.*

   (b) What is the degeneracy of the n^{th} energy level (that is, how many physically distinct states share the same energy)?

Now suppose that, in addition to its angular momentum along the sphere, our planet also carries its own internal[1] angular momentum, $\vec{S}$, so that, as for $\vec{L}$,

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y\,.$$

Importantly, $\vec{S}$ is completely independent of $\vec{L}$, ie

$$[L_i, S_j] = 0\,.$$

The total angular momentum of the full system is then[3]

$$\vec{J} = \vec{L} + \vec{S} \qquad J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}\,.$$

(c) Is it possible to find a simultaneous eigenstate of L^2, L_z, S^2 and S_z?
Hint: This should take zero calculation.

(d) Is it possible to find a simultaneous eigenstate of J^2 and L^2? J^2 and S^2?

[1]For example, $\vec{S}$ could represent the fact that the planet is itself, like the rotating earth, a tiny rigid rotor, or that the planet in fact moves on a small epicycle[2], etc. The details of $\vec{S}$ do not matter for the problem at hand, all you need to know is that it is an angular momentum vector and is independent of $\vec{L}$

[2]Do not scorn epicycles! They're nothing but a truncated fourier series.

[3]For example, $J_x = L_x + S_x$, etc, while $\vec{L}\cdot\vec{S} = L_x S_x + L_y S_y + L_z S_z$.

(e) Is it possible to find a simultaneous eigenstate of J^2 and L_z? J^2 and S_z?

(f) Is it possible to find a simultaneous eigenstate of J^2 and J_z?

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8.04 Quantum Physics I
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