LECTURE 4

- **Readings:** Section 1.6

Lecture outline

- Principles of counting
- Many examples
  - permutations
  - k-permutations
  - combinations
  - partitions
- Binomial probabilities

Discrete uniform law

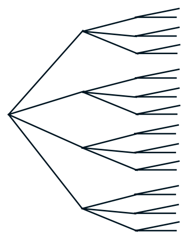
- Let all sample points be equally likely
- Then,

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}} = \frac{|A|}{|\Omega|}$$

- Just count...

Basic counting principle

- r stages
- n_i choices at stage i



- Number of choices is: $n_1 n_2 \cdots n_r$
- Number of license plates with 3 letters and 4 digits $=$
- ... if repetition is prohibited $=$
- **Permutations:** Number of ways of ordering n elements is:
- Number of subsets of $\{1, \ldots, n\}$ $=$

Example

- Probability that six rolls of a six-sided die all give different numbers?
  - Number of outcomes that make the event happen:
  - Number of elements in the sample space:
  - Answer:

Combinations

- $\binom{n}{k}$: number of k-element subsets of a given n-element set
- Two ways of constructing an ordered sequence of k **distinct** items:
  - Choose the k items one at a time:
    $n(n-1)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$ choices
  - Choose k items, then order them ($k!$ possible orders)
- Hence:

$$\binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\sum_{k=0}^{n} \binom{n}{k} =$$

Binomial probabilities

- n independent coin tosses
  - $\mathbf{P}(H) = p$
- $\mathbf{P}(HTTHHH) =$
- $\mathbf{P}(\text{sequence}) = p^{\#\text{ heads}}(1-p)^{\#\text{ tails}}$

$$\mathbf{P}(k \text{ heads}) = \sum_{k-\text{head seq.}} \mathbf{P}(\text{seq.})$$

$$= (\#\text{ of } k-\text{head seqs.}) \cdot p^k(1-p)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Coin tossing problem

- event B: 3 out of 10 tosses were "heads".
  - Given that B occurred, what is the (conditional) probability that the first 2 tosses were heads?
- All outcomes in set B are equally likely: probability $p^3(1-p)^7$
  - Conditional probability law is uniform
- Number of outcomes in B:
- Out of the outcomes in B, how many start with HH?

Partitions

- 52-card deck, dealt to 4 players
- Find $\mathbf{P}(\text{each gets an ace})$
- Outcome: a partition of the 52 cards
  - number of outcomes:

$$\frac{52!}{13!\,13!\,13!\,13!}$$

- Count number of ways of distributing the four aces: $4 \cdot 3 \cdot 2$
- Count number of ways of dealing the remaining 48 cards

$$\frac{48!}{12!\,12!\,12!\,12!}$$

- Answer:

$$\frac{4 \cdot 3 \cdot 2 \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$