6.041/6.431 Probabilistic Systems Analysis

Quiz II Review
Fall 2010

1 Probability Density Functions (PDF)

For a continuous RV X with PDF $f_X(x)$,

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$
$$P(X \in A) = \int_A f_X(x)dx$$

Properties:

- Nonnegativity:
$$f_X(x) \geq 0 \ \forall x$$
- Normalization:
$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

2 PDF Interpretation

Caution: $f_X(x) \neq P(X = x)$

- if X is continuous, $P(X = x) = 0 \ \ \forall x$!!
- $f_X(x)$ can be ≥ 1

Interpretation: "probability per unit length" for "small" lengths around x

$$P(x \leq X \leq x + \delta) \approx f_X(x)\delta$$

3 Mean and variance of a continuous RV

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
\text{Var}(X) &= E\left[(X - E[X])^2\right] \\
&= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx \\
&= E[X^2] - (E[X])^2 \ (\geq 0) \\
E[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)dx \\
E[aX + b] &= aE[X] + b \\
\text{Var}(aX + b) &= a^2 \text{Var}(X)
\end{aligned}$$

4 Cumulative Distribution Functions

Definition:

$$F_X(x) = P(X \leq x)$$

monotonically increasing from 0 (at $-\infty$) to 1 (at $+\infty$).

- Continuous RV (CDF is continuous in x):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

- Discrete RV (CDF is piecewise constant):

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

$$p_X(k) = F_X(k) - F_X(k-1)$$

5 Uniform Random Variable

If X is a uniform random variable over the interval [a,b]:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{otherwise } (x > b) \end{cases}$$

$$E[X] = \frac{b-a}{2}$$

$$\text{var}(X) = \frac{(b-a)^2}{12}$$

6 Exponential Random Variable

X is an exponential random variable with parameter λ:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad \text{var}(X) = \frac{1}{\lambda^2}$$

Memoryless Property: Given that $X > t$, $X - t$ is an exponential RV with parameter λ

7 Normal/Gaussian Random Variables

General normal RV: $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu, \quad \text{Var}(X) = \sigma^2$$

Property: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$

then $Y \sim N(a\mu + b, a^2\sigma^2)$

8 Normal CDF

Standard Normal RV: $N(0,1)$
CDF of standard normal RV Y at y: $\Phi(y)$
- given in tables for $y \geq 0$
- for $y < 0$, use the result: $\Phi(y) = 1 - \Phi(-y)$

To evaluate CDF of a general standard normal, express it as a function of a standard normal:

$$X \sim N(\mu, \sigma^2) \Leftrightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$P(X \leq x) = P\Big(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\Big) = \Phi\Big(\frac{x-\mu}{\sigma}\Big)$$

9 Joint PDF

Joint PDF of two continuous RV X and Y: $f_{X,Y}(x,y)$

$$P(A) = \int\int_A f_{X,Y}(x,y)dxdy$$

Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$
$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$
Joint CDF: $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

10 Independence

By definition,

$$X,\ Y \text{ independent} \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \forall (x,y)$$

If X and Y are independent:

- $E[XY]=E[X]E[Y]$
- $g(X)$ and $h(Y)$ are independent
- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

11 Conditioning on an event

Let X be a continuous RV and A be an event with $P(A) > 0$,

$$\begin{aligned} f_{X|A}(x) &= \begin{cases} \frac{f_X(x)}{P(X\in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \\ P(X \in B | X \in A) &= \int_B f_{X|A}(x)dx \\ E[X|A] &= \int_{-\infty}^{\infty} x f_{X|A}(x)dx \\ E[g(X)|A] &= \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx \end{aligned}$$

If $A_1, \ldots, A_n$ are disjoint events that form a partition of the sample space,

$$\begin{aligned} f_X(x) &= \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x) \text{ } (\approx \text{total probability theorem}) \\ E[X] &= \sum_{i=1}^{n} P(A_i) E[X|A_i] \text{ (total expectation theorem)} \\ E[g(X)] &= \sum_{i=1}^{n} P(A_i) E[g(X)|A_i] \end{aligned}$$

12 Conditioning on a RV

$X,\ Y$ continuous RV

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ f_X(x) &= \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy \quad (\approx totalprobthm) \end{aligned}$$

Conditional Expectation:

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \\ E[g(X)|Y=y] &= \int_{-\infty}^{\infty} g(X) f_{X|Y}(x|y) dx \\ E[g(X,Y)|Y=y] &= \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y) dx \end{aligned}$$

Total Expectation Theorem:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y) dy \\ E[g(X)] &= \int_{-\infty}^{\infty} E[g(X)|Y=y] f_Y(y) dy \\ E[g(X,Y)] &= \int_{-\infty}^{\infty} E[g(X,Y)|Y=y] f_Y(y) dy \end{aligned}$$

13 Continuous Bayes' Rule

X, Y continuous RV, N discrete RV, A an event.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) f_X(t) dt} \\ P(A|Y=y) &= \frac{P(A) f_{Y|A}(y)}{f_Y(y)} = \frac{P(A) f_{Y|A}(y)}{f_{Y|A}(y) P(A) + f_{Y|A^c}(y) P(A^c)} \end{aligned}$$

$$P(N=n|Y=y) = \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)} = \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)}$$

14 Derived distributions

Def: PDF of a *function* of a RV X with known PDF: $Y = g(X)$.
Method:

- Get the CDF:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{x|g(x)\leq y} f_X(x)dx$$

- Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Special case: if $Y = g(X) = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X(\frac{x-b}{a})$

15 Convolution

$W = X + Y$, with X, Y independent.

- Discrete case:

$$p_W(w) = \sum_x p_X(x) p_Y(w - x)$$

- Continuous case:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\ dx$$

Graphical Method:

- put the PMFs (or PDFs) on top of each other
- flip the PMF (or PDF) of Y
- shift the flipped PMF (or PDF) of Y by w
- cross-multiply and add (or evaluate the integral)

In particular, if X, Y are independent and normal, then
$W = X + Y$ is normal.

16 Law of iterated expectations

$E[X|Y = y] = f(y)$ is a number.
$E[X|Y] = f(Y)$ is a random variable
(the expectation is taken with respect to X).
To compute $E[X|Y]$, first express $E[X|Y = y]$ as a function of y.

Law of iterated expectations:

$$E[X] = E[E[X|Y]]$$

(equality between two real numbers)

17 Law of Total Variance

Var$(X|Y)$ is a random variable that is a function of Y (the variance is taken with respect to X).
To compute Var$(X|Y)$, first express

$$\mathrm{Var}(X|Y=y) = E[(X - E[X|Y=y])^2|Y=y]$$

as a function of y.

Law of conditional variances:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

(equality between two real numbers)

18 Sum of a random number of iid RVs

N discrete RV, X_i i.i.d and independent of N.
$Y = X_1 + \ldots + X_N$. Then:

$$\begin{aligned} E[Y] &= E[X]E[N] \\ \mathrm{Var}(Y) &= E[N]\mathrm{Var}(X) + (E[X])^2\mathrm{Var}(N) \end{aligned}$$

19 Covariance and Correlation

$$\begin{aligned} \mathrm{Cov}(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

- By definition, X, Y are uncorrelated $\Leftrightarrow$ $\mathrm{Cov}(X,Y) = 0$.
- If X, Y independent $\Rightarrow$ X and Y are uncorrelated. (the converse is not true)
- In general, Var(X+Y)= Var(X)+ Var(Y)+ 2 Cov(X,Y)
- If X and Y are uncorrelated, Cov(X,Y)=0 and Var(X+Y)= Var(X)+Var(Y)

Correlation Coefficient: (dimensionless)

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \quad \in [-1, 1]$$

$\rho = 0 \Leftrightarrow$ X and Y are uncorrelated.
$|\rho| = 1 \Leftrightarrow \quad X - E[X] = c[Y - E[Y]]$ (linearly related)