Chapter 7: TEM Transmission Lines

7.1 TEM waves on structures

7.1.1 Introduction

Transmission lines typically convey electrical signals and power from point to point along arbitrary paths with high efficiency, and can also serve as circuit elements. In most transmission lines, the electric and magnetic fields point purely transverse to the direction of propagation; such waves are called transverse electromagnetic or *TEM waves*, and such transmission lines are called *TEM lines*. The basic character of TEM waves is discussed in Section 7.1, the effects of junctions are introduced in Section 7.2, and the uses and analysis of TEM lines with junctions are treated in Section 7.3. Section 7.4 concludes by discussing TEM lines that are terminated at both ends so as to form resonators.

Transmission lines in communications systems usually exhibit frequency-dependent behavior, so complex notation is commonly used. Such lines are the subject of this chapter. For broadband signals such as those propagating in computers, complex notation can be awkward and the physics obscure. In this case the signals are often analyzed in the time domain, as introduced in Section 7.1.2 and discussed further in Section 8.1. Non-TEM transmission lines are commonly called waveguides; usually the waves propagate inside some conducting envelope, as discussed in Section 9.3, although sometimes they propagate partly outside their guiding structure in an "open" waveguide such as an optical fiber, as discussed in Section 12.2.

7.1.2 TEM waves between parallel conducting plates

The sinusoidal uniform plane wave of equations (7.1.1) and (7.1.2) is consistent with the presence of thin parallel conducting plates orthogonal to the electric field $\overline{E}(z,t)$, as illustrated in Figure 7.1.1(a)[31].

$$\overline{E}(z,t) = \hat{x}E_o \cos(\omega t - kz) \ \ [V/m] \tag{7.1.1}$$

$$\overline{H}(z,t) = \hat{y}\frac{E_o}{\eta_o}\cos(\omega t - kz) \ \ [A/m] \tag{7.1.2}$$

Although perfect consistency requires that the plates be infinite, there is approximate consistency so long as the plate separation d is small compared to the plate width W and the fringing fields outside the structure are negligible. The more general wave $\overline{E}(z,t) = \hat{x}E_x(z - ct)$, $\overline{H}(z,t) = \hat{z} \times \overline{E}(z,t)/\eta_o$ is also consistent [see (2.2.13), (2.2.18)], since any arbitrary waveform E(z - ct) can be expressed as the superposition of sinusoidal waves at all frequencies. In both cases all boundary conditions of Section 2.6 are satisfied because $\overline{E}_{//} = \overline{H}_{\perp} = 0$ at the

[31] See Section 2.3.1 for an introduction to uniform sinusoidal electromagnetic plane waves.

conductors. The voltage between two plates v(z,t) for this sinusoidal wave can be found by integrating $\overline{E}(z,t)$ over the distance d from the lower plate, which we associate here with the voltage +v, to the upper plate:

$$v(t,z) = \hat{x}\bullet\overline{E}(z,t)d = E_o d\cos(\omega t - kz) \quad [V] \tag{7.1.3}$$

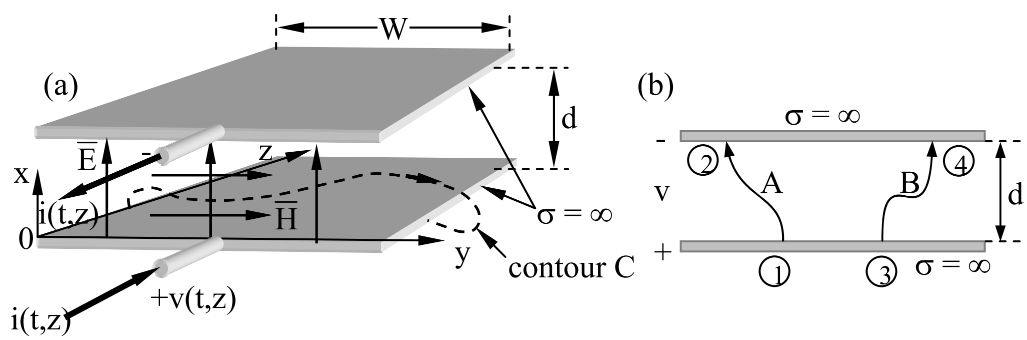

Figure 7.1.1 Parallel-plate TEM transmission line.

Although this computed voltage v(t,z) does not depend on the path of integration connecting the two plates, provided it is at constant z, it does depend on z itself. Thus there can be two different voltages between the same pair of plates at different positions z. Kirchoff's voltage law says that the sum of voltage drops around a loop is zero; this law is violated here because such a loop in the x-z plane encircles time varying magnetic fields, $\overline{H}(z,t)$, as illustrated. In contrast, the sum of voltage drops around a loop confined to constant z is zero because it circles no $\partial\overline{H}/\partial t$; therefore the voltage v(z,t), computed by integrating $\overline{E}(z)$ between the two plates, does not depend on the path of integration at constant z. For example, the integrals of $\overline{E}\bullet d\overline{s}$ along contours A and B in Figure 7.1.1(b) must be equal because the integral around the loop 1, 2, 4, 3, 1 is zero and the path integrals within the perfect conductors both yield zero.

If the electric and magnetic fields are zero outside the two plates and uniform between them, then equal and opposite currents i(t,z) flow in the two plates in the ±z direction. The surface current is determined by the boundary condition (2.6.17): $\overline{J}_s = \hat{n}\times\overline{H}$ [A m^{-1}]. If the two conducting plates are spaced close together compared to their widths W so that d << W, then the fringing fields at the plate edges can be neglected and the total current flowing in the plates can be found from the given magnetic field $\overline{H}(z,t) = \hat{y}(E_o/\eta_o)\cos(\omega t - kz)$, and the integral form of Ampere's law:

$$\int_C \overline{H}\bullet d\overline{s} = \iint_A \left[\overline{J} + \left(\partial\overline{D}/\partial t\right)\right]\bullet\hat{n}\ da \tag{7.1.4}$$

If the integration contour C encircles the lower plate and surface A at constant z in a clockwise (right-hand) sense with respect to the +z axis as illustrated in Figure 7.1.1, then $\overline{D}\bullet\hat{n} = 0$ and the current flowing in the +z direction in the lower plate is simply:

$$i(z,t) = W\ J_{sz}(z,t) = W\ H_y(z,t) = (WE_o/\eta_o)\cos(\omega t - kz)\ [A] \tag{7.1.5}$$

An equal and opposite current flows in the upper plate.

Note that the computed current does not depend on the integration contour C chosen so long as C circles the plate at constant z. Also, the current flowing into a section of conducting plate at z_1 does not generally equal the current flowing out at z_2, seemingly violating Kirchoff's current law (the sum of currents flowing into a node is zero). This inequality exists because any section of parallel plates exhibits capacitance that conveys a displacement current $\partial\overline{D}/\partial t$ between the two plates; the right-hand side of Equation (2.1.6) suggests the equivalent nature of the conduction current density $\overline{J}$ and the displacement current density $\partial\overline{D}/\partial t$.

Such a two-conductor structure conveying waves that are purely transverse to the direction of propagation, i.e., $E_z = H_z = 0$, is called a *TEM transmission line* because it is propagating transverse electromagnetic waves (*TEM waves*). Such lines generally have a physical cross-section that is independent of z. This particular TEM transmission line is called a *parallel-plate TEM line*.

Because there are no restrictions on the time structure of a plane wave, any v(t) can propagate between parallel conducting plates. The ratio between v(z,t) and i(z,t) for this or any other sinusoidal or non-sinusoidal forward traveling wave is the *characteristic impedance Z_o* of the TEM structure:

$$v(z,t)/i(z,t) = \eta_o d/W = Z_o\ [\text{ohms}] \qquad \text{(characteristic impedance)} \tag{7.1.6}$$

In the special case d = W, Z_o equals the characteristic impedance η_o of free space, 377 ohms. Usually W >> d in order to minimize fringing fields, yielding Z_o << 377.

Since the two parallel plates can be perfectly conducting and lossless, the physical significance of Z_o ohms may be unclear. Z_o is defined as the ratio of line voltage to line current for a forward wave only, and is non-zero because the plates have inductance L per meter associated with the magnetic fields within the line. The value of Z_o also depends on the capacitance C per meter of this structure. Section 7.1.3 shows (7.1.59) that $Z_o = (L/C)^{0.5}$ for any lossless TEM line and (7.1.19) shows it for a parallel-plate line. The product of voltage and current v(z,t)i(z,t) represents power P(z,t) flowing past any point z toward infinity; this power is not being converted to heat by resistive losses, it is simply propagating away without reflections.

It is easy to demonstrate that the power P(z,t) carried by this forward traveling wave is the same whether it is computed by multiplying v and i, or by integrating the Poynting vector $\overline{S} = \overline{E} \times \overline{H}$ [W m^{-2}] over the cross-sectional area Wd of the TEM line:

$$P(z,t) = v(z,t)i(z,t) = [E(z,t)d][H(z,t)W] = [E(z,t)H(z,t)]Wd = S\ Wd \tag{7.1.7}$$

The differential equations governing v and i on TEM lines are easily derived from Faraday's and Ampere's laws for the fields between the plates of this line:

$$\nabla\times\overline{E} = -(\partial/\partial t)\mu\overline{H} = \hat{y}(\partial/\partial z)E_x(z,t) \tag{7.1.8}$$

$$\nabla\times\overline{H} = (\partial/\partial t)\varepsilon\overline{E} = -\hat{x}(\partial/\partial z)H_y(z,t) \tag{7.1.9}$$

Because all but one term in the curl expressions are zero, these two equations are quite simple. By substituting $v = E_x d$ (7.1.3) and $i = H_y W$ (7.1.5), (7.1.8) and (7.1.9) become:

$$dv/dz = -(\mu d/W)(di/dt) = -L\,di/dt \tag{7.1.10}$$

$$di/dz = -(\varepsilon W/d)(dv/dt) = -C\,dv/dt \tag{7.1.11}$$

where we have used the expressions for *inductance per meter* L [Hy m^{-1}] and *capacitance per meter* C [F m^{-1}] of a parallel-plate TEM line [see (3.2.11)[32] and (3.1.10)]. This form of the differential equations in terms of L and C applies to any lossless TEM line, as shown in Section 7.1.3.

These two differential equations can be solved for v by eliminating i. The current i can be eliminated by differentiating (7.1.10) with respect to z, and (7.1.11) with respect to t, thus introducing $d^2i/(dt\ dz)$ into both expressions permitting its substitution. That is:

$$d^2v/dz^2 = -L\,d^2i/(dt\ dz) \tag{7.1.12}$$

$$d^2i/(dz\ dt) = -C\,d^2v/dt^2 \tag{7.1.13}$$

Combining these two equations by eliminating $d^2i/(dt\ dz)$ yields the wave equation:

$$d^2v/dz^2 = LC\,d^2v/dt^2 = \mu\varepsilon\,d^2v/dt^2 \qquad \text{(wave equation)} \tag{7.1.14}$$

Wave equations relate the second spatial derivative to the second time derivative of the same variable, and the solution therefore can be any arbitrary function of an argument that has the same dependence on space as on time, except for a constant multiplier. That is, one solution to (7.1.14) is:

$$v(z,t) = v_+(z-ct) \tag{7.1.15}$$

where v_+ is an arbitrary function of the argument $(z-ct)$ and is associated with waves propagating in the +z direction at velocity c. This is directly analogous to the propagating waves

[32] Note: (3.2.11) gives the total inductance L for a length D of line, where area A = Dd. The inductance per unit length L = μd/W in both cases.

characterized in Figure 2.2.1 and in Equation (2.2.9). Demonstration that (7.1.15) satisfies (7.1.14) for $c = (\mu\varepsilon)^{-0.5}$ follows the same proof provided for (2.2.9) in (2.2.10–12).

The general solution to (7.1.14) is any arbitrary waveform of the form (7.1.15) plus an independent arbitrary waveform propagating in the -z direction:

$$v(z,t) = v_+(z-ct) + v_-(z+ct) \tag{7.1.16}$$

The general expression for current i(z,t) on a TEM line can be found, for example, by substituting (7.1.16) into the differential equation (7.1.11) and integrating over z. Thus, using the notation that $v'(q) \equiv dv(q)/dq$:

$$di/dz = -C\,dv/dt = cC\left[v'_+(z-ct) - v'_-(z+ct)\right] \tag{7.1.17}$$

$$i(z,t) = cC\left[\, v_+(z-ct) - v_-(z+ct)\,\right] = Z_o^{-1}\left[\, v_+(z-ct) - v_-(z+ct)\,\right] \tag{7.1.18}$$

Equation (7.1.18) defines the characteristic impedance $Z_o = (cC)^{-1} = \sqrt{L/C}$ for the TEM line. Both the forward and backward waves alone have the ratio Z_o between v and i, although the sign of i is reversed for the negative-propagating wave because a positive voltage then corresponds to a negative current. These same TEM results are derived differently in Sections 7.1.3 and 8.1.1.

The characteristic impedance Z_o of a parallel-plate line can be usefully related using (7.1.18) to the capacitance C and inductance L per meter, where $C = \varepsilon W/d$ and $L = \mu d/W$ for parallel-plate structures (7.1.10–11):

$$Z_o = \sqrt{\frac{L}{C}}\ [\text{ohms}] = \frac{d}{c\varepsilon W} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{d}{W} \qquad \text{(characteristic impedance)} \tag{7.1.19}$$

All lossless TEM lines have this simple relationship, as seen in (8.3.9) for R = G = 0. It is also consistent with (7.1.6), where $\eta_o = 1/c\varepsilon = (\mu_o/\varepsilon_o)^{0.5}$.

The electric and magnetic energies per meter on a parallel-plate TEM line of plate separation d and plate width W are:[33]

$$\mathcal{W}_e(t,z) = \frac{1}{2}\,\varepsilon\left|\overline{E}(t,z)\right|^2 = \frac{1}{2}\varepsilon\left(\frac{v(t,z)}{d}\right)^2 Wd\ \ \left[\text{J m}^{-1}\right] \tag{7.1.20}$$

$$\mathcal{W}_m(t,z) = \frac{1}{2}\,\mu\left|\overline{H}(t,z)\right|^2 = \frac{1}{2}\mu\left(\frac{i(t,z)}{d}\right)^2 Wd\ \ \left[\text{J m}^{-1}\right] \tag{7.1.21}$$

[33] Italicized symbols for $\mathcal{W}_e$ and $\mathcal{W}_m$ [J m^{-1}] distinguish them from W_e and W_m [J m^{-3}].

Substituting C = εW/d and L = μd/W into (7.1.20) and (7.1.21) yields:

$$W_e(t,z) = \frac{1}{2}Cv^2 \left[Jm^{-1}\right] \qquad \text{(TEM electric energy density)} \qquad (7.1.22)$$

$$W_m(t,z) = \frac{1}{2}Li^2 \left[Jm^{-1}\right] \qquad \text{(TEM magnetic energy density)} \qquad (7.1.23)$$

If there is only a forward-moving wave, then $v(t,z) = Z_o i(t,z)$ and so:

$$W_e(t,z) = \frac{1}{2}Cv^2 = \frac{1}{2}CZ_o{}^2 i^2 = \frac{1}{2}Li^2 = W_m(t,z) \qquad (7.1.24)$$

These relations (7.1.22) to (7.1.24) are true for any TEM line.

The same derivations can be performed using complex notation. Thus (7.1.10) and (7.1.11) can be written:

$$\frac{d\underline{V}(z)}{dz} = -\frac{\mu d}{W} j\omega \underline{I}(z) = -j\omega L \underline{I}(z) \qquad (7.1.25)$$

$$\frac{dI(z)}{dz} = -\frac{\varepsilon W}{d} j\omega \underline{V}(z) = -j\omega C \underline{V}(z) \qquad (7.1.26)$$

Eliminating I(z) from this pair of equations yields the wave equation:

$$\left(\frac{d^2}{dz^2} + \omega^2 LC\right)\underline{V}(z) = 0 \qquad \text{(wave equation)} \qquad (7.1.27)$$

The solution to the wave equation (7.1.27) is the sum of forward and backward propagating waves with complex magnitudes that indicate phase:

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz} \qquad (7.1.28)$$

$$\underline{I}(z) = Y_o(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}) \qquad (7.1.29)$$

where the *wavenumber* k follows from $k^2 = \omega^2 LC$, which is obtained by substituting (7.1.28) into (7.1.27):

$$k = \omega\sqrt{LC} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad (7.1.30)$$

The characteristic impedance of the line, as seen in (7.1.19) is:

$$Z_o = \sqrt{\frac{L}{C}} = \frac{1}{Y_o} \text{ [ohms]} \tag{7.1.31}$$

and the time average stored electric and magnetic energy densities are:

$$W_e = \frac{1}{4} C |\underline{V}|^2 \text{ [J/m]}, \qquad W_m = \frac{1}{4} L |\underline{I}|^2 \text{ [J/m]} \tag{7.1.32}$$

The behavior of these arbitrary waveforms at TEM junctions is discussed in the next section and the practical application of these general solutions for arbitrary waveforms is discussed further in Section 8.1. Their practical application to sinusoidal waveforms is discussed in Sections 7.2–4.

Example 7.1A

A certain TEM line consists of two parallel metal plates that are 10 cm wide, separated in air by d = 1 cm, and extremely long. A voltage v(t) = 10 cos ωt volts is applied to the plates at one end (z = 0). What currents i(t,z) flow? What power P(t) is being fed to the line? If the plate resistance is zero, where is the power going? What is the inductance L per unit length for this line?

Solution: In a TEM line the ratio v/i = Z_o for a single wave, where $Z_o = \eta_o d/W$ [see (7.1.6)], and $\eta_o = (\mu/\varepsilon)^{0.5} \cong 377$ ohms in air. Therefore $i(t,z) = Z_o^{-1}v(t,z) = (W/d\eta_o)10\cos(\omega t - kz) \cong [0.1/(0.01\times377)]\ 10\cos(\omega t - kz) \cong 0.27\cos[\omega(t - z/c)]$ [A]. $P = vi = v^2/Z_o \cong 2.65\cos^2[\omega(t - z/c)]$ [W]. The power is simply propagating losslessly along the line toward infinity. Since $c = (LC)^{-0.5} = 3\times10^8$, and $Z_o = (L/C)^{0.5} \cong 37.7$, therefore $L = Z_o/c = 1.3\times10^{-7}$ [Henries m^{-1}].

7.1.3 TEM waves in non-planar transmission lines

TEM waves can propagate in any perfectly conducting structure having at least two non-contacting conductors with an arbitrary cross-section independent of z, as illustrated in Figure 7.1.2, if they are separated by a uniform medium characterized by ε, μ, and σ. The parallel plate TEM transmission line analyzed in Section 7.1.2 is a special case of this configuration, and we shall see that the behavior of non-planar TEM lines is characterized by the same differential equations for v(z,t) and i(z,t), (7.1.10) and (7.1.11), when expressed in terms of L and C. This result follows from the derivation below.

We first divide the del operator into its transverse and longitudinal (z-axis) components:

$$\nabla = \nabla_T + \hat{z}\partial/\partial z \tag{7.1.33}$$

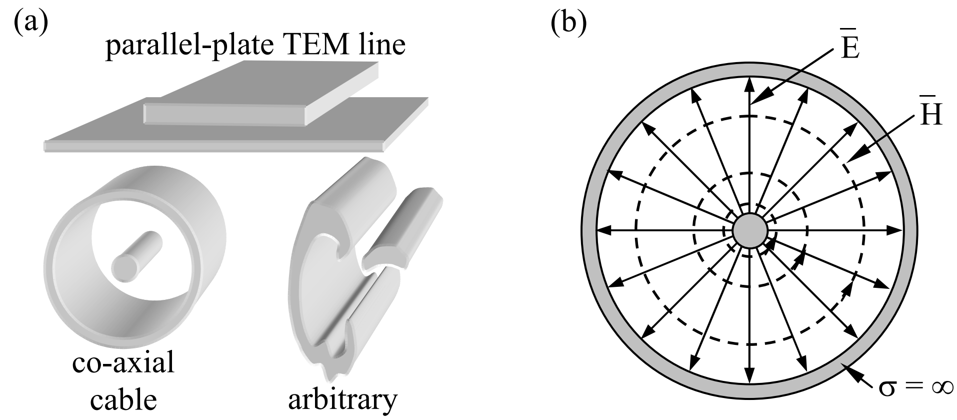

Figure 7.1.2 TEM lines with arbitrary cross-sections.

where $\nabla_T \equiv \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$. Faraday's and Ampere's laws then become:

$$\nabla \times \overline{E} = \nabla_T \times \overline{E}_T + (\partial/\partial z)(\hat{z} \times \overline{E}_T) = -\mu \partial \overline{H}_T/\partial t \tag{7.1.34}$$

$$\nabla \times \overline{H} = \nabla_T \times \overline{H}_T + (\partial/\partial z)(\hat{z} \times \overline{H}_T) = \sigma \overline{E}_T + \varepsilon \partial \overline{E}_T/\partial t \tag{7.1.35}$$

The right-hand sides of these two equations have no $\hat{z}$ components, and therefore the transverse curl components on the left-hand side are zero because they lie only along the z axis:

$$\nabla_T \times \overline{E}_T = \nabla_T \times \overline{H}_T = 0 \tag{7.1.36}$$

Moreover, the divergences of $\overline{E}_T$ and $\overline{H}_T$ are also zero since $\hat{z} \bullet \overline{H}_T = \hat{z} \bullet \overline{E}_T = 0$, and:

$$\nabla \bullet \overline{H} = 0 = \nabla_T \bullet \overline{H}_T + (\partial/\partial z)(\hat{z} \bullet \overline{H}_T) \tag{7.1.37}$$

$$\nabla \bullet \overline{E} = \rho/\varepsilon = 0 = \nabla_T \bullet \overline{E}_T + (\partial/\partial z)(\hat{z} \bullet \overline{E}_T) \tag{7.1.38}$$

Since the curl and divergence of $\overline{E}_T$ and $\overline{H}_T$ are zero, both these fields must independently satisfy Laplace's equation (4.5.7), which governs electrostatics and magnetostatics; these field solutions will differ because their boundary conditions differ. Thus we can find the transverse electric and magnetic fields for TEM lines with arbitrary cross-sections using the equation-solving and field mapping methods described in Sections 4.5 and 4.6.

The behavior of $\overline{E}$ and $\overline{H}$ for an arbitrary TEM line can be expressed more simply if we first define the line's *capacitance per meter* C and the *inductance per meter* L. C is the charge Q' per unit length divided by the voltage v between the two conductors of interest, and L is the flux linkage Λ' per unit length divided by the current i. Capacitance, inductance, and flux linkage are discussed more fully in Sections 3.1 and 3.2.

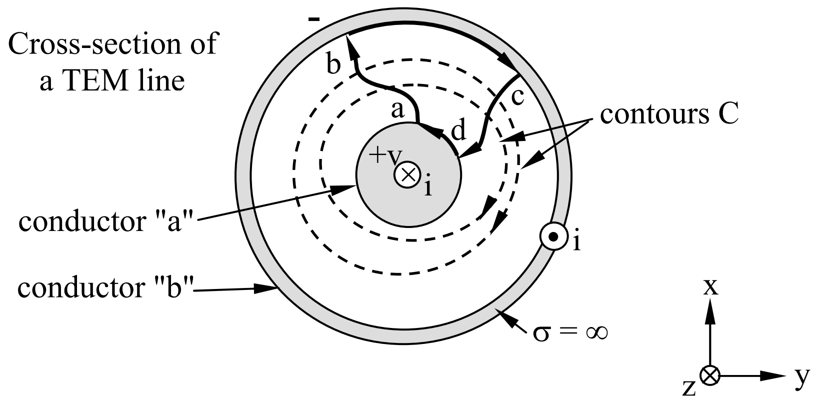

Figure 7.1.3 Integration paths for computing TEM line voltages and currents.

To compute Q' and Λ' we consider a differential element of length δ along the z axis of the TEM line illustrated in Figure 7.1.3, and then compute for Q' and Λ', respectively, surface and line integrals encircling the central positively charged conducting element "a" in a right-hand sense relative to $\hat{z}$. To compute the voltage v we integrate $\overline{E}_T$ from element a to element b, and to compute the current i we integrate $\overline{H}_T$ in a right-hand sense along the contour C circling conductor a:

$$C = Q'/v = \left(\delta^{-1}\iint_A \varepsilon\overline{E}_T \bullet \hat{n}\ da\right)\Big/\left(\int_a^b \overline{E}_T \bullet d\bar{s}\right) \qquad \text{(capacitance/m)} \qquad (7.1.39)$$

$$= \left[\oint_C \varepsilon\hat{z} \bullet \left(\overline{E}_T \times d\bar{s}\right)\right]\Big/\left(\int_a^b \overline{E}_T \bullet d\bar{s}\right) \quad \left[\text{Fm}^{-1}\right]$$

$$L = \Lambda'/i = \left[-\int_a^b \mu\hat{z} \bullet \left(\overline{H}_T \times d\bar{s}\right)\right]\Big/\left(\oint_C \overline{H}_T \bullet d\bar{s}\right) \qquad \text{(inductance/m)} \qquad (7.1.40)$$

$$= \left[\int_a^b \mu\overline{H}_T \bullet \left(\hat{z} \times d\bar{s}\right)\right]\Big/\left(\oint_C \overline{H}_T \bullet d\bar{s}\right) \quad \left[\text{Hm}^{-1}\right]$$

It is also useful to define G, the line *conductance per meter*, in terms of the leakage current density J_σ' [A m^{-1}] conveyed between the two conductors by the conductivity σ of the medium, where we can use (7.1.39) to show:

$$G = J_\sigma'/v = \left(\delta^{-1}\iint_A \sigma\overline{E}_T \bullet \hat{n}\ da\right)\Big/\left(\int_a^b \overline{E}_T \bullet d\bar{s}\right) = C\sigma/\varepsilon \qquad (7.1.41)$$

We can readily prove that the voltage and current computed using line integrals in (7.1.39–41) do not depend on the integration path. Figure 7.1.3 illustrates two possible paths of integration for computing v within a plane corresponding to a single value of z, the paths ab and dc. Since the curl of $\overline{E}_T$ is zero in the transverse plane we have:

$$\oint_C \overline{E}_T \bullet d\overline{s} = \int_a^b \overline{E}_T \bullet d\overline{s} + \int_b^c \overline{E}_T \bullet d\overline{s} + \int_c^d \overline{E}_T \bullet d\overline{s} + \int_d^a \overline{E}_T \bullet d\overline{s} = 0 \qquad (7.1.42)$$

The line integrals along the conductors are zero (paths bc and da), and the cd path is the reverse of the dc path. Therefore voltage is uniquely defined because for any path dc we have:

$$\int_a^b \overline{E}_T \bullet d\overline{s} = \int_d^c \overline{E}_T \bullet d\overline{s} = v(z,t) \qquad (7.1.43)$$

The current i(z,t) is also uniquely defined because all possible contours C in Figure 7.1.3 circle the same current flowing in conductor a:

$$i(z,t) = \oint_C \overline{H}_T \bullet d\overline{s} \qquad (7.1.44)$$

To derive the differential equations governing v(z,t) and i(z,t) we begin with (7.1.34) and (7.1.35), noting that $\nabla_T \times \overline{E}_T = \nabla_T \times \overline{H}_T = 0$:

$$(\partial/\partial z)(\hat{z} \times \overline{E}_T) = -\mu \partial \overline{H}_T / \partial t \qquad (7.1.45)$$

$$(\partial/\partial z)(\hat{z} \times \overline{H}_T) = (\sigma + \varepsilon \partial/\partial t)\overline{E}_T \qquad (7.1.46)$$

To convert (7.1.45) into an equation in terms of v we can compute the line integral of $\overline{E}_T$ from a to b: the first step is to use the identity $\overline{A} \times (\overline{B} \times \overline{C}) = \overline{B}(\overline{A} \bullet \overline{C}) - \overline{C}(\overline{A} \bullet \overline{B})$ to show $(\hat{z} \times \overline{E}_T) \times \hat{z} = \overline{E}_T$. Using this we operate on (7.1.45) to yield:

$$\begin{aligned}(\partial/\partial z)\int_a^b \left[(\hat{z} \times \overline{E}_T) \times \hat{z}\right] \bullet d\overline{s} &= (\partial/\partial z)\int_a^b \overline{E}_T \bullet d\overline{s} \\ &= \partial v(z,t)/\partial z \\ &= -\mu(\partial/\partial t)\int_a^b (\overline{H}_T \times \hat{z}) \bullet d\overline{s}\end{aligned} \qquad (7.1.47)$$

Then the right-hand integral in (7.1.47), in combination with (7.1.40) and (7.1.44), becomes:

$$\int_a^b (H_T \times \hat{z}) \bullet d\overline{s} = \int_a^b \overline{H}_T \bullet (\hat{z} \times d\overline{s}) = \mu^{-1} L \oint_C \overline{H}_T \bullet d\overline{s} = \mu^{-1} L i(z,t) \qquad (7.1.48)$$

Combining (7.1.47) and (7.1.48) yields:

$$\partial v(z,t)/\partial z = -L\,\partial i(z,t)/\partial t \qquad (7.1.49)$$

A similar contour integration of $\overline{H}_T$ to yield i(z,t) simplifies (7.1.46):

$$(\partial/\partial z)\int_C\left[(\hat{z}\times\overline{H}_T)\times\hat{z}\right]\bullet d\bar{s}=(\partial/\partial z)\oint_C\overline{H}_T\bullet d\bar{s}=\partial i/\partial z=(\sigma+\varepsilon\partial/\partial z)\oint_C(\overline{E}_T\times\hat{z})\bullet d\bar{s} \qquad (7.1.50)$$

The definitions of C (7.1.39) and G (7.1.41), combined with $(\overline{E}\times\hat{z})\bullet d\bar{s}=(\overline{E}_T\times d\bar{s})\bullet\hat{z}$ and the definition (7.1.43) of v, yields:

$$\partial i(z,t)/\partial z=-(G+C\,\partial/\partial t)\,v(z,t) \qquad (7.1.51)$$

This pair of equations, (7.1.49) and (7.1.51), can then be combined to yield a more complete description of wave propagation on general TEM lines.

Because the characteristic impedance and phase velocity for general TEM lines are frequency dependent, the simple solutions (7.1.49) and (7.1.51) are not convenient. Instead it is useful to express them as complex functions of ω:

$$\partial\underline{V}(z)/\partial z=-j\omega L\underline{I}(z) \qquad (7.1.52)$$

$$\partial\underline{I}(z)/\partial z=-(G+j\omega C)\underline{V}(z) \qquad (7.1.53)$$

Combining this pair of equations yields the wave equation:

$$\partial^2\underline{V}(z)/\partial z^2=j\omega L(G+j\omega C)\underline{V}(z) \qquad \text{(TEM wave equation)} \qquad (7.1.54)$$

The solution to this *TEM wave equation* must be a function that equals a constant times its own second derivative, such as:

$$\underline{V}(z)=\underline{V}_+e^{-j\underline{k}z}+\underline{V}_-e^{+j\underline{k}z} \qquad \text{(wave equation solution)} \qquad (7.1.55)$$

Substituting this assumed solution into the wave equation yields the *dispersion relation* for general TEM lines made with perfect conductors:

$$\underline{k}^2=-j\omega L(G+j\omega C) \qquad \text{(TEM dispersion relation)} \qquad (7.1.56)$$

This equation yields a complex value for the *TEM propagation constant* $\underline{k}$ = k' - jk", the significance of which is that the forward (V_+) and backward (V_-) propagating waves are exponentially attenuated with distance:

$$\underline{V}(z)=\underline{V}_+e^{-jk'z-k''z}+\underline{V}_-e^{+j\underline{k}'z+k''z} \qquad (7.1.57)$$

The current can be found by substituting (7.1.57) into (7.1.53) to yield:

$$\underline{I}(z)=(\underline{k}/j\omega L)\left(\underline{V}_+e^{-j\underline{k}z}-\underline{V}_-e^{+j\underline{k}z}\right)=\left(\underline{V}_+e^{-j\underline{k}z}-\underline{V}_-e^{+j\underline{k}z}\right)/\underline{Z}_o \qquad (7.1.58)$$

$$\underline{Z}_o = \left[j\omega L / (G + j\omega C) \right]^{0.5} \tag{7.1.59}$$

These expressions reduce to those for lossless TEM lines as $G \rightarrow 0$.

Another consequence of this dispersion relation (7.1.56) is that the *TEM phase velocity* v_p is frequency dependent and thus most lossy lines are dispersive:

$$v_p = \omega / k' = (LC)^{-0.5} (1 - jG/\omega C)^{-0.5} \tag{7.1.60}$$

Although most TEM lines also have resistance R per unit length, this introduces $E_z \neq 0$, so analysis becomes much more complex. In this case the approximate Telegrapher's equations (8.3.3–4) are often used.

Example 7.1B

What is the characteristic impedance Z_o for the air-filled *co-axial cable* illustrated in Figure 7.1.3 if the relevant diameters for the inner and outer conductors are a and b, respectively, where b/a = e? "Co-axial" means cylinders a and b share the same axis of symmetry.

Solution: $Z_o = (L/C)^{0.5}$ from (7.1.59). Since $c = (LC)^{-0.5}$ it follows that $L = (c^{-2}C^{-1})$ and $Z_o = 1/cC$ ohms. C follows from (7.1.39), which requires knowledge of the transverse electric field $\overline{E}_T$ (for TEM waves, there are no non-transverse fields). Symmetry in this cylindrical geometry requires $\overline{E}_T = \hat{r}E_o/r$. Thus

$$C = Q'/v = \left[\oiint_A \varepsilon_o \overline{E}_T \bullet \hat{r}\, da \right] \Big/ \left[\int_a^b \overline{E}_T \bullet d\overline{s} \right] = a^{-1}\left[\varepsilon_o E_o 2\pi a\right] \Big/ \left[\int_a^b E_o r^{-1} dr \right]$$

$= \varepsilon_o 2\pi \ln(b/a) = 2\pi\varepsilon_o = 56 \times 10^{-12}$ [F]. Therefore $Z_o = (56 \times 10^{-12} \times 3 \times 10^8)^{-1}$ $\cong 60$ ohms, and $L \cong 2 \times 10^{-7}$ [H].

7.1.4 Loss in transmission lines

Transmission line losses can be computed in terms of the resistance R, Ohms per meter, of TEM line length, or conductance G, Siemens/m, of the medium separating the two conductors. As discussed in Section 8.3.1, the time average power P_d dissipated per meter of length is simply the sum of the two contributions from the series and parallel conductances:

$$P_d(z)\,[\text{W/m}] = |\underline{I}(z)|^2 R/2 + |\underline{V}(z)|^2 G/2 \tag{7.1.61}$$

When R and G are unknown, resistive losses in transmission lines can be estimated by integrating $|\overline{J}|^2/2\sigma$ [W m^{-3}] over the volume of interest, where σ is the material conductivity [S m^{-1}] and $\overline{J}$ is the current density [A m^{-2}]. This surface loss density P_d [W m^{-2}] is derived for good conductors in Section 9.2 and is shown in (9.2.61) to be equal to the power dissipated by the same surface current $\underline{J}_s$ flowing uniformly through a slab of thickness δ, where $\delta = (2/\omega\mu\sigma)^{0.5}$ is the skin depth. The surface current $\bar{J}_s$ equals $|\underline{H}_s|$, which is the magnetic field parallel to the conductor surface. Therefore:

$$P_d \cong |\underline{H}_s|^2 \sqrt{\frac{\omega\mu}{8\sigma}} \ \left[W/m^2\right] \qquad \text{(power dissipation in conductors)} \qquad (7.1.62)$$

For example, it is easy to compute with (7.1.62) the power dissipated in a 50-ohm copper TEM coaxial cable carrying P_o = 10 watts of entertainment over a 500-MHz band with an inner conductor diameter of one millimeter. First we note that $|\underline{H}_s| = \underline{I}/2\pi r$ [A/m] where $|\underline{I}^2|Z_o/2 = P_o$ = 10, and $2r = 10^{-3}$ [m]. Therefore $|\underline{H}_s| = (P_o/Z_o)^{0.5}/2\pi r$ [A/m] ≅ 142. Also, since the diameter of the outer sheath is typically ~5 times that of the inner conductor, the surface current density there, J_s, is one fifth that for the inner conductor, and the power dissipation per meter length is also one fifth. Therefore the total power dissipated per meter, P_L, in both conductors is ~1.2 times that dissipated in the inner conductor alone. If we consider only the highest and most lossy frequency, and assume $\sigma = 5\times10^7$, then substituting $|\underline{H}_s|$ into (7.1.62) and integrating over both conductors yields the power loss:

$$\begin{aligned} P_L &\cong 1.2\times 2\pi r|\underline{H}_s|^2 (\omega\mu_o/4\sigma)^{0.5} = 1.2\times 2\pi r\left[(2P_o/Z_o)^{0.5}/2\pi r\right]^2 (\omega\mu_o/8\sigma)^{0.5} \\ &= 1.2\times P_o (Z_o\pi r)^{-1}(\omega\mu_o/2\sigma)^{0.5} = 12\left(50\pi 10^{-3}\right)^{-1}\left(2\pi\times5\times10^8\times4\pi\times10^{-7}/10^8\right)^{0.5} \\ &= 0.48 \ \text{ watts / meter} \end{aligned} \qquad (7.1.63)$$

The loss L [dB m^{-1}] is proportional to the ratio of P_L [W m^{-1}] to P_o [W]:

$$L\left[\text{dB m}^{-1}\right] = 4.34 \ P_L/P_o \qquad (7.1.64)$$

Thus P_L is 0.48 watts/meter, a large fraction of the ten watts propagating on the line. This loss of 4.8 percent of the power per meter, including the outer conductor, corresponds to 10 $\log_{10}(1 - 0.048) \cong -0.21$dB per meter. If we would like amplifiers along a cable to provide no more than ~50 dB gain, we need amplifiers every ~234 meters. Dropping the top frequency to 100 MHz, or increasing the diameter of the central wire could reduce these losses by perhaps a factor of ~4. These loss issues and desires for broad bandwidth are motivating substitution of low-loss optical fiber over long cable lines, and use of co-axial cables only for short hops from a local fiber to the home or business.

Example 7.1C

A perfectly conducting 50-ohm coaxial cable is filled with slightly conducting dielectric that gives the line a shunt conductivity $G = 10^{-6}$ Siemens m^{-1} between the two conductors. What is the attenuation of this cable (dB m^{-1})?

Solution: The attenuation L[dB m^{-1}] = 4.34 P_d/P_o (7.1.64), where the power on the line P_o [W] = $|\underline{V}|^2/2Z_o$, and the dissipation here is P_d [W m^{-1}] = $|\underline{V}|^2G/2$ (7.1.61); see Figure 8.3.1 for the incremental model of a lossy TEM transmission line. Therefore L = 4.34 GZ_o = 2.2×10^{-4} dB m^{-1}. This is generally independent of frequency and therefore might dominate at lower frequencies if the frequency-dependent dissipative losses in the wires become sufficiently small.

7.2 TEM lines with junctions

7.2.1 Boundary value problems

A junction between two transmission lines forces the fields in the first line to conform to the fields at the second line at the boundary between the two. This is a simple example of a broad class of problems called boundary value problems. The general electromagnetic *boundary value problem* involves determining exactly which, if any, combination of waves matches any given set of *boundary conditions*, which generally includes both active and passive boundaries, the active boundaries usually being sources. Boundary conditions generally constrain $\overline{E}$ and/or $\overline{H}$ for all time on the boundary of the one-, two- or three-dimensional region of interest.

The uniqueness theorem presented in Section 2.8 states that only one solution satisfies all Maxwell's equations if the boundary conditions are sufficient. Therefore we may solve boundary value problems simply by hypothesizing the correct combination of waves and testing it against Maxwell's equations. That is, we leave undetermined the numerical constants that characterize the chosen combination of waves, and then determine which values of those constraints satisfy Maxwell's equations. This strategy eases the challenge of hypothesizing the final answer directly. Moreover, symmetry and other considerations often suggest the nature of the wave combination required by the problem, thus reducing the numbers of unknown constants that must be determined.

The four basic steps for solving boundary value problems are:

1) Determine the natural behavior of each homogeneous section of the system without the boundaries.

2) Express this general behavior as the superposition of waves or static fields characterized by unknown constants; symmetry and other considerations can minimize the number of waves required. Here our basic building blocks are TEM waves.

3) Write equations for the boundary conditions that must be satisfied by these sets of superimposed waves, and then solve for the unknown constants.

4) Test the resulting solution against any of Maxwell's equations that have not already been imposed.

Variations of this four-step procedure can be used to solve almost any problem by replacing Maxwell's equations with their approximate equivalent for the given problem domain[34]. For example, profitability, available capital, technological constraints, employee capabilities, and customer needs are often "boundary conditions" when deriving strategies for start-up enterprises, while "natural behavior" could include the probable family of behaviors of the entrepreneurial team and its customers, financiers, and suppliers.

7.2.2 Waves at TEM junctions in the time domain

The boundary value problem approach described in Section 7.2.1 can be used for waves at TEM junctions. We assume that an arbitrary incident wave will produce both reflected and transmitted waves. For this introductory problem we also assume that no waves are incident from the other direction, for their solution could be superimposed later. Section 7.2.3 treats the same problem in the complex domain. We represent TEM lines graphically by parallel lines and their characteristic impedance Z_o, as illustrated in Figure 7.2.1 for lines a and b.

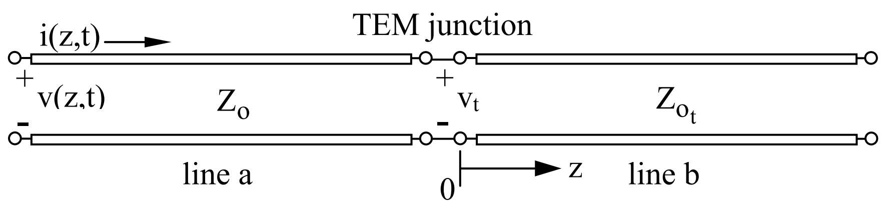

Figure 7.2.1 Junction of two TEM transmission lines.

Step one of the boundary value method involves characterizing the natural behavior of waves in the two media of interest, lines a and b. This follows from (7.1.16) for v(z,t) and (7.1.18) for i(z,t). Step two involves hypothesizing the form of the reflected and transmitted waves, $v_-(z,t)$ and $v_t(z,t)$. For simplicity we assume the source $v_+(z,t)$ is on the left, the TEM junction is at z = 0, and the line impedances Z_o are constants independent of time and frequency. Step three is to write the boundary conditions for the waves with unknown constants; v and i must both be constant across the junction at z = 0:

$$v(z,t) = v_+(z,t) + v_-(z,t) = v_t(z,t) \qquad \text{(at } z = 0\text{)} \qquad (7.2.1)$$

$$i(z,t) = Z_o^{-1}\left[v_+(z,t) - v_-(z,t)\right] = Z_t^{-1} v_t(t) \qquad \text{(at } z = 0\text{)} \qquad (7.2.2)$$

Step four involves solving (7.2.1) and (7.2.2) for the unknown waves $v_-(z,t)$ and $v_t(z,t)$. We can simplify the problem by taking the ratios of reflection and transmission relative to the incident wave and provide its amplitude later. If we regard the arguments (z=0, t) as understood, then (7.2.1) and (7.2.2) become:

[34] A key benefit of a technical education involves learning precise ways of thinking and solving problems; this procedure, when generalized, is an excellent example applicable to almost any career.

$$1+(v_-/v_+) = v_t/v_+ \tag{7.2.3}$$

$$1-(v_-/v_+) = (Z_o/Z_t)\, v_t/v_+ \tag{7.2.4}$$

To make the algebra for these two equations still more transparent it is customary to define v_-/v_+ as the *reflection coefficient* Γ, v_t/v_+ as the *transmission coefficient* T, and $Z_t/Z_o = Z_n$ as the *normalized impedance* for line b. Note that v_-, v_+, Z_o, and Z_t are real, and the fraction of incident power that is reflected from a junction is $|\Gamma|^2$. Equations (7.2.3) and (7.2.4) then become:

$$1+\Gamma = T \tag{7.2.5}$$

$$1-\Gamma = T/Z_n \tag{7.2.6}$$

Multiplying (7.2.6) by Z_n and subtracting the result from (7.2.5) eliminates T and yields:

$$\Gamma = \frac{v_-}{v_+} = \frac{Z_n - 1}{Z_n + 1} \tag{7.2.7}$$

$$v_-(0,t) = \left[(Z_n - 1)/(Z_n + 1)\right] v_+(0,t) \tag{7.2.8}$$

$$v_-(0+ct) = \left[(Z_n - 1)/(Z_n + 1)\right] v_+(0+ct) \tag{7.2.9}$$

$$v_-(z+ct) = \left[(Z_n - 1)/(Z_n + 1)\right] v_+(z+ct) \tag{7.2.10}$$

The transitions to (7.2.9) and (7.2.10) utilized the fact that if two functions of two arguments are equal for all values of their arguments, then the functions remain equal as their arguments undergo the same numerical shifts. For example, if X(a) = Y(b) where a and b have the same units, then X(a + c) = Y(b + c). Combining (7.2.3) and (7.2.7) yields the transmitted voltage v_t in terms of the source voltage v_+:

$$v_t(z-ct) = \left[2Z_n/(Z_n + 1)\right] v_+(z-ct) \tag{7.2.11}$$

This completes the solution for signal behavior at single TEM junctions.

Example 7.2A

Two parallel plates of width W and separation d_1 = 1 cm are connected at z = D to a similar pair of plates spaced only d_2 = 2 mm apart. If the forward wave on the first line is $V_o \cos(\omega t - kz)$, what voltage $v_t(t,z)$ is transmitted beyond the junction at z = D?

Solution: $v_t(t,z) = Tv_+(t,z) = (1 + \Gamma)v_+(t,z) = 2Z_n\ v_+(t,z)/(Z_n + 1)$, where $Z_n = Z_t/Z_o = \eta_o d_2 W/\eta_o d_1 W = d_2/d_1 = 0.2$. Therefore for z > D, $v_t(t,z) = v_+(t,z)2\times 0.2/(0.2 + 1) = (V_o/3)\cos(\omega t - kz)$ [V].

7.2.3 Sinusoidal waves on TEM transmission lines and at junctions

The basic equations characterizing lossless TEM lines in the sinusoidal steady state correspond to the pair of differential equations (7.1.25) and (7.1.26):

$$d\underline{V}(z)/dz = -j\omega L\underline{I}(z) \qquad (7.2.12)$$

$$d\underline{I}(z)/dz = -j\omega C\underline{V}(\ ) \qquad (7.2.13)$$

L and C are the inductance and capacitance of the line per meter, respectively.

This pair of equations leads easily to the *transmission line wave equation*:

$$d^2\underline{V}(z)/dz^2 = -\omega^2 LC\underline{V}(z) \qquad \text{(wave equation)} \qquad (7.2.14)$$

The solution $\underline{V}(z)$ to this wave equation involves exponentials in z because the second derivative of $\underline{V}(z)$ equals a constant times $\underline{V}(z)$. The exponents can be + or -, so in general a sum of these two alternatives is possible, where $\underline{V}_+$ and $\underline{V}_-$ are complex constants determined later by boundary conditions and k is given by (7.1.30):

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}\ [V] \qquad \text{(TEM voltage)} \qquad (7.2.15)$$

The corresponding current is readily found using (7.2.12):

$$\underline{I}(z) = (j/\omega L)d\underline{V}(z)/dz = (j/\omega L)\left(-jk\underline{V}_+ e^{-jkz} + jk\underline{V}_- e^{+jkz}\right) \qquad (7.2.16)$$

$$\underline{I}(z) = (1/Z_o)\left(\underline{V}_+^{-jkz} - \underline{V}_- e^{+jkz}\right) \qquad \text{(TEM current)} \qquad (7.2.17)$$

where the *characteristic impedance* Z_o of the line is:

$$Z_o = Y_o^{-1} = \omega L/k = cL = (L/C)^{0.5}\ [\text{ohms}] \qquad \text{(characteristic impedance)} \qquad (7.2.18)$$

The *characteristic admittance* Y_o of the line is the reciprocal of Z_o, and has units of Siemens or ohms^{-1}. It is important to appreciate the physical significance of Z_o; it is simply the ratio of voltage to current for a wave propagating in one direction only on the line, e.g., for the + wave only. This ratio does not correspond to dissipative losses in the line, although it is related to the power traveling down the line for any given voltage across the line.

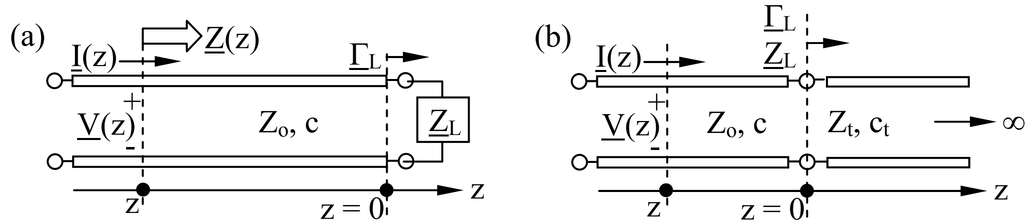

Figure 7.2.2 TEM transmission line impedances and coupling.

When there are both forward and backward waves on a line, the voltage/current ratio is called the complex impedance and varies with position, as suggested in Figure 7.2.2(a). The *impedance* at any point along the line is defined as:

$$\underline{Z}(z) \equiv \underline{V}(z)/\underline{I}(z) = Z_o\left[1+\underline{\Gamma}(z)\right]/\left[1-\underline{\Gamma}(z)\right] \\ = Z_o\left[1+\underline{\Gamma}(z)\right]/\left[1-\underline{\Gamma}(z)\right] \text{ ohms} \qquad \text{(line impedance)} \qquad (7.2.19)$$

The complex *reflection coefficient* $\underline{\Gamma}(z)$ is defined as:

$$\underline{\Gamma}(z) \equiv \underline{V}_- e^{+jkz} \Big/ \underline{V}_+ e^{-jkz} = \left(\underline{V}_-/\underline{V}_+\right)e^{2jkz} = \underline{\Gamma}_L e^{2jkz} \qquad \text{(reflection coefficient)} \qquad (7.2.20)$$

When z = 0 at the load, then $\underline{V}_-/\underline{V}_+$ is defined at the load and $\underline{\Gamma}_L$ is the load reflection coefficient, denoted by the subscript L.

Equation (7.2.20) leads to a simple algorithm for relating impedances at different points along the line. We first define normalized impedance $\underline{Z}_n$ and relate it to the reflection coefficient $\underline{\Gamma}(z)$ using (7.2.19); (7.2.22) follows from (7.2.21):

$$\underline{Z}_n(z) \equiv \frac{\underline{Z}(z)}{Z_o} = \frac{1+\underline{\Gamma}(z)}{1-\underline{\Gamma}(z)} \qquad \text{(normalized impedance)} \qquad (7.2.21)$$

$$\underline{\Gamma}(z) = \frac{\underline{Z}_n(z)-1}{\underline{Z}_n(z)+1} \qquad (7.2.22)$$

For example, we can see the effect of the load impedance $\underline{Z}_L$ (z = 0) at some other point z on the line by using (7.2.20–22) in an appropriate sequence:

$$\underline{Z}_L \rightarrow \underline{Z}_{Ln} \rightarrow \underline{\Gamma}_L \rightarrow \underline{\Gamma}(z) \rightarrow \underline{Z}_n(z) \rightarrow \underline{Z}(z) \qquad \text{(impedance transformation)} \qquad (7.2.23)$$

A simple example of the use of (7.2.23) is the transformation of a 50-ohm resistor by a 100-ohm line λ/4 long. Using (7.2.23) in sequence, we see $\underline{Z}_L$ = 50, $\underline{Z}_{Ln}$ = 50/100 = 0.5, $\underline{\Gamma}_L$ = -1/3 from (7.2.22), $\underline{\Gamma}(z = -\lambda/4)$ = +1/3 from (7.2.20) where $e^{+2jkz} = e^{2j(2\pi/\lambda)(-\lambda/4)} = e^{-j\pi} = -1$, $\underline{Z}_n(-\lambda/4)$ = 2 from (7.2.21), and therefore $\underline{Z}(-\lambda/4)$ = 200 ohms.

Two other impedance transformation techniques are often used instead: a direct equation and the Smith chart (Section 7.3). The direct equation (7.2.24) can be derived by first substituting $\underline{\Gamma}_L = (\underline{Z}_L - Z_o)/(\underline{Z}_L + Z_o)$, i.e. (7.2.22), into $\underline{Z}(z) = \underline{V}(z)/\underline{I}(z)$, where $\underline{V}(z)$ and $\underline{I}(z)$ are given by (7.2.15) and (7.2.17), respectively, and $\underline{V}_-/\underline{V}_+ = \underline{\Gamma}_L$. The next step involves grouping the exponentials to yield sin kz and cos kz, and then dividing sin by cos to yield tan and the solution:

$$\underline{Z}(z) = Z_o \frac{\underline{Z}_L - jZ_o \tan kz}{Z_o - j\underline{Z}_L \tan kz} \qquad \text{(transformation equation)} \qquad (7.2.24)$$

A closely related problem is illustrated in Figure 7.2.2(b) where two transmission lines are connected together and the right-hand line presents the impedance $\underline{Z}_t$ at z = 0. To illustrate the general method for solving boundary value problems outlined in Section 7.2.1, we shall use it to compute the reflection and transmission coefficients at this junction. The expressions (7.2.15) and (7.2.17) nearly satisfy the first two steps of that method, which involve writing trial solutions composed of superimposed waves with unknown coefficients that satisfy the wave equation within each region of interest. The third step is to write equations for these waves that satisfy the boundary conditions, and then to solve for the unknown coefficients. Here the boundary conditions are that both $\underline{V}$ and $\underline{I}$ are continuous across the junction at z = 0; the subscript t corresponds to the transmitted wave. The two waves on the left-hand side have amplitudes $\underline{V}_+$ and $\underline{V}_-$, whereas the wave on the right-hand side has amplitude $\underline{V}_t$. We assume no energy enters from the right. Therefore:

$$\underline{V}(0) = \underline{V}_+ + \underline{V}_- = \underline{V}_t \qquad (7.2.25)$$

$$\underline{I}(0) = \left(\underline{V}_+ - \underline{V}_-\right)/Z_o = \underline{V}_t/\underline{Z}_t \qquad (7.2.26)$$

We define the complex reflection and transmission coefficients at the junction (z = 0) to be $\underline{\Gamma}$ and $\underline{T}$, respectively, where:

$$\underline{\Gamma} = \underline{V}_-/\underline{V}_+ \qquad \text{(complex reflection coefficient)} \qquad (7.2.27)$$

$$\underline{T} = \underline{V}_t/\underline{V}_+ \qquad \text{(complex transmission coefficient)} \qquad (7.2.28)$$

We may solve for $\underline{\Gamma}$ and $\underline{T}$ by first dividing (7.2.25) and (7.2.26) by $\underline{V}_+$:

$$1 + \underline{\Gamma} \quad \underline{T} \qquad (7.2.29)$$

$$1 - \underline{\Gamma} = \left(Z_o/\underline{Z}_t\right)\underline{T} \qquad (7.2.30)$$

This pair of equations is readily solved for $\underline{\Gamma}$ and $\underline{T}$:

$$\underline{\Gamma} = \frac{\underline{Z}_t - Z_o}{\underline{Z}_t + Z_o} = \frac{\underline{Z}_n - 1}{\underline{Z}_n + 1} \qquad (7.2.31)$$

$$\underline{T} = \underline{\Gamma} + 1 = \frac{2\underline{Z}_n}{\underline{Z}_n + 1} \qquad (7.2.32)$$

where normalized impedance was defined in (7.2.21) as $\underline{Z}_n \equiv \underline{Z}_t/Z_o$. For example, (7.2.31) says that the reflection coefficient $\underline{\Gamma}$ is zero when the normalized impedance is unity and the line impedance is matched, so $\underline{Z}_t = Z_o$; (7.2.32) then yields $\underline{T} = 1$.

The complex coefficients $\underline{\Gamma}$ and $\underline{T}$ refer to wave amplitudes, but often it is power that is of interest. In general the time-average power incident upon the junction is:

$$P_+ = \underline{V}_+\underline{I}_+^*/2 = |\underline{V}_+|^2/2Z_o \text{ [W]} \qquad \text{(incident power)} \qquad (7.2.33)$$

Similarly the reflected and transmitted powers are P_- and P_t, where $P_- = |\underline{V}_-|^2/2Z_o$ and $P_t = |\underline{V}_t|^2/2Z_t$ [W].

Another consequence of having both forward and backward moving waves on a TEM line is that the magnitudes of the voltage and current vary along the length of the line. The expression for voltage given in (7.2.15) can be rearranged as:

$$|\underline{V}(z)| = \left|\underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}\right| = \left|\underline{V}_+ e^{-jkz}\right| \left|1 + \underline{\Gamma}(z)\right| \qquad (7.2.34)$$

The magnitude of $|\underline{V}_+ e^{-jkz}|$ is independent of z, so the factor $|1 + \underline{\Gamma}(z)|$ controls the magnitude of voltage on the line, where $\underline{\Gamma}(z) = \underline{\Gamma}_L e^{2jkz}$ (7.2.20). Figure 7.2.3(a) illustrates the behavior of $|V(z)|$; it is quasi-sinusoidal with period $\lambda/2$ because of the 2jkz in the exponent. The maximum value $|V(z)|_{max} = |V_+| + |V_-|$ occurs when $\underline{\Gamma}(z) = |\underline{\Gamma}|$; the minimum occurs when $\underline{\Gamma}(z) = -|\underline{\Gamma}|$.

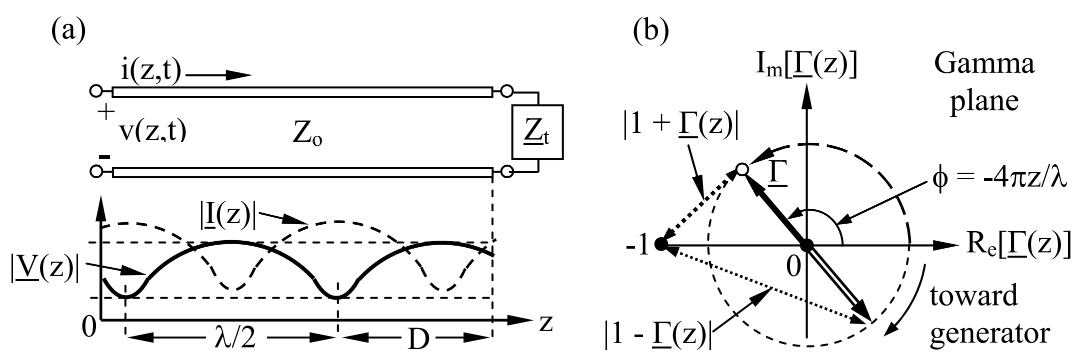

Figure 7.2.3 Standing waves on a TEM line and the Gamma plane.

The origins of this behavior of $|\underline{V}(z)|$ is suggested in Figure 7.2.3(b), which illustrates the z dependence of $\underline{\Gamma}(z)$ in the complex *gamma plane*, where the horizontal and vertical axes are the real and imaginary parts of $\underline{\Gamma}(z)$, respectively. Increases in z simply rotate the vector $\underline{\Gamma}(z)$ clockwise, preserving its magnitude [see (7.2.20) and Figure 7.2.3(b)].

The quasi-sinusoidal form of $|\underline{V}(z)|$ arises because $|\underline{V}(z)| \propto |1 + \underline{\Gamma}(z)|$, which is the length of the vector linking $\underline{\Gamma}(z)$ with the point -1 on the gamma plane, as illustrated in Figure 7.2.3(b). As the phase ϕ of $\underline{\Gamma}$ varies with z and circles the diagram, the vector $1 + \underline{\Gamma}(z)$ varies as might an arm turning a crank, and so it is sometimes called the "crank diagram". When $|\underline{\Gamma}| << 1$ then $|\underline{V}(z)|$ resembles a weak sinusoid oscillating about a mean value of $|\underline{V}_+|$, whereas when $|\underline{\Gamma}| \cong 1$ then $|\underline{V}(z)|$ resembles a fully rectified sinusoid. The voltage envelope $|\underline{V}(z)|$ is called the standing-wave pattern, and fields have a standing-wave component when $|\underline{\Gamma}| > 0$. The figure also illustrates how $|\underline{I}(z)| \propto |1 - \underline{\Gamma}(z)|$ exhibits the same quasi-sinusoidal variation as $|\underline{V}(z)|$, but 180 degrees out of phase.

Because |V(z)| and |I(z)| are generally easy to measure along any transmission line, it is useful to note that such measurements can be used to determine not only the fraction of power that has been reflected from any load, and thus the efficiency of any connection, but also the impedance of the load itself. First we define the *voltage standing wave ratio* or *VSWR* as:

$$\text{VSWR} \equiv |\underline{V}(z)|_{max} / |\underline{V}(z)|_{min} = \left(|\underline{V}_+| + |\underline{V}_-|\right) / \left(|\underline{V}_+| - |\underline{V}_-|\right) = \left(1 + |\underline{\Gamma}|\right) / \left(1 - |\underline{\Gamma}|\right) \qquad (7.2.35)$$

Therefore:

$$|\underline{\Gamma}| = (\text{VSWR} - 1)/(\text{VSWR} + 1) \qquad (7.2.36)$$

$$P_-/P_+ = |\underline{\Gamma}|^2 = \left[(\text{VSWR} - 1)/(\text{VSWR} + 1)\right]^2 \qquad (7.2.37)$$

This simple relation between VSWR and fractional power reflected (P_-/P_+) helped make VSWR a common specification for electronic equipment.

To find the load impedance $\underline{Z}_L$ from observations of $|\underline{V}(z)|$ such as those plotted in Figure 7.2.3(a) we first associate any voltage minimum with that point on the gamma plane that corresponds to $-|\underline{\Gamma}|$. Then we can rotate on the gamma plane counter-clockwise (toward the load) an angle $\phi = 2kD = 4\pi D/\lambda$ radians that corresponds to the distance D between that voltage minimum and the load, where a full revolution in the gamma plane corresponds to $D = \lambda/2$. Once $\underline{\Gamma}$ for the load is determined, it follows from (7.2.21) that:

$$\underline{Z}_L = Z_o[1 + \underline{\Gamma}]/[1 - \underline{\Gamma}] \qquad (7.2.38)$$

If more than two TEM lines join a single junction then their separate impedances combine in series or parallel, as suggested in Figure 7.2.4. The impedances add in parallel for Figure 7.2.4(a) so the impedance at the junction as seen from the left would be:

$$\underline{Z}_{parallel} = \underline{Z}_a \underline{Z}_b / (\underline{Z}_a + \underline{Z}_b) \qquad (7.2.39)$$

For Figure 7.2.4(b) the lines are connected in series so the impedance seen from the left would be $\underline{Z}_a + \underline{Z}_b$.

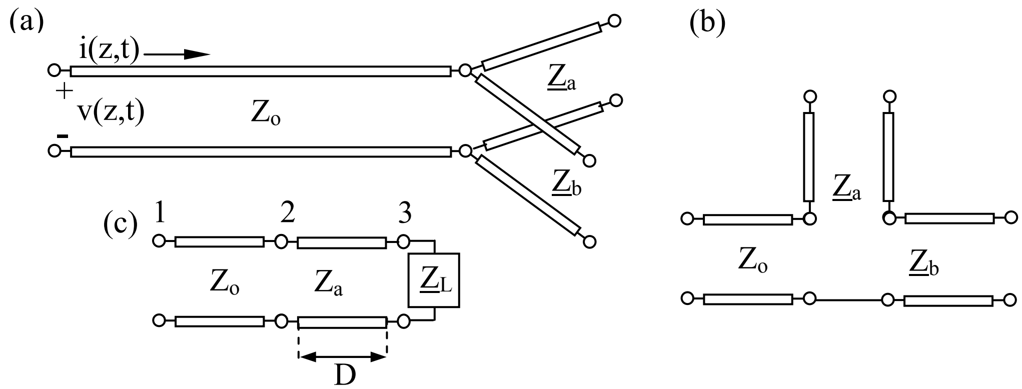

Figure 7.2.4 Multiple connected TEM lines.

Figure 7.2.4(c) illustrates how TEM lines can be concatenated. In this case the impedance $\underline{Z}_1$ seen at the left-hand terminals could be determined by transforming the impedance $\underline{Z}_L$ at terminals (3) to the impedance $\underline{Z}_2$ that would be seen at terminals (2). The impedance seen at (2) could then be transformed a second time to yield the impedance seen at the left-hand end. The algorithm for this might be:

$$\underline{Z}_L \to \underline{Z}_{Ln} \to \underline{\Gamma}_3 \to \underline{\Gamma}_2 \to \underline{Z}_{n2} \to \underline{Z}_2 \to \underline{Z}_{n2'} \to \underline{\Gamma}_{2'} \to \underline{\Gamma}_1 \to \underline{Z}_{n1} \to \underline{Z}_1 \qquad (7.2.40)$$

Note that Z_{n2} is normalized with respect to Z_a and Z_{n2}' is normalized with respect to Z_o; both are defined at junction (2). Also, $\underline{\Gamma}_2$ is the reflection coefficient at junction (2) within the line Z_a, and $\underline{\Gamma}_2'$ is the reflection coefficient at junction (2) within the line Z_o.

Example 7.2B

A 100-ohm air-filled TEM line is terminated at z = 0 with a capacitor $C = 10^{-11}$ farads. What is $\underline{\Gamma}(z)$? At what positions z < 0 are voltage minima located on the line when f = 1/2π GHz? What is the VSWR? At z = - λ/4, what is the equivalent impedance?

Solution: The normalized load impedance $Z_L/Z_o \equiv Z_{Ln} = 1/j\omega C Z_o = -j/(10^9 \times 10^{-11} \times 100) = -j$, and (7.2.22) gives $\underline{\Gamma}_L = (\underline{Z}_{Ln} - 1)/(\underline{Z}_{Ln} + 1) = -(1+j)/(1-j) = -j$. $\underline{\Gamma}(z) = \underline{\Gamma}_L e^{2jkz} = -je^{2jkz}$. (7.2.34) gives $|\underline{V}(z)| \propto |1 + \underline{\Gamma}(z)| = |1 - je^{2jkz}| = 0$ when $e^{2jkz} = -j = e^{-j(\pi/2 + n2\pi)}$, where n =0,1,2… Therefore $2jkz = -j(\pi/2 + n2\pi)$, so z(nulls) = $-(\pi/2 + n2\pi)\lambda/4\pi = -(\lambda/8)(1 + 4n)$. But $f = 10^9/2\pi$, and so $\lambda = c/f = 2\pi c \times 10^{-9} = 0.6\pi$ [m]. (7.2.34) gives VSWR = $(1 + |\underline{\Gamma}|)/(1 - |\underline{\Gamma}|) = \infty$. At $z = -\lambda/4$, $\underline{\Gamma} \to -\underline{\Gamma}_L = +j$ via (7.2.20), so by (7.2.38) $\underline{Z} = Z_o[1 + \underline{\Gamma}]/[1 - \underline{\Gamma}] = 100[1 + j]/[1 - j] = j100 = j\omega L_o \Rightarrow L_o = 100/\omega = 100/10^9 = 10^{-7}$ [H].

Example 7.2C

The VSWR observed on a 100-ohm air-filled TEM transmission line is 2. The voltage minimum is 15 cm from the load and the distance between minima is 30 cm. What is the frequency of the radiation? What is the impedance $\underline{Z}_L$ of the load?

Solution: The distance between minima is $\lambda/2$, so $\lambda = 60$ cm and $f = c/\lambda = 3\times10^8/0.6 = 500$ MHz. The load impedance is $\underline{Z}_L = Z_o[1 + \underline{\Gamma}_L]/[1 - \underline{\Gamma}_L]$ (7.2.38) where $|\underline{\Gamma}_L| = (\text{VSWR} - 1)/(\text{VSWR} + 1) = 1/3$ from (5.2.83). $\underline{\Gamma}_L$ is rotated on the Smith chart 180 degrees counter-clockwise (toward the load) from the voltage minimum, corresponding to a quarter wavelength. The voltage minimum must lie on the negative real $\underline{\Gamma}$ axis, and therefore $\underline{\Gamma}_L$ lies on the positive real $\underline{\Gamma}$ axis. Therefore $\underline{\Gamma}_L = 1/3$ and $\underline{Z}_L = 100(1 + 1/3)/(1 - 1/3) = 200$ ohms.

7.3 Methods for matching transmission lines

7.3.1 Frequency-dependent behavior

This section focuses on the frequency-dependent behavior introduced by obstacles and impedance transitions in transmission lines, including TEM lines, waveguides, and optical systems. Frequency-dependent transmission line behavior can also be introduced by loss, as discussed in Section 8.3.1, and by the frequency-dependent propagation velocity of waveguides and optical fibers, as discussed in Sections 9.3 and 12.2.

The basic issue is illustrated in Figure 7.3.1(a), where an obstacle reflects some fraction of the incident power. If we wish to eliminate losses due to reflections we need to cancel the reflected wave by adding another that has the same magnitude but is 180° out of phase. This can easily be done by adding another obstacle in front of or behind the first with the necessary properties, as suggested in (b). However, the reflections from the further obstacle can bounce between the two obstacles multiple times, and the final result must consider these additional rays too. If the reflections are small the multiple reflections become negligible. This strategy works for any type of transmission line, including TEM lines, waveguides and optical systems.

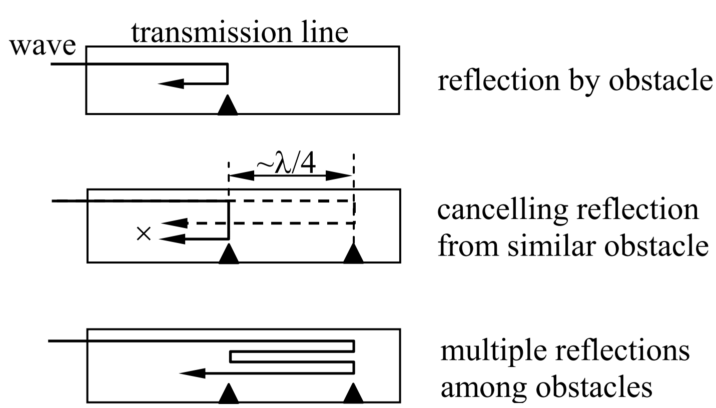

Figure 7.3.1 Cancellation of reflections on transmission lines.

The most important consequence of any such tuning strategy to eliminate reflections is that the two reflective sources are often offset spatially, so the relative phase between them is wavelength dependent. If multiple reflections are important, this frequency dependence can increase substantially. Rather than consider all these reflections in a tedious way, we can more directly solve the equations by extending the analysis of Section 7.2.3, which is summarized below in the context of TEM lines having characteristic admittance Y_o and a termination of complex impedance $\underline{Z}_L$:

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz} \text{ [V]} \tag{7.3.1}$$

$$\underline{I}(z) = Y_o\left(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz}\right) \text{ [A]} \tag{7.3.2}$$

$$\underline{\Gamma}(z) \equiv \left(\underline{V}_- e^{jkz}\right)/\left(\underline{V}_+ e^{-jkz}\right) = \left(\underline{V}_-/\underline{V}_+\right)e^{2jkz} = \left(\underline{Z}_n - 1\right)/\left(\underline{Z}_n + 1\right) \tag{7.3.3}$$

The normalized impedance $\underline{Z}_n$ is defined as:

$$\underline{Z}_n \equiv \underline{Z}/Z_o = \left[1 + \underline{\Gamma}(z)\right]/\left[1 - \underline{\Gamma}(z)\right] \tag{7.3.4}$$

$\underline{Z}_n$ can be related to $\underline{\Gamma}(z)$ by dividing (7.3.1) by (7.3.2) to find $\underline{Z}(z)$, and the inverse relation (7.3.3) follows. Using (7.3.3) and (7.3.4) in the following sequences, the impedance $\underline{Z}(z_2)$ at any point on an unobstructed line can be related to the impedance at any other point z_1:

$$\underline{Z}(z_1) \Leftrightarrow \underline{Z}_n(z_1) \Leftrightarrow \underline{\Gamma}(z_1) \Leftrightarrow \underline{\Gamma}(z_2) \Leftrightarrow \underline{Z}_n(z_2) \Leftrightarrow \underline{Z}(z_2) \tag{7.3.5}$$

The five arrows in (7.3.5) correspond to application of equations (7.3.3) and (7.3.4) in the following left-to-right sequence: (4), (3), (3), (4), (4), respectively.

One standard problem involves determining $\underline{Z}(z)$ (for $z < 0$) resulting from a load impedance $\underline{Z}_L$ at $z = 0$. One approach is to replace the operations in (7.3.5) by a single equation, derived in (7.2.24):

$$\underline{Z}(z) = Z_o\left(\underline{Z}_L - jZ_o \tan kz\right)/\left(Z_o - j\underline{Z}_L \tan kz\right) \qquad (\textit{impedance transformation}) \tag{7.3.6}$$

For example, if $\underline{Z}_L = 0$, then $\underline{Z}(z) = -jZ_o \tan kz$, which means that $\underline{Z}(z)$ can range between $-j\infty$ and $+j\infty$, depending on z, mimicking any reactance at a single frequency. The impedance repeats at distances of $\Delta z = \lambda$, where $k(\Delta z) = (2\pi/\lambda)\Delta z = 2\pi$. If $\underline{Z}_L = Z_o$, then $\underline{Z}(z) = Z_o$ everywhere.

Example 7.3A

What is the impedance at 100 MHz of a 100-ohm TEM line $\lambda/4$ long and connected to a: 1) short circuit? 2) open circuit? 3) 50-ohm resistor? 4) capacitor $C = 10^{-10}$ F?

Solution: In all four cases the relation between $\underline{\Gamma}(z = 0) = \underline{\Gamma}_L$ at the load, and $\underline{\Gamma}(z = -\lambda/4)$ is the same [see (7.3.3)]: $\underline{\Gamma}(z = -\lambda/4) = \underline{\Gamma}_L e^{2jkz} = \underline{\Gamma}_L e^{2j(2\pi/\lambda)(-\lambda/4)} = -\underline{\Gamma}_L$. Therefore in all four cases we see from (7.3.4) that $\underline{Z}_n(z = -\lambda/4) = (1 - \underline{\Gamma}_L)/(1 + \underline{\Gamma}_L) = 1/\underline{Z}_n(0)$. $\underline{Z}_n(z = 0)$ for these four cases is: 0, ∞, 0.5, and $1/j\omega C Z_o = 1/(j2\pi 10^8 10^{-10} 100) = 1/j2\pi$, respectively. Therefore $\underline{Z}(z = -\lambda/4) = 100\underline{Z}_n^{-1}$ ohms, which for these four cases equals: ∞, 0, 200, and $j200\pi$ ohms, respectively. Since the impedance of an inductor is $\underline{Z} = j\omega L$, it follows that $j200\pi$ is equivalent at 100 MHz to $L = 200\pi/\omega = 200\pi/200\pi 10^8 = 10^{-8}$ [Hy].

7.3.2 Smith chart, stub tuning, and quarter-wave transformers

A common problem is how to cancel reflections losslessly, thus forcing all incident power into a load. This requires addition of one or more reactive impedances in series or in parallel with the line so as to convert the impedance at that point to Z_o, where it must remain for all points closer to the source or next obstacle. Software tools to facilitate this have been developed, but a simple graphical tool, the *Smith chart*, provides useful insight into what can easily be matched and what cannot. Prior to computers it was widely used to design and characterize microwave systems.

The key operations in (7.3.5) are rotation on the *gamma plane* $[\underline{\Gamma}(z_1) \Leftrightarrow \underline{\Gamma}(z_2)]$ and the conversions $\underline{Z}_n \Leftrightarrow \underline{\Gamma}$, given by (7.3.3–4). Both of these operations can be accommodated on a single graph that maps the one-to-one relationship $\underline{Z}_n \Leftrightarrow \underline{\Gamma}$ on the complex gamma plane, as suggested in Figure 7.3.2(a). Conversions $\underline{\Gamma}(z_1) \Leftrightarrow \underline{\Gamma}(z_2)$ are simply rotations on the gamma plane. The gamma plane was introduced in Figure 7.2.3. The Smith chart simply overlays the equivalent normalized impedance values $\underline{Z}_n$ on the gamma plane; only a few key values are indicated in the simplified version shown in (a). For example, the loci for which the real R_n and imaginary parts X_n of $\underline{Z}_n$ remain constant lie on segments of circles ($\underline{Z}_n \equiv R_n + jX_n$) .

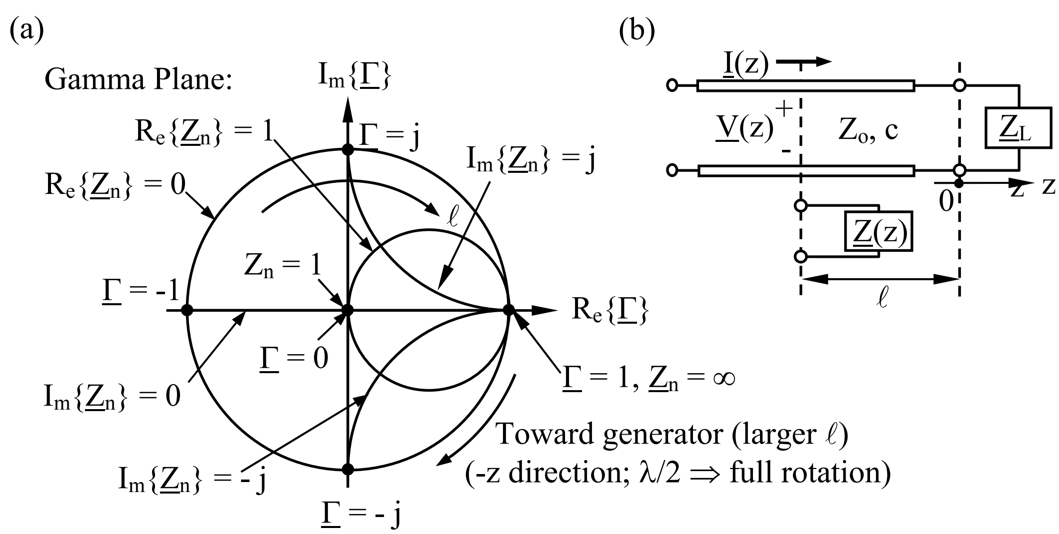

Figure 7.3.2 Relation between the gamma plane and the Smith chart.

Rotation on the gamma plane relates the values of $\underline{Z}_n$ and $\underline{\Gamma}$ at one z to their values at another, as suggested in Figure 7.3.2(b). Since $\underline{\Gamma}(z) = (\underline{V}_-/\underline{V}_+)e^{2jkz} = \underline{\Gamma}_L e^{2jkz} = \underline{\Gamma}_L e^{-2jk\ell}$, and since $e^{j\phi}$ corresponds to counter-clockwise rotation as ϕ increases, movement toward the generator (-z direction) corresponds to clockwise rotation in the gamma plane. The exponent of $e^{-2jk\ell}$ is $-j4\pi\ell/\lambda$, so a full rotation on the gamma plane corresponds to movement ℓ down the line of only $\lambda/2$.

A simple example illustrates the use of the Smith chart. Consider an inductor having $j\omega L = j100$ on a 100-ohm line. Then $\underline{Z}_n = j$, which corresponds to a point at the top of the Smith chart where $\underline{\Gamma} = +j$ (normally $\underline{Z}_n \neq \underline{\Gamma}$). If we move toward the generator $\lambda/4$, corresponding to rotation of $\underline{\Gamma}(z)$ half way round the Smith chart, then we arrive at the bottom where $\underline{Z}_n = -j$ and $\underline{Z} = Z_o\underline{Z}_n = -j100 = 1/j\omega C$. So the equivalent capacitance C at the new location is $1/100\omega$ farads.

The Smith chart has several other interesting properties. For example, rotation half way round the chart (changing $\underline{\Gamma}$ to $-\underline{\Gamma}$) converts any normalized impedance into the corresponding normalized admittance. This is easily proved: since $\underline{\Gamma} = (\underline{Z}_n - 1)/(\underline{Z}_n + 1)$, conversion of $\underline{Z}_n \rightarrow \underline{Z}_n^{-1}$ yields $\underline{\Gamma}' = (\underline{Z}_n^{-1} - 1)/(\underline{Z}_n^{-1} + 1) = (1 - Z_n)/(Z_n + 1) = -\underline{\Gamma}$ [Q.E.D.][35] Pairs of points with this property include $\underline{Z}_n = \pm j$ and $\underline{Z}_n = (0,\infty)$.

Another useful property of the Smith chart is that the voltage-standing-wave ratio (VSWR) equals the maximum positive real value $R_{n\,max}$ of $\underline{Z}_n$ lying on the circular locus occupied by $\underline{\Gamma}(z)$. This is easily shown from the definition of VSWR:

$$\begin{aligned}\text{VSWR} &\equiv |\underline{V}_{max}|/|\underline{V}_{min}| = \left(\left|\underline{V}_+e^{-jkz}\right| + \left|\underline{V}_-e^{+jkz}\right|\right)\Big/\left(\left|\underline{V}_+e^{-jkz}\right| - \left|\underline{V}_-e^{+jkz}\right|\right) \\ &\equiv \left(1+|\underline{\Gamma}|\right)\big/\left(1-|\underline{\Gamma}|\right) = R_{n\,max}\end{aligned} \quad (7.3.7)$$

A more important use of the Smith chart is illustrated in Figure 7.3.3, where the load 60 + j80 is to be matched to a 100-ohm TEM line so all the power is dissipated in the 60-ohm resistor. In particular the length ℓ of the transmission line in Figure 7.3.3(a) is to be chosen so as to transform $\underline{Z}_L = 60 + 80j$ so that its real part becomes Z_o. The new imaginary part can be cancelled by a reactive load (L or C) that will be placed either in position M or N. The first step is to locate $\underline{Z}_n$ on the Smith chart at the intersection of the $R_n = 0.6$ and $X_n = 0.8$ circles, which happen to fall at $\underline{\Gamma} = 0.5j$. Next we locate the gamma circle $\underline{\Gamma}(z)$ along which we can move by varying ℓ. This intersects the $R_n = 1$ circle at point "a" after rotating toward the generator "distance A". Next we can add a negative reactance to cancel the reactance $jX_n = +1.18j$ at point "a" to yield $\underline{Z}_n(a) = 1$ and $\underline{Z} = Z_o$. A negative reactance is a capacitor C in series at location M in the circuit. Therefore $1/j\omega C = -1.18jZ_o$ and $C = (1.18\omega Z_o)^{-1}$. The required line length ℓ corresponds to $\sim 0.05\lambda$, a scale for which is printed on the perimeter of official charts as illustrated in Figure 7.3.4.

---

[35] Q.E.D. is an abbreviation for the Latin phrase "quod erat demonstratum", or "that which was to be demonstrated".

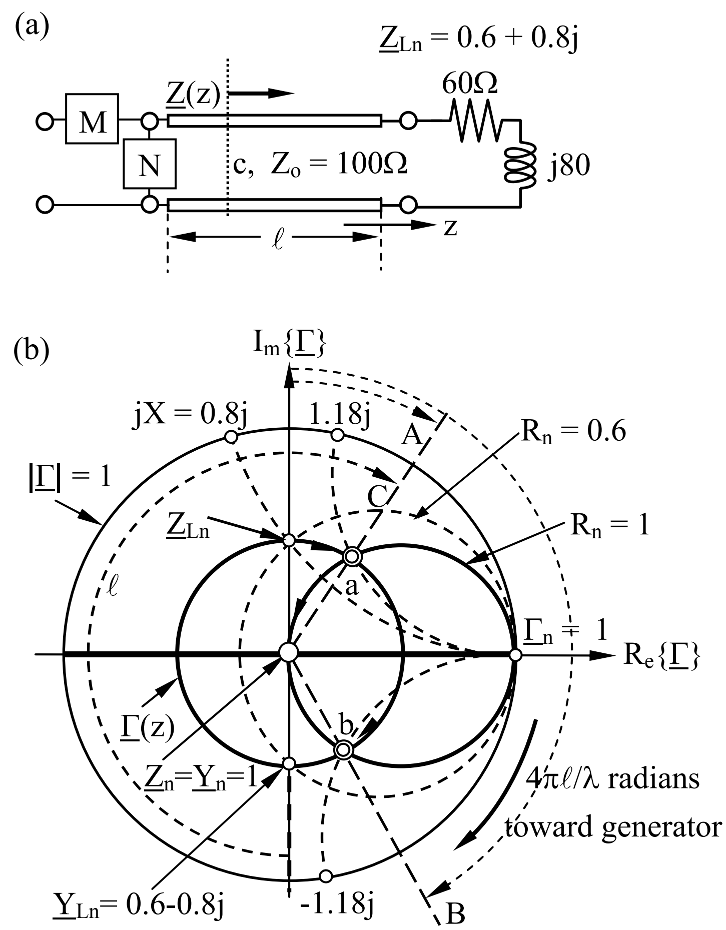

Figure 7.3.3 Matching a reactive load using the Smith chart.

More than three other matching schemes can be used here. For example, we could lengthen ℓ to distance "B" and point "b", where a positive reactance of $X_n = 1.18$ could be added in series at position M to provide a match. This requires an inductor $L = 1.18Z_o/\omega$.

Alternatively, we could note that $\underline{Z}_{Ln}$ corresponds to $\underline{Y}_{Ln} = 0.6 - 0.8j$ on the opposite side of the chart ($\underline{\Gamma} \rightarrow -\underline{\Gamma}$), where the fact that both $\underline{Z}_{Ln}$ and $\underline{Y}_{Ln}$ have the same real parts is a coincidence limited to cases where $\underline{\Gamma}_L$ is pure imaginary. Rotating toward the generator distance C again puts us on the $G_n = 1$ circle ($\underline{Y}_n \equiv G_n + jB_n$), so we can add a negative admittance B_n of -1.18j to yield Y_o. Adding a negative admittance in parallel at $z = -\ell$ corresponds to adding an inductor L in position N, where $-jZ_oX_n = 1/j\omega L$, so $L = (1.18\omega Z_o)^{-1}$. By rotating further to point "b" a capacitor could be added in parallel instead of the inductor. Generally one uses the shortest line length possible and the smallest, lowest-cost, lowest-loss reactive element.

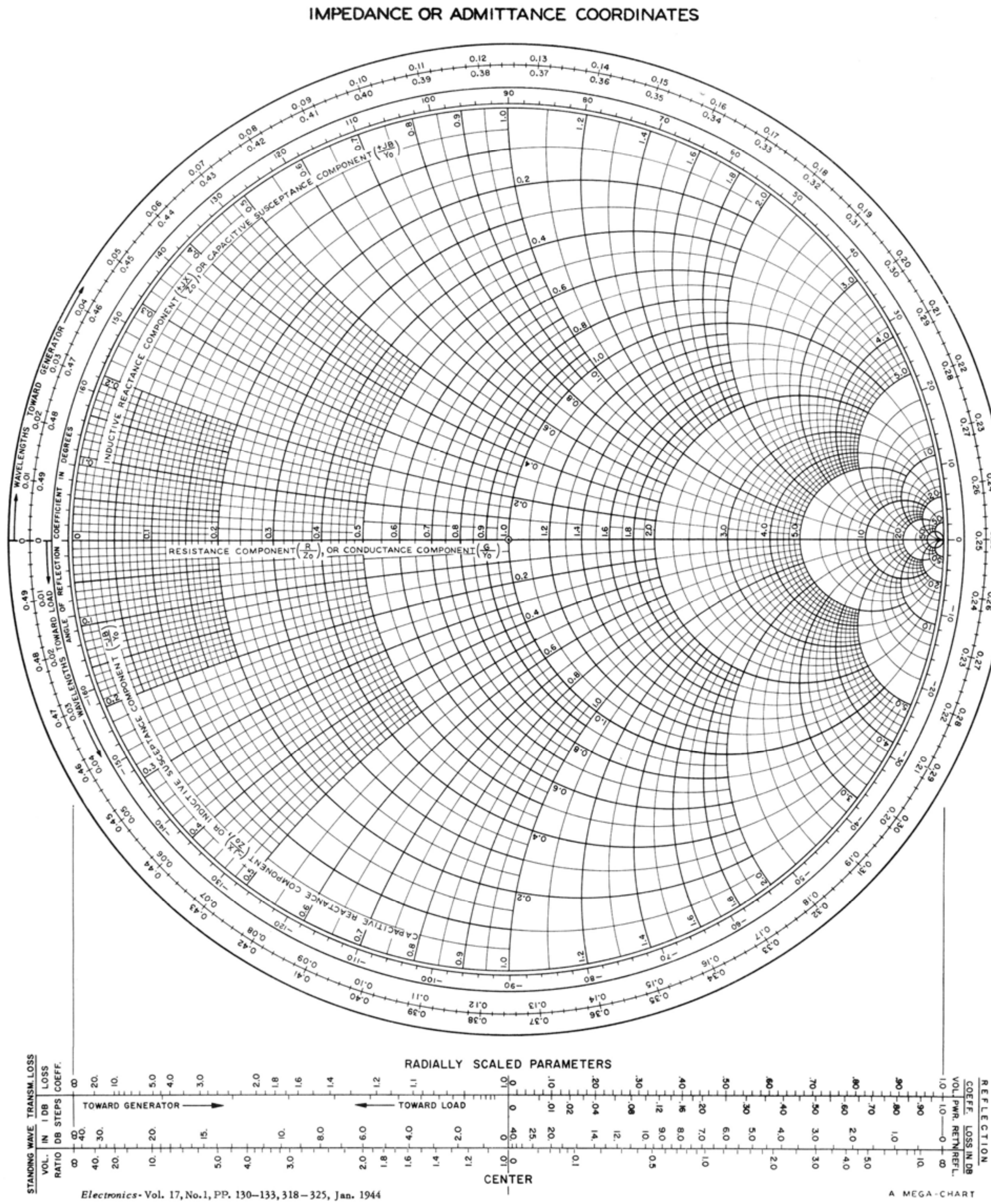

Figure 7.3.4 Smith chart.

Often printed circuits do not add capacitors or inductors to tune devices, but simply print an extra TEM line on the circuit board that is open- or short-circuit at its far end and is cut to a length that yields the desired equivalent L or C at the given frequency ω.

One useful approach to matching resistive loads is to insert a quarter-wavelength section of TEM line of impedance Z_A between the load Z_L and the feed line impedance Z_o. Then $Z_{Ln} = Z_L/Z_A$ and one quarter-wave-length down the TEM line where $\underline{\Gamma}$ becomes - $\underline{\Gamma}$, the normalized impedance becomes the reciprocal, $Z'_n = Z_A/Z_L$ and the total impedance there is $Z' = Z_A^2/Z_L$. If this matches the output transmission line impedance Z_o so that $Z_o = Z_A^2/Z_L$ then there are no reflections. The quarter-wavelength section is called a *quarter-wave transformer* and has the impedance $Z_A = (Z_L Z_o)^{0.5}$. A similar technique can be used if the load is partly reactive without the need for L's or C's, but the length and impedance of the transformer must be adjusted. For example, any line impedance Z_A will yield a normalized load impedance that can be rotated on a Smith chart to become a real impedance; if Z_A and the transformer length are chosen correctly, this real impedance will match Z_o. Matching usually requires iteration with a Smith chart or a numerical technique.

7.4 TEM resonances

7.4.1 Introduction

Resonators are widely used for manipulating signals and power, although unwanted resonances can sometimes limit system performance. For example, resonators can be used either as band-pass filters that remove all frequencies from a signal except those near the desired resonant frequency ω_n, or as band-stop filters that remove unwanted frequencies near ω_n and let all frequencies pass. They can also be used effectively as step-up transformers to increase voltages or currents to levels sufficient to couple all available energy into desired loads without reflections. That is, the matching circuits discussed in Section 7.3.2 can become sufficiently reactive for badly mismatched loads that they act like band-pass resonators that match the load only for a narrow band of frequencies. Although simple RLC resonators have but one natural resonance and complex RLC circuits have many, distributed electromagnetic systems can have an infinite number.

A *resonator* is any structure that can trap oscillatory electromagnetic energy so that it escapes slowly or not at all. Section 7.4.2 discusses energy trapped in TEM lines terminated so that outbound waves are reflected back into the resonator, and Section 9.4 treats cavity resonators formed by terminating rectangular waveguides with short circuits that similarly reflect and trap otherwise escaping waves. In each of these cases boundary conditions restricted the allowed wave structure inside to patterns having integral numbers of half- or quarter-wavelengths along any axis of propagation, and thus only certain discrete resonant frequencies ω_n can be present.

All resonators dissipate energy due to resistive losses, leakages, and radiation, as discussed in Section 7.4.3. The rate at which this occurs depends on where the peak currents or voltages in the resonator are located with respect to the resistive or radiating elements. For example, if the resistive element is in series at a current null or in parallel at a voltage null, there is no dissipation. Since dissipation is proportional to resonator energy content and to the squares of current or voltage, the decay of field strength and stored energy is generally exponential in time. Each resonant frequency f_n has its own rate of energy decay, characterized by the dimensionless *quality factor* Q_n, which is generally the number of radians $\omega_n t$ required for the total energy w_{Tn} stored in mode n to decay by a factor of 1/e. More importantly, $Q \cong f_o/\Delta f$, where f_n is the resonant frequency and Δf_n is the half-power full-width of resonance n.

Section 7.4.4 then discusses how resonators can be coupled to circuits for use as filters or transformers, and Section 7.4.5 discusses how arbitrary waveforms in resonators are simply a superposition of orthogonal modes, each decaying at its own rate.

7.4.2 TEM resonator frequencies

A *resonator* is any structure that traps electromagnetic radiation so it escapes slowly or not at all. Typical *TEM resonators* are terminated at their ends with lossless elements such as short- or open-circuits, inductors, or capacitors. Complex notation is used because resonators are strongly frequency-dependent. We begin with the expressions (7.1.55) and (7.1.58) for voltage and current on TEM lines:

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz} \quad [V] \tag{7.4.1}$$

$$\underline{I}(z) = Y_o\left[\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}\right] \quad [A] \tag{7.4.2}$$

For example, if both ends of a TEM line of length D are open-circuited, then $\underline{I}(z) = 0$ at $z = 0$ and $z = D$. Evaluating (7.4.1) at $z = 0$ yields $\underline{V}_- = \underline{V}_+$. At the other boundary:[36]

$$\underline{I}(D) = 0 = Y_o \underline{V}_+\left(e^{-jkD} - e^{+jkD}\right) = -2jY_o \underline{V}_+ \sin(kD) = -2jY_o \underline{V}_+ \sin(2\pi D/\lambda) \tag{7.4.3}$$

To satisfy (7.4.3), $\sin(2\pi D/\lambda) = 0$, and so λ is restricted to specific resonances:

$$\lambda_n = 2D/n = c/f_n \text{ for } n = 0,1,2,3,... \tag{7.4.4}$$

That is, at resonance the length of this open-circuited line is $D = n\lambda_n/2$, as suggested in Figure 7.4.1(a) for n = 1. The corresponding resonant frequencies are:

$$f_n = c/\lambda_n = nc/2D \text{ [Hz]} \tag{7.4.5}$$

By our definition, static storage of electric or magnetic energy corresponds to a resonance at zero frequency. For example, in this case the line can hold a static charge and store electric energy at zero frequency (n = 0) because it is open-circuited at both ends. Because the different modes of a resonator are spatially orthogonal, the total energy stored in a resonator is the sum of the energies stored in each of the resonances separately, as shown later in (7.4.20).

---

[36] We use the identity $\sin\phi = (e^{j\phi} - e^{-j\phi})/2j$.

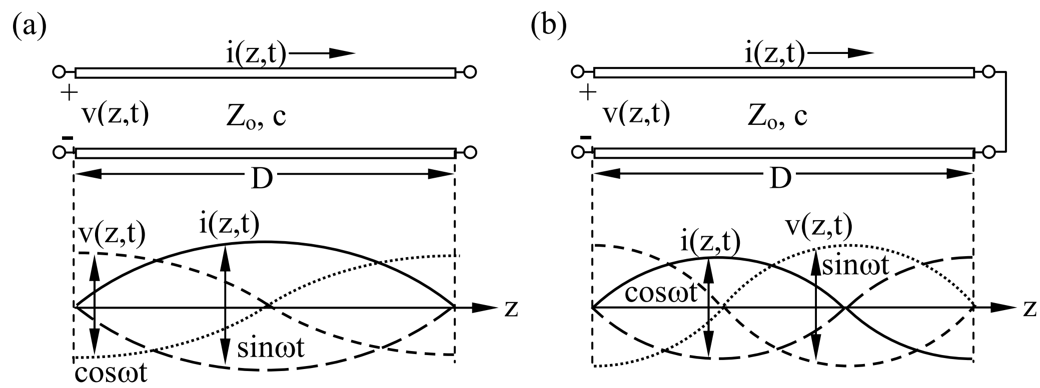

Figure 7.4.1 Voltage and current on TEM resonators.

The time behavior corresponding to (7.4.2) when $2Y_o\underline{V}_+ = I_o$ is:

$$i(t,z) = R_e\{\underline{I}e^{j\omega t}\} = I_o \sin\omega t \sin(2\pi z/\lambda) \tag{7.4.6}$$

where $\omega = 2\pi c/\lambda$. The corresponding voltage v(t,z) follows from (7.4.1), $\underline{V}_- = \underline{V}_+$, and our choice that $2Y_o\underline{V}_+ = I_o$:

$$\underline{V}(z) = \underline{V}_+\left[e^{-jkz} + e^{+jkz}\right] = 2\underline{V}_+\cos(2\pi z/\lambda) \tag{7.4.7}$$

$$v(t,z) = \mathrm{Re}\{\underline{V}e^{j\omega t}\} = Z_o I_o \cos\omega t \cos(2\pi z/\lambda) \tag{7.4.8}$$

Both v(z,t) and i(z,t) are sketched in Figure 7.4.1(a) for n = 1. The behavior of i(t,z) resembles the motion of a piano string at resonance and is 90° out of phase with v (z, t) in both space and time.

Figure 7.4.1(b) illustrates one possible distribution of voltage and current on a TEM resonator short-circuited at one end and open-circuited at the other. Since i(t) = 0 at the open circuit and v(t) = 0 at the short circuit, boundary conditions are satisfied by the illustrated i(t,z) and v(t,z). In this case:

$$D = (\lambda_n/4)(2n+1) \text{ for } n = 0,1,2,... \tag{7.4.9}$$

$$f_n = c/\lambda_n = c(2n+1)/4D \text{ [Hz]} \tag{7.4.10}$$

For n = 0 the zero-frequency solution for Figure 7.4.1(a) corresponds to the line being charged to a DC voltage V_o with zero current. The electric energy stored on the line is then $DCV_o^2/2$ [J], where the electric energy density on a TEM line (7.1.32) is:

$$W_e = C\langle v^2(t,z)\rangle/2 = C|\underline{V}|^2/4\ [\mathrm{J\ m^{-1}}] \quad (7.4.11)$$

The extra factor of 1/2 in the right-hand term of (7.4.11) results because $\langle\cos^2\omega t\rangle = 0.5$ for non-zero frequencies. A transmission line short-circuited at both ends also has a zero-frequency resonance corresponding to a steady current flowing around the line through the two short circuits at the ends, and the voltage across the line is zero everywhere.[37] The circuit of Figure 7.4.1(b) cannot store energy at zero frequency, however, and therefore has no zero-frequency resonance.

There is also a simple relation between the electric and magnetic energy storage in resonators because $Z_o = (L/C)^{0.5}$ (7.1.31). Using (7.4.11), (7.4.6), and (7.4.8) for n > 0:

$$\langle W_e\rangle = C\langle v^2(t,z)\rangle/2 = C\langle[Z_o I_o \cos\omega_n t\cos(2\pi z/\lambda_n)]^2\rangle/2 \quad (7.4.12)$$

$$= (Z_o{}^2 I_o{}^2 C/4)\cos^2(2\pi z/\lambda_n) = (L I_o{}^2/4)[\cos^2(2\pi z/\lambda_n)]\ [\mathrm{J\ m^{-1}}] \quad (7.4.13)$$

$$\langle W_m\rangle = L\langle i^2(t,z)\rangle/2 = L\langle[I_o \sin\omega_n t\sin(2\pi z/\lambda_n)]^2\rangle/2 \quad (7.4.14)$$

$$= (L I_o{}^2/4)\sin^2(2\pi z/\lambda_n)\ [\mathrm{J\ m^{-1}}] \quad (7.4.15)$$

Integrating these two time-average energy densities $\langle W_e\rangle$ and $\langle W_m\rangle$ over the length of a TEM resonator yields the important result that at any resonance the total time-average stored electric and magnetic energies w_e and w_m are equal; the fact that the lengths of all open- and/or short-circuited TEM resonators are integral multiples of a quarter wavelength λ_n is essential to this result. Energy conservation also requires this because periodically the current or voltage is everywhere zero together with the corresponding energy; the energy thus oscillates between magnetic and electric forms at twice f_n.

All resonators, not just TEM, exhibit equality between their time-average stored electric and magnetic energies. This can be proven by integrating Poynting's theorem (2.7.24) over the volume of any resonator for the case where the surface integral of $\underline{\bar{S}}\bullet\hat{n}$ and the power dissipated P_d are zero:[38]

$$0.5\oint\!\!\oint_A \underline{\bar{S}}\bullet\hat{n}\,da + \oint\!\!\oint\!\!\oint_V [\langle P_d(t)\rangle + 2j\omega(W_m - W_e)]dv = 0 \quad (7.4.16)$$

$$\therefore w_m \equiv \oint\!\!\oint\!\!\oint_V W_m\,dv = \oint\!\!\oint\!\!\oint_V W_e\,dv = w_e \qquad \text{(energy balance at resonance)} \quad (7.4.17)$$

[37] Some workers prefer not to consider the zero-frequency case as a resonance; by our definition it is.

[38] We assume here that μ and ε are real quantities so W is real too.

This proof also applies, for example, to TEM resonators terminated by capacitors or inductors, in which case the reactive energy in the termination must be balanced by the line, which then is not an integral number of quarter wavelengths long.

Any system with spatially distributed energy storage exhibits multiple resonances. These resonance modes are generally orthogonal so the total stored energy is the sum of the separate energies for each mode, as shown below for TEM lines.

Consider first the open-ended TEM resonator of Figure 7.4.1(a), for which the voltage of the nth mode, following (7.4.7), might be:

$$\underline{V}_n(z) = \underline{V}_{no}\cos(n\pi z/D) \tag{7.4.18}$$

The total voltage is the sum of the voltages associated with each mode:

$$\underline{V}(z) = \sum_{n=0}^{\infty} \underline{V}(n) \tag{7.4.19}$$

The total electric energy on the TEM line is:

$$\begin{aligned} w_{eT} &= \int_0^D \left(C|\underline{V}(z)|^2/4\right)dz = (C/4)\int_0^D \sum_m \sum_n \left(\underline{V}_m(z)\underline{V}_n^*(z)\right)dz \\ &= (C/4)\int_0^D \sum_m \sum_n \left[\underline{V}_{mo}\cos(m\pi z/D)\underline{V}_{no}^*\cos(n\pi z/D)\right]dz \\ &= (C/4)\sum_n \int_0^D |\underline{V}_{no}|^2\cos^2(n\pi z/D)\,dz = (CD/8)\sum_n |\underline{V}_{no}|^2 \\ &= \sum_n w_{eTn} \end{aligned} \tag{7.4.20}$$

where the total electric energy stored in the nth mode is:

$$w_{eTn} = CD|\underline{V}_{no}|^2/8 \text{ [J]} \tag{7.4.21}$$

Since the time average electric and magnetic energies in any resonant mode are equal, the total energy is twice the value given in (7.4.21). Thus the total energy w_T stored on this TEM line is the sum of the energies stored in each resonant mode separately because all m≠n cross terms in (7.4.20) integrate to zero. Superposition of energy applies here because all TEM_m resonant modes are spatially orthogonal. The same is true for any TEM resonator terminated with short or open circuits. Although spatial orthogonality may not apply to the resonator of Figure 7.4.2(a), which is terminated with a lumped reactance, the modes are still orthogonal because they have different frequencies, and integrating $v_m(t)v_n(t)$ over time also yields zero if m≠n.

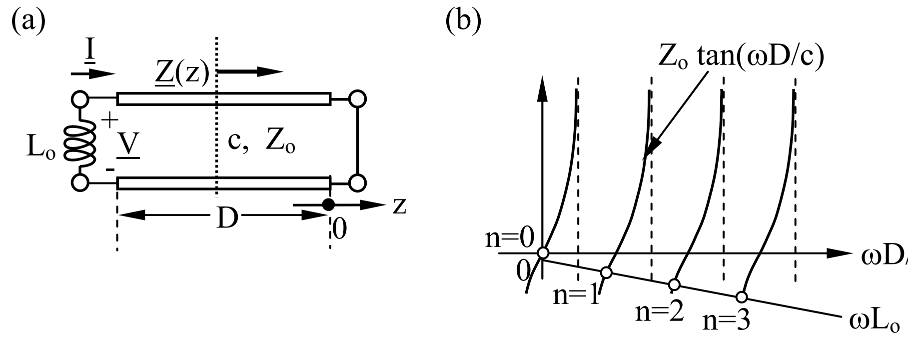

Figure 7.4.2 Inductively loaded TEM transmission line resonator.

Other types of resonator also generally have orthogonal resonant modes, so that in general:

$$w_T = \sum_n w_{Tn} \tag{7.4.22}$$

If a TEM resonator is terminated with a reactive impedance such as $j\omega L$ or $1/j\omega C$, then energy is still trapped but the resonant frequencies are non-uniformly distributed. Figure 7.4.2(a) illustrates a lossless short-circuited TEM line of length D that is terminated with an inductor L_o. Boundary conditions immediately yield an expression for the resonant frequencies ω_n. The impedance of the inductor is $j\omega L_o$ and that of the TEM line follows from (7.3.6) for $Z_L = 0$:

$$\underline{Z}(z) = Z_o\left(\underline{Z}_L - jZ_o \tan kz\right) / \left(\underline{Z}_o - jZ_L \tan kz\right) = -jZ_o \tan kz \tag{7.4.23}$$

Since the current $\underline{I}$ and voltage $\underline{V}$ at the inductor junction are the same for both the transmission line and the inductor, their ratios must also be the same except that we define $\underline{I}$ to be flowing out of the inductor into the TEM line, which changes the sign of $+j\omega L_o$; so:

$$\underline{V}/\underline{I} = -j\omega L_o = jZ_o \tan kD \tag{7.4.24}$$

$$\omega_n = -\frac{Z_o}{L_o} \tan kD = -\frac{Z_o}{L_o} \tan\left(\omega_n D/c\right) > 0 \tag{7.4.25}$$

The values of ω_n that satisfy (7.4.25) are represented graphically in Figure 7.4.2(b), and are spaced non-uniformly in frequency. The resonant frequency $\omega_o = 0$ corresponds to direct current and pure magnetic energy storage. Figure 7.4.2(b) yields ω_n for a line shorted at both ends when $L_o = 0$, and shows that for small values of L_o (perturbations) that the shift in resonances $\Delta\omega_n$ are linear in L_o.

We generally can tune resonances to nearby frequencies by changing the resonator slightly. Section 9.4.2 derives the following expression (7.4.26) for the fractional change $\Delta f/f$ in any resonance f as a function of the incremental increases in average electric ($\Delta\omega_e$) and magnetic

($\Delta\omega_m$) energy storage and, equivalently, in terms of the incremental volume that was added to or subtracted from the structure, where W_e and W_m are the electric and magnetic energy densities in that added ($+\Delta v_{vol}$) or removed ($-\Delta v_{vol}$) volume, and w_T is the total energy associated with f. The energy densities can be computed using the unperturbed values of field strength to obtain approximate answers.

$$\Delta f/f = \left(\Delta w_e - \Delta w_m\right)/w_T = \Delta v_{ol}\left(W_e - W_m\right)/w_T \qquad \text{(frequency perturbation)} \qquad (7.4.26)$$

A simple example illustrates its use. Consider the TEM resonator of Figure 7.4.2(a), which is approximately short-circuited at the left end except for a small tuning inductance L_o having an impedance $|j\omega L| << Z_o$. How does L_o affect the resonant frequency f_1? One approach is to use (7.4.25) or Figure 7.4.2(b) to find w_n. Alternatively, we may use (7.4.26) to find $\Delta f = - f_1 \times \Delta w_m/w_T$, where $f_1 \cong c/\lambda \cong c/2D$ and $\Delta w_m = L_o|\underline{I}'|^2/4 = |\underline{V}'|^2/4\omega^2 L$, where $\underline{I}'$ and $\underline{V}'$ are exact. But the unperturbed voltage at the short-circuited end of the resonator is zero, so we must use $\underline{I}'$ because perturbation techniques require that only small fractional changes exist in parameters to be computed, and a transition from zero to any other value is not a perturbation. Therefore $\Delta w_m = L_o|\underline{I}_o|^2/4$. To cancel $|\underline{I}_o|^2$ in the expression for Δf, we compute w_T in terms of voltage: $w_T = 2w_m = 2\int_0^D \left(L\left|\underline{I}(z)\right|^2/4\right)dz = DLI_o^2/4$. Thus:

$$\Delta f = \Delta f_n = -f_n\left(\frac{\Delta w_m}{w_T}\right) = -f_n\frac{|\underline{I}_o|^2/4}{DL|\underline{I}_o|^2/4} = -f_n\frac{L_o}{LD} \qquad (7.4.27)$$

7.4.3 Resonator losses and Q

All resonators dissipate energy due to resistive losses, leakage, and radiation. Since dissipation is proportional to resonator energy content and to the squares of current or voltage, the decay of field strength and stored energy is generally exponential in time. Each resonant frequency f_n has its own rate of energy decay, characterized by the dimensionless *quality factor Q_n*, which is generally the number of radians $\omega_n t$ required for the total energy w_{Tn} stored in mode n to decay by a factor of 1/e:

$$w_{Tn}(t) = w_{Tn0}e^{-\omega_n t/Q_n} \;[\mathrm{J}] \qquad (7.4.28)$$

Q_n is easily related to P_n, the power dissipated by mode n:

$$P_n \cong -dw_{Tn}/dt = \omega_n w_{Tn}/Q_n \;[\mathrm{W}] \qquad (7.4.29)$$

$$Q_n \cong \omega_n w_{Tn}/P_n \qquad \text{(quality factor Q)} \qquad (7.4.30)$$

The rate of decay for each mode depends on the location of the resistive or radiating elements relative to the peak currents or voltages for that mode. For example, if a resistive element experiences a voltage or current null, there is no dissipation. These relations apply to all resonators, for example, RLC resonators: (3.5.20–23).

Whether a resonator is used as a band-pass or band-stop filter, it has a bandwidth $\Delta\omega$ within which more than half the peak power is passed or stopped, respectively. This half-power bandwidth $\Delta\omega$ is simply related to Q by (3.5.36):

$$Q_n \cong \omega_n/\Delta\omega_n \tag{7.4.31}$$

The concept and utility of Q and the use of resonators in circuits are developed further in Section 7.4.4.

Loss in TEM lines arises because the wires are resistive or because the medium between the wires conducts slightly. In addition, lumped resistances may be present, as suggested in Figure 7.4.3(b) and (d). If these resistances do not significantly perturb the lossless voltage and current distributions, then the power dissipated and Q of each resonance ω_n can be easily estimated using *perturbation techniques*. The perturbation method simply involves computing power dissipation using the voltages or currents appropriate for the lossless case under the assumption that the fractional change induced by the perturbing element is small (perturbations of zero-valued parameters are not allowed). The examples below illustrate that perturbing resistances can be either very large or very small.

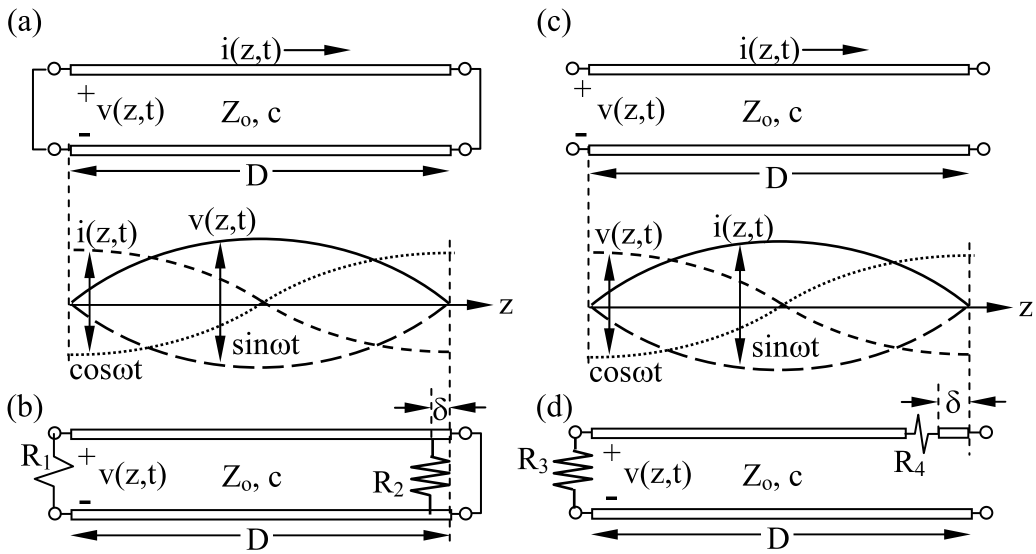

Figure 7.4.3 TEM resonators perturbed by loss.

Consider first the illustrated ω_1 resonance of Figure 7.4.3(a) as perturbed by the small resistor $R_1 << Z_o$; assume R_2 is absent. The nominal current on the TEM line is:

$$i(t,z) = R_e\{\underline{I}e^{j\omega t}\} = I_o \sin\omega t \cos(\pi z/D) \tag{7.4.32}$$

The power P_1 dissipated in R_1 at z = 0 using the unperturbed current is:

$$P_1 = \langle i^2(t, z=0)\rangle R_1 = I_o{}^2 R_1/2 \tag{7.4.33}$$

The corresponding total energy w_{T1} stored in this unperturbed resonance is twice the magnetic energy:

$$w_{T1} = 2\int_0^D \left(L\langle i^2(t)\rangle/2\right)dz = DLI_o{}^2/4 \text{ [J]} \tag{7.4.34}$$

Using (7.4.30) for Q and (7.4.5) for ω we find:

$$Q_1 \cong \omega_1 w_{T1}/P_1 = (\pi c/D)\left(DLI_o{}^2/4\right)/\left(I_o{}^2 R_1/2\right) = \pi cL/2R_1 = (Z_o/R_1)\pi/2 \tag{7.4.35}$$

Thus $Q_1 \cong Z_o/R_1$ and is high when $R_1 << Z_o$; in this case R_1 is truly a perturbation, so our solution is valid.

A more interesting case involves the loss introduced by R_2 in Figure 7.4.3(b) when R_1 is zero. Since the unperturbed shunting current at that position on the line is zero, we must use instead the unperturbed voltage v(z,t) to estimate P_1 for mode 1, where that nominal line voltage is:

$$v(z,t) = V_o \sin \omega t \sin(\pi z/D) \tag{7.4.36}$$

The associated power P_1 dissipated at position δ, and total energy w_{T1} stored are:

$$P_1 \cong \langle v(\delta,t)^2\rangle/R_2 = V_o{}^2 \sin^2(\pi\delta/D)/2R_2 \cong (V_o\pi\delta/D)^2/2R_2 \text{ [W]} \tag{7.4.37}$$

$$w_{T1} \cong 2\int_0^D \left(C\langle v^2(z,t)\rangle/2\right)dz = DCV_o{}^2/4 \text{ [J]} \tag{7.4.38}$$

Note that averaging $v^2(z,t)$ over space and time introduces two factors of 0.5. Using (7.4.35) for Q and (7.4.5) for ω we find:

$$Q_1 \cong \omega_1 w_{T1}/P_1 = (\pi c/D)\left(DCV_o^2/4\right)/\left[(V_o\pi\delta/D)^2/2R_2\right] = (D/\delta)^2(R_2/2\pi Z_o) \tag{7.4.39}$$

Thus Q_1 is high and R_2 is a small perturbation if D >> δ, even if $R_2 < Z_o$. This is because a leakage path in parallel with a nearby short circuit can be a perturbation even if its conductance is fairly high.

In the same fashion Q can be found for the loss perturbations of Figure 7.4.3(d). For example, if $R_4 = 0$, then [following (7.4.39)] the effect of R3 is:

$$Q_1 \cong \omega_1 w_{T1}/P_1 = (\pi c/D)\left(DCV_o^2/4\right)\Big/\left(V_o^2/2R_3\right) = (\pi/2)(R_3/Z_o) \qquad (7.4.40)$$

In this case R_3 is a perturbation if $R_3 >> Z_o$. Most R_4 values are also perturbations provided $\delta <<$ D, similar to the situation for R_2, because any resistance in series with a nearby open circuit will dissipate little power because the currents there are so small.

<u>Example 7.4A</u>

What is the Q of a TEM resonator of length D characterized by ω_o, C, and G?

Solution: Equation (7.4.40) says $Q = \omega_o w_T/P_d$, where the power dissipated is given by (7.1.61): $P_d = \int_0^D \left(G|\underline{V}(z)|^2/2\right)dz$. The total energy stored w_T is twice the average stored electric energy $w_T = 2w_e = 2\int_0^D \left(C|\underline{V}(z)|^2/4\right)dz$ [see 7.1.32)]. The voltage distribution $|\underline{V}(z)|$ in the two integrals cancels in the expression for Q, leaving $Q = 2\omega_o C/G$.

7.4.4 <u>Coupling to resonators</u>

Depending on how resonators are coupled to circuits, they can either pass or stop a band of frequencies of width $\sim\Delta\omega_n$ centered on a resonant frequency ω_n. This effect can be total or partial; that is, there might be total rejection of signals either near resonance or far away, or only a partial enhancement or attenuation. This behavior resembles that of the series and parallel RLC resonators discussed in Section 3.5.2.

Figure 7.4.4 shows how both series and parallel RLC resonators can block all the available power to the load resistor R_L near resonance, and similar behavior can be achieved with TEM resonators as suggested below; these are called *band-stop filters*. Alternatively, both series and parallel RLC resonators can pass to the load resistor the band near resonance, as suggested in Figure 3.5.3; these are called *band-pass filters*. In Figure 7.4.4(a) the series LC resonator shorts out the load R near resonance, while in (b) the parallel LC resonator open-circuits the load conductance G; the resonant band is stopped in both cases.

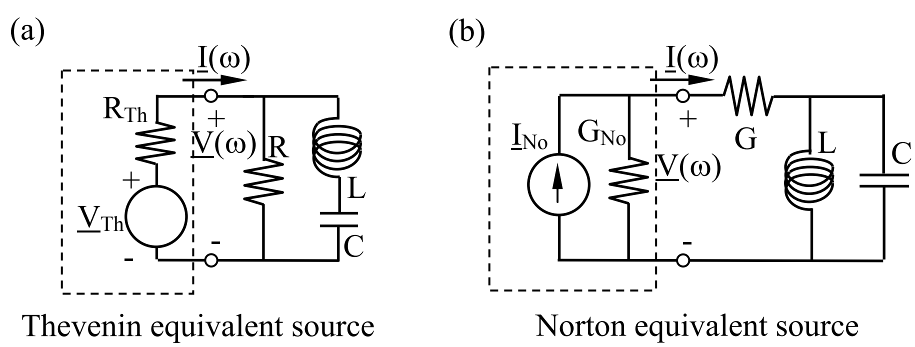

Figure 7.4.4 Band-stop RLC resonators.

The half-power width $\Delta\omega$ of each resonance is inversely proportional to the *loaded Q*, where Q_L was defined in (3.5.40), P_{DE} is the power dissipated externally (in the source resistance R_{Th}), and P_{DI} is the power dissipated internally (in the load R):

$$Q_L \equiv \omega w_T/(P_{DI} + P_{DE}) \qquad \text{(loaded Q)} \qquad (7.4.41)$$

$$\Delta\omega_n = \omega_n/Q_n \ [\text{radians s}^{-1}] \qquad \text{(half-power bandwidth)} \qquad (7.4.42)$$

When $\omega = (LC)^{-0.5}$ the LC resonators are either open- or short-circuit, leaving only the source and load resistors, R_{Th} and R_L. At the frequency f of maximum power transfer the fraction of the available power that can be passed to the load is determined by the ratio $Z_n' = R_L/R_{Th}$. For example, if the power source were a TEM transmission line of impedance $Z_o \equiv R_{Th}$, then the minimum fraction of incident power reflected from the load (7.2.22) would be:

$$|\underline{\Gamma}|^2 = |(\underline{Z}_n'-1)/(\underline{Z}_n'+1)|^2 \qquad (7.4.43)$$

The fraction reflected is zero only when the normalized load resistance $Z_n' = 1$, i.e., when $R_L = R_{Th}$. Whether the maximum transfer of power to the load occurs at resonance ω_n (band-pass filter) or only at frequencies removed more than $\sim\Delta\omega$ from ω_n (band-stop filter) depends on whether the current is blocked or passed at ω_n by the LC portion of the resonator. For example, Figures 3.5.3 and 7.4.4 illustrate two forms of band-pass and band-stop filter circuits, respectively.

Resonators can be constructed using TEM lines simply by terminating them at both ends with impedances that reflect most or all incident power so that energy remains largely trapped inside, as illustrated in Figure 7.4.5(a). Because the load resistance R_L is positioned close to a short circuit ($\delta << \lambda/4$), the voltage across R_L is very small and little power escapes, even if $R_L \cong Z_o$. The Q for the ω_1 resonance is easily calculated by using (7.4.40) and the expression for line voltage (7.4.36):

$$v(z,t) = V_o \sin\omega t \sin(\pi z/D) \qquad (7.4.44)$$

$$\begin{aligned} Q &\equiv \omega_1 w_T/P_D = (\pi c/D)\left(DCV_o^{\,2}/4\right)/\left[V_o^{\,2}\sin^2(\pi\delta/D)/2R_L\right] \\ &= (\pi R_L/2Z_o)/\sin^2(\pi\delta/D) \cong (D/\delta)^2 R_L/2\pi Z_o \quad (\text{for } \delta << D) \end{aligned} \qquad (7.4.45)$$

Adjustment of δ enables achievement of any desired Q for any given R_L in an otherwise lossless system. If we regard R_L as internal to the resonator then the Q calculated above is the *internal Q*, Q_I.

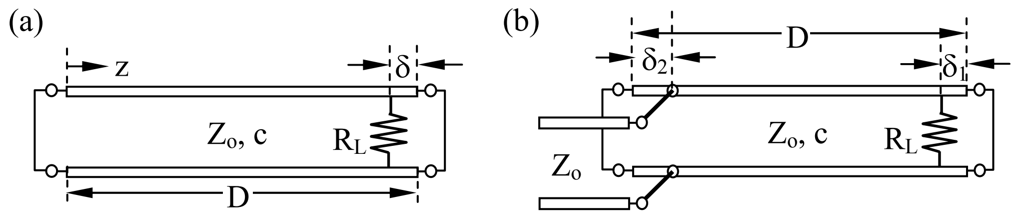

Figure 7.4.5 Coupled TEM resonator.

We may connect this resonator externally by adding a feed line at a short distance δ_2 from its left end, as illustrated in Figure 7.4.5(b). If the feed line is matched at its left end then the *external Q*, Q_E, associated with power dissipated there is given by (7.4.45) for $\delta = \delta_2$ and $R_L = Z_o$. By adjusting δ_2 any Q_E can be obtained. Figures 3.5.3 and 7.4.4 suggest how the equivalent circuits for either band-pass or band-stop filters can match all the available power to the load if $R_{Th} = R_L$ and therefore $Q_E = Q_I$. Thus all the available power can be delivered to R_L in Figure 7.4.5(b) for any small δ_1 by selecting δ_2 properly; if δ_2 yields a perfect match at resonance, we have a *critically coupled resonator*. If δ_2 is larger than the critically coupled value, then the input transmission line is too strongly coupled, $Q_E < Q_I$, and we have an *over-coupled resonator*; conversely, smaller values of δ_2 yield $Q_E > Q_I$ and undercoupling. The bandwidth of this band-pass filter $\Delta\omega$ is related to the *loaded Q*, Q_L, as defined in (7.4.41) where:

$$Q_L^{-1} = Q_I^{-1} + Q_E^{-1} = \Delta\omega/\omega \qquad (7.4.46)$$

If the band-pass filter of Figure 7.4.5(b) is matched at resonance so $Q_E = Q_I$, it therefore has a bandwidth $\Delta\omega = 2\omega/Q_I$, where Q_I is given by (7.4.39) and is determined by our choice of δ_1. Smaller values of δ_1 yield higher values for Q_L and narrower bandwidths $\Delta\omega$. In the special case where R_L corresponds to another matched transmission line with impedance Z_o, then a perfect match at resonance results here when $\delta_1 = \delta_2$.

Many variations of the coupling scheme in Figure 7.4.5 exist. For example, the feed line and resonator can be isolated by a shunt consisting of a large capacitor or a small inductor, both approximating short circuits relative to Z_o, or by a high-impedance block consisting of a small capacitor or large inductor in series. Alternatively, an external feed line can be connected in place of R_4 in Figure 7.4.3(d). In each weakly coupled case perturbation methods quickly yield Q_I and Q_E, and therefore Q_L, $\Delta\omega$, and the impedance at resonance.

The impedance at resonance can be found once Q_E, Q_I, and Z_o for the feedline are known, and once it is known whether the resonance is a series or parallel resonance. Referring to Figures 3.5.3 and 7.4.4 for equivalent circuits for band-pass and band-stop filters, respectively, it is clear that if $Q_E = Q_I$, then band-pass resonators are matched at resonance while band-stop series-resonance resonators are short circuits and parallel-resonance resonators are open circuits. Away from resonance band-pass resonators become open circuits for series resonances and short circuits for parallel resonances, while both types of band-stop resonator become matched loads if $Q_E = Q_I$. At resonance all four types of resonator have purely real impedances and reflection coefficients Γ that can readily be found by examining the four equivalent circuits cited above.

Sometimes unintended resonances can disrupt systems. For example, consider a waveguide that can propagate two modes, only one of which is desired. If a little bit of the unwanted mode is excited at one end of the waveguide, but cannot escape through the lines connected at each end, then the second mode is largely trapped and behaves as a weakly coupled resonator with its own losses. At each of its resonances it will dissipate energy extracted from the main waveguide. If the internal losses happen to cause $Q_E = Q_I$ for these parasitic resonances, no matter how weakly coupled they are, they can appear as a matched load positioned across the main line; dissipation by parasitic resonances declines as their internal and external Q's increasingly differ.

The ability of a weakly coupled resonance to have a powerful external effect arises because the field strengths inside a low-loss resonator can rise to values far exceeding those in the external circuit. For example, the critically coupled resonator of Figure 7.4.5(b) for $R_L = Z_o$ and $\delta_1 = \delta_2$, has internal voltages $v(z,t) = V_o \sin\omega t \sin(\pi z/D)$ given by (7.4.44), where the maximum terminal voltage is only $V_o \sin(\pi\delta/D) \cong V_o \pi\delta/D << V_o$. Thus a parasitic resonance can slowly absorb energy from its surroundings at its resonant frequency until its internal fields build to the point that even with weak coupling it has a powerful effect on the external fields and thus reaches an equilibrium value. It is these potentially extremely strong resonant fields that enables critically coupled resonators to couple energy into poorly matched loads--the fields in the resonator build until the power dissipated in the load equals the available power provided. In some cases the fields can build to the point where the resonator arcs internally, as can happen with an empty microwave oven without an extra internal load to prevent it.

This analysis of the resonant behavior of TEM lines is approximate because the resonator length measured in wavelengths is a function of frequency within $\Delta\omega$, so exact answers require use the TEM analysis methods of Sections 7.2–3, particularly when $\Delta\omega$ becomes a non-trivial fraction of the frequency difference between adjacent resonances.

Example 7.4B

Consider a variation of the coupled resonator of Figure 7.4.5(b) where the resonator is open-circuited at both ends and the weakly coupled external connections at δ_1 and δ_2 from the ends are in series with the 100-ohm TEM resonator line rather than in parallel. Find δ_1 and δ_2 for: $Q_L = 100$, $Z_o = 100$ ohms for both the feed line and resonator, $R_L = 50$ ohms, and the resonator length is $D \cong \lambda/2$, where λ is the wavelength within the resonator.

Solution: For critical coupling, $Q_E = Q_I$, so the resonator power lost to the input line, $|\underline{I}_2|^2 Z_o/2$, must equal that lost to the load, $|\underline{I}_1|^2 R_L/2$, and therefore $|\underline{I}_1|/|\underline{I}_2| = (Z_o/R_L)^{0.5} = 2^{0.5}$. Since the $\lambda/2$ resonance of an open-circuited TEM line has $I(z) \cong I_o \sin(\pi z/D) \cong \pi z/D$ for $\delta << D/\pi$ (high Q), therefore $|\underline{I}_1|/|\underline{I}_2| = [\sin(\pi\delta_1/D)]/[\sin(\pi\delta_2/D)] \cong \delta_1/\delta_2 \cong 2^{0.5}$. Also, $Q_L = 100 = 0.5\times Q_I = 0.5\omega_o w_T/P_{DI}$, where: $\omega_o = 2\pi f_o = 2\pi c/\lambda = \pi c/D$; $w_T = 2w_m = 2\int_0^D (L|I|^2/4)dz \cong LI_o^2 D/4$; and $P_{DI} = |I(\delta_1)|^2 R_L/2 = I_o^2 \sin^2(\pi\delta_1/D)R_L/2 \cong (I_o\pi\delta_1/D)^2 R_L/2$. Therefore $Q_I = \omega_o w_T/P_{DI} = 200 = (\pi c/D)(LI_o^2 D/4)/[(I_o\pi\delta_1/D)^2 R_L/2] = (D/\delta_1)^2(Z_o/R_L)/\pi$, where $cL = Z_o = 100$. Thus $\delta_1 = \pi^{-0.5}D/10$ and $\delta_2 = \delta_1 2^{-0.5} = (2\pi)^{-0.5}D/10$.

7.4.5 Transients in TEM resonators

TEM and cavity resonators have many resonant modes, all of which can be energized simultaneously, depending on initial conditions. Because Maxwell's equations are linear, the total fields can be characterized as the linear superposition of fields associated with each excited mode. This section illustrates how the relative excitation of each TEM resonator mode can be determined from any given set of initial conditions, e.g. from v(z, t = 0) and i(z, t = 0), and how the voltage and current subsequently evolve. The same general method applies to modal excitation of cavity resonators. By using a similar orthogonality method to match boundary conditions in space rather than in time, the modal excitation of waveguides and optical fibers can be found, as discussed in Section 9.3.3.

The central concept developed below is that any initial condition in a TEM resonator at time zero can be replicated by superimposing some weighted set of voltage and current modes. Once the phase and magnitudes of those modes are known, the voltage and current are then known for all time. The key solution step uses the fact that the mathematical functions characterizing any two different modes a and b, e.g. the voltage distributions $\underline{V}_a(z)$ and $\underline{V}_b(z)$, are spatially othogonal: $\int \underline{V}_a(z)\underline{V}_b{}^*(z)\,dz = 0$.

Consider the open-circuited TEM resonator of Figure 7.4.3(c), for which $\underline{V}_{n-} = \underline{V}_{n+}$ for any mode n because the reflection coefficient at the open circuit at z = 0 is +1. The resulting voltage and current on the resonator for mode n are:[39]

$$\underline{V}_n(z) = \underline{V}_{n+}e^{-jk_n z} + \underline{V}_{n-}e^{+jk_n z} = 2\underline{V}_{n+}\cos k_n z \tag{7.4.47}$$

$$\underline{I}_n(z) = Y_o\left(\underline{V}_{n+}e^{-jk_n z} - \underline{V}_{n-}e^{+jk_n z}\right) = -2jY_o\underline{V}_{n+}\sin k_n z \tag{7.4.48}$$

where $k_n = \omega_n/c$ and (7.4.5) yields $\omega_n = n\pi c/D$. We can restrict the general expressions for voltage and current to the moment t = 0 when the given voltage and current distributions are $v_o(z)$ and $i_o(z)$:

$$v(z,t=0) = v_o(z) = \sum_{n=0}^{\infty} R_e\left\{\underline{V}_n(z)e^{j\omega_n t}\right\}_{t=0} = \sum_{n=0}^{\infty} R_e\left\{2\underline{V}_{n+}\cos k_n z\right\} \tag{7.4.49}$$

$$i(z,t=0) = i_o(z) = \sum_{n=0}^{\infty} R_e\left\{\underline{I}_n(z)e^{j\omega_n t}\right\}_{t=0} = Y_o\sum_{n=0}^{\infty} I_m\left\{2\underline{V}_{n+}\sin k_n z\right\} \tag{7.4.50}$$

[39] Where we recall $\cos\phi = (e^{j\phi} + e^{-j\phi})/2$ and $\sin\phi = (e^{j\phi} - e^{-j\phi})/2j$.

We note that these two equations permit us to solve for both the real and imaginary parts of $\underline{V}_{n+}$, and therefore for v(z,t) and i(z,t). Using spatial orthogonality of modes, we multiply both sides of (7.4.49) by cos(mπz/D) and integrate over the TEM line length D, where $k_n = n\pi z/D$:

$$\begin{aligned}\int_0^D v_o(z)\cos(m\pi z/D)\,dz &= \int_0^D \sum_{n=0}^{\infty} R_e\{2\underline{V}_{n+}\cos k_n z\}\cos(m\pi z/D)\,dz \\ &= \sum_{n=0}^{\infty} R_e\{2\underline{V}_{n+}\}\int_0^D \cos(n\pi z/D)\cos(m\pi z/D)\,dz = 2R_e\{\underline{V}_{n+}\}(D/2)\delta_{mn}\end{aligned} \tag{7.4.51}$$

where $\delta_{mn} \equiv 0$ if $m \neq n$, and $\delta_{mn} \equiv 1$ if $m = n$. Orthogonality of modes thus enables this integral to single out the amplitude of each mode separately, yielding:

$$R_e\{\underline{V}_{n+}\} = D^{-1}\int_0^D v_o(z)\cos(n\pi z/D)\,dz \tag{7.4.52}$$

Similarly, we can multiply (7.4.50) by sin(mπz/D) and integrate over the length D to yield:

$$Im\{\underline{V}_{n+}\} = Z_o D^{-1}\int_0^D i_o(z)\sin(n\pi z/D)\,dz \tag{7.4.53}$$

Once $\underline{V}_n$ is known for all n, the full expressions for voltage and current on the TEM line follow, where $\omega_n = \pi nc/D$:

$$v(z,t) = \sum_{n=0}^{\infty} R_e\{\underline{V}_n(z)e^{j\omega_n t}\} \tag{7.4.54}$$

$$i(z,t) = \sum_{n=0}^{\infty} R_e\{\underline{I}_n(z)e^{j\omega_n t}\} = Y_o\sum_{n=0}^{\infty} I_m\{2\_V_{n+}\sin k_n z\} \tag{7.4.55}$$

In general, each resonator mode decays exponentially at its own natural rate, until only the longest-lived mode remains.

As discussed in Section 9.3.3, the relative excitation of waveguide modes by currents can be determined in a similar fashion by expressing the fields in a waveguide as the sum of modes, and then matching the boundary conditions imposed by the given excitation currents at the spatial origin (not time origin). The real and imaginary parts of the amplitudes characterizing each waveguide propagation mode can then be determined by multiplying both sides of this boundary equation by spatial sines or cosines corresponding to the various modes, and integrating over the surface defining the boundary at z = 0. Arbitrary spatial excitation currents generally excite both propagating and evanescent modes in some combination. Far from the excitation point only the propagating modes are evident, while the evanescent modes are evident principally as a reactance seen by the current source.

6.013 Electromagnetics and Applications
Spring 2009