Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.007 Electromagnetic Energy: From Motors to Lasers

Supplemental Class Notes

Energy Method for Finding Forces *

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1 Introduction

In this short note we describe a method for calculating forces of electrical and magnetic origin. This method, sometimes called "The Principal of Virtual Work" (POVW) derives from the First Law of Thermodynamics (conservation of energy), and should be fairly intuitive. The method succeeds in many cases where other methods are difficult or unwieldy to use, but we can show (and do here) that in some cases that are straightforward to work, the results of POVW are consistent with direct calculation of force using fields and charges or the Lorentz Force Law.

Electromechanical energy converters such as motors, generators, loudspeakers, and so forth are described conceptually in Figure 1. Electric power goes in one side (wires), the intended output (mechanical power) goes out the other (in the case of a motor, this is the shaft), and then there are attendant losses that must be accounted for. In electric motors those losses come out as heat from resistive drops in the wires, as heat from friction and windage, as noise from vibration and unintended torques and from turbulence in the air gap, and from other mechanisms that we won't deal with here.

The key to the POVW method for estimating forces is to isolate the force producing mechanisms, identify the energy going into the electromagnetic fields and then identifying how that energy varies with the mechanical variables such as position.

2 Force and Potential Energy

Consider the situation shown in Figure 2. Were we to lift a weight with mass M from a position on the floor to some position above the floor of height x (with respect to the floor), we would have to overcome the gravity force $F_g = Mg$, where g is called the 'acceleration due to gravity', and this amounts to about 9.812 Newtons per kilogram near the surface of the earth. To lift the mass
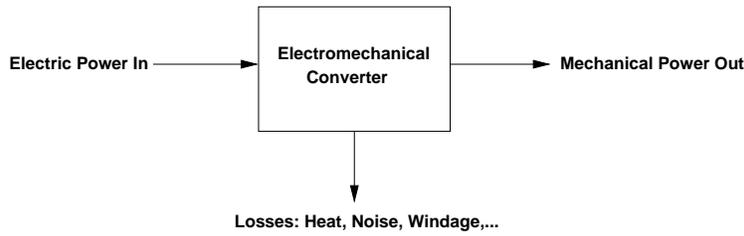

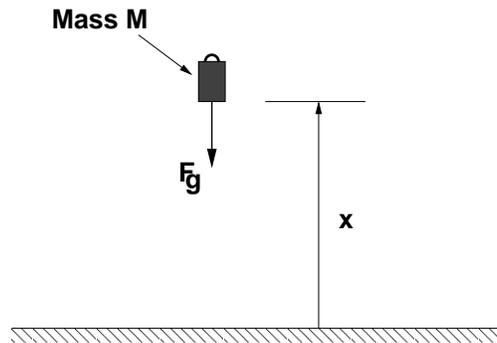Figure 1: Conceptual Electromechanical Converter

Figure 2: Force of Gravity on a Mass

we must do work on it, and this adds to its 'potential energy': to go to height x we must give it potential energy:

$$E_p = Mgx$$

The force of gravity, which we knew in the first place, is also seen to be the derivative of that potential energy with respect to height:

$$F_g = -\frac{\partial E_p}{\partial x}$$

The Principal of Virtual Work is almost that simple, once the variation in stored energy with position can be identified.

3   Electric Field Systems

We start with an electroquasistatic system: one in which electric fields are dominant and in which we can generally ignore magnetic fields. We know that, were we able to identify the electric fields acting on all of the charges of this system we could use the fact that force on a charge is:

$$\vec{F} = q\vec{E}$$

Simply adding up all of the forces on the individual charges should give us the total force produced by the system. This may or may not be straightforward, and it is usually not.
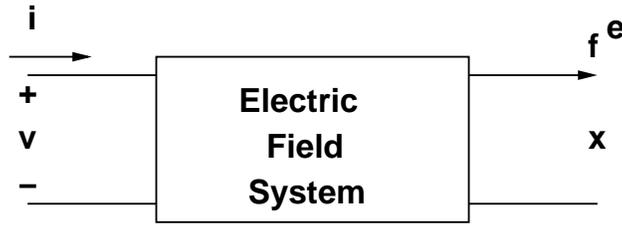
Figure 3: Conceptual Electric Field System

Figure 3 shows, conceptually, the kind of system we will be working with. The system has one set of electrical terminals, characterized by voltage and current, and one set of mechanical terminals, characterized by a force and a displacement. Note that in this picture, the force is imposed by the electrical system on whatever it might be driving, while the electrical terminals show power flow *into* the electric field system.

Power flow into the system is voltage times current:

$$P_e = vi = v\frac{dq}{dt}$$

where q is charge stored in the system.

We can identify the mechanical flow of energy *out* of this system through the mechanical terminals is force times velocity.

$$P_m = f_e u = f_e \frac{dx}{dt}$$

One caution we must state at this point is that all systems of the type we are dealing with will have losses. Electrical losses will be incurred because dielectrics are 'leaky', so some of that charge will be lost by conduction. Similarly, there will be mechanical losses due to friction. However, we can always identify the 'conservative' part of the system, the lossless part, and we claim (you should examine this claim carefully!) that it is the conservative part that makes the force of electric origin, so that losses can be identified separately. Some reflection on this should reveal that the notion of a 'conservative' system is equivalent to the notion that there is a uniquely defined *state* of the system determined by the two variables, in this case the charge q and the mechanical position variable x. That is:

$$W_e = W_e(q, x)$$

If this stored energy is a function *only* of the state of the system, it is independent of how the system got to that state. Once the electrical power input to the system and mechanical power output from the system have been identified, it is possible to calculate the rate of change of energy stored in the electric field system:

$$\frac{dW_e}{dt} = P_e - P_m = v\frac{dq}{dt} - f^e\frac{dx}{dt}$$

Then the change in energy between any two states of the system, call them a and b would be:

$$W_e(a) - W_e(b) = \int_b^a v\frac{dq}{dt}dt - f_e\frac{dx}{dt}dt = \int_b^a vdq - f_e dx$$
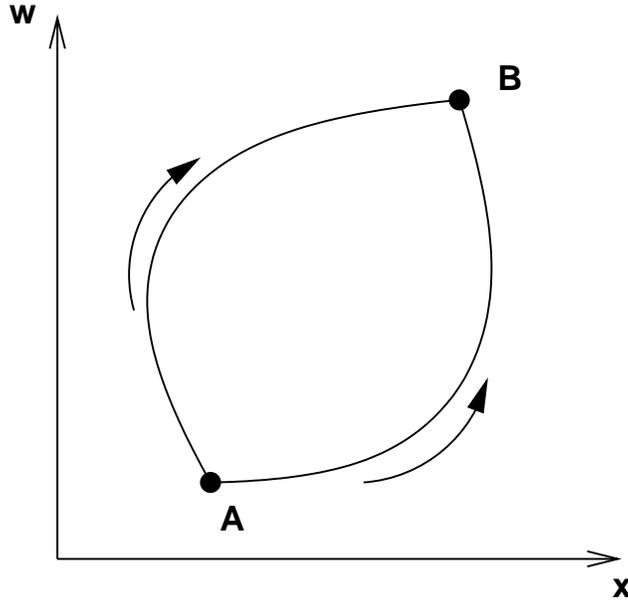
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Figure 4: Difference in Energy is Independent of Path Between Points

This is just like noting the change in energy stored by that weight in the gravity example is $W_g(a) - W_g(b) = \int_b^a Mg\,dx$. In the case of the electric field system, however, it is a two-dimensional, or vector integral.

Note that if the energy stored in the electric field system is uniquely determined by the two variables, charge and displacement (q and x), the rate of change of energy stored in the field is:

$$\frac{dW_e}{dt} = \frac{\partial W_e}{\partial q}\frac{dq}{dt} + \frac{\partial W^e}{dx}\frac{dx}{dt}$$

Clearing the derivatives, we can find the total differential of energy stored:

$$dW^e = \frac{\partial W_e}{\partial q}dq + \frac{\partial W_e}{\partial x}dx$$

Comparing the two expressions for the differentials of energy, we must come to the conclusion that:

$$v = \frac{\partial W_e}{\partial q} \qquad\qquad f_e = -\frac{\partial W_e}{\partial x}$$

3.1   Simple example: Capacitor with Variable Spacing

As an example of this force determination, consider a capacitor consisting of two metal plates separated by a variable spacing g, as shown in Figure 5. We know that the capacitance of this thing is:

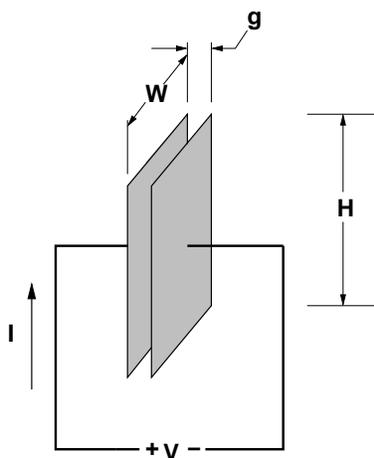$$C = \frac{\epsilon_0 W H}{g}$$

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Figure 5: Force Between Capacitor Plates

So that the voltage can be expressed as a simple function of charge:

$$v = \frac{q}{C} = \frac{qg}{\epsilon_0 W H}$$

To find energy stored in this thing, assume that the position x is fixed with no charge on the plates, so that no work must be done to position it. Then when the charge is Q:

$$W_e = \int_0^Q v dq = \int_0^Q \frac{qg}{\epsilon_0 W H} dq = \frac{1}{2} \frac{Q^2 g}{\epsilon_0 W H}$$

According to our analysis, the force acting on the gap is:

$$f_e = -\frac{\partial W_e}{\partial x} = -\frac{1}{2} \frac{Q^2}{\epsilon_0 W H}$$

This force can be seen to be trying to make the dimensional variable g *smaller* (negative force). This could be made a bit more tractable by noting that total charge is, in terms of voltage V:

$$Q = \epsilon_0 W H E_x = \frac{\epsilon_0 W H V}{g}$$

And then the force is easily written in terms of voltage as:

$$f_e = -\frac{1}{2} \frac{\epsilon_0 W H V^2}{g^2}$$

3.2   Reconciliation with charge times electric field

We should be able to show that the energy method gives the same answer as multiplying charge times electric field.

The situation can be demonstrated by looking a bit more closely at the capacitor. It will help to think about the units of charge. The charge on the capacitor plates (positive on the left, negtive
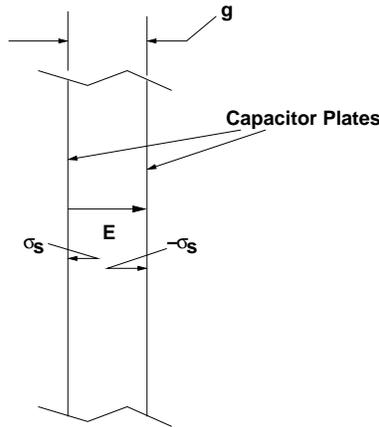
Figure 6: Details of Capacitor Plates

on the right) is Q, which has units of Coulombs. If we can assume that this charge is distributed uniformly over the plates, we would have a *surface* charge density, with units of Coulombs per square meter:

$$\sigma_s = \frac{Q}{WH}$$

Figure 6 shows a small section of the capacitor. We assume here that the plates are parallel to each other. Since the plates are made of a metal, any electric field must be perpendicular to the surface. If the horizontal direction in Figure 6 is noted as x, then the electric field in the gap must be:

$$\vec{E} = \hat{x}\frac{V}{g} = \hat{x}\frac{\sigma_s}{\epsilon_0}$$

Simply multiplying surface charge times electric field would yield an expression that is actually twice the real force.

We know that the force on a single charge q is $\vec{f_e} = q\vec{E}$, for which the units are Newtons. Consequently the *volume* force on a charge density is $\vec{F} = \rho\vec{E}$, for which the units are Newtons per cubic meter. Our difficulty here is that the field $\vec{E}$ arises from the surface charge σ_s, and that surface charge is a singularity. To resolve this we engage in what is not exactly a ruse, as shown in Figure 7. In this picture, we assume that the charge is distributed over a volume, resolving the singularity. That volume charge (units are Coulombs per cubic meter) is:

$$\rho = \frac{Q}{\delta WH} = \frac{\sigma_s}{\delta}$$

Given this distribution of charge over a volume, we no longer have a singularity and it is straightforward to estimate electric field as a continuous function of space. As shown in Figure 8, looking at the charge and electric field about the left-hand plate, the field is zero at the metal surface but rises over the assumed charge distribution:

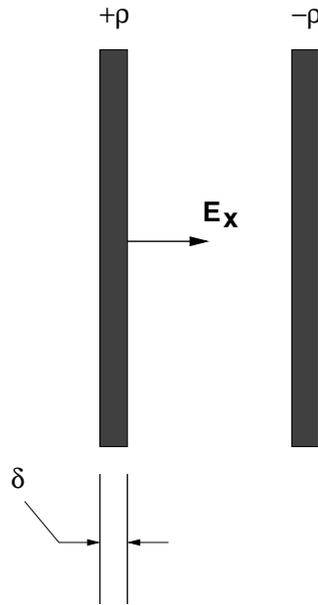$$E_x(x) = \frac{\sigma_s}{\epsilon_0 \delta}x$$

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Figure 7: Charge distributed over a volume

Note that as the assumed dimension δ becomes smaller, the slope of field magnitude increases, but the electric field in the gap (to the right of the charge is independent of the dimension δ.

Volume force density is

$$\vec{F} = \rho\vec{E} = \hat{x}\left(\frac{\sigma_s}{\epsilon_0\delta}\right)^2 x$$

Total force is directly calculatee by integrating over the volume of the charge distribution:

$$\vec{f} = WH\int_0^{\delta} \hat{x}\left(\frac{\sigma_s}{\epsilon_0\delta}\right)^2 xdx = WH\frac{\sigma_s^2}{2\epsilon_0}$$

As we expect, the actual value assumed for the thickness of the current distribution is immaterial. The answer is also consistent with what we computed from the Principal of Virtual Work.

4   Forces in Magnetic Field Systems

By 'Magnetic Field Systems' we mean magnetoquasistatic systems, in which charges do not accumulate and time variations of electric fields are not important sources of magnetic field. Consider a magnetic field system with one electrical terminal pair and one mechanical terminal, as shown in Figure 9.

As with the electric field system, all converters have loss mechanisms and so are not themselves conservative. However, the magnetic field system that produces force is, in principle, conservative in the sense that its state and stored energy can be described by only two variables. The 'history' of the system, meaning the path by which it was put into its current state is not important.
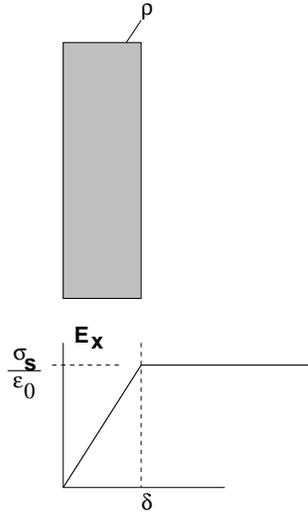
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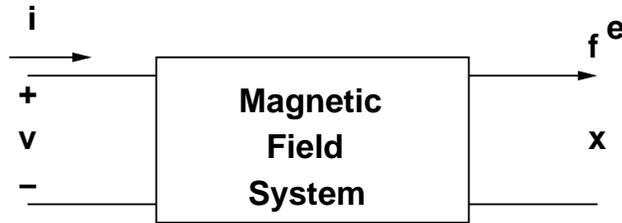Figure 8: Electric Field Developed from Distributed Charge

Figure 9: Conceptual Magnetic Field System

It is possible to chose the variables in such a way that electrical power *into* this conservative system is:

$$P_e = vi = i\frac{d\lambda}{dt}$$

Similarly, mechanical power *out of* the system is:

$$P_m = f_e\frac{dx}{dt}$$

The difference between these two is the rate of change of energy stored in the system:

$$\frac{dW_m}{dt} = P_e - P_m$$

It is then possible to compute the change in energy required to take the system from one state to another by:

$$W_m(a) - W_m(b) = \int_b^a id\lambda - f_e dx$$

where the two states of the system are described by $a = (\lambda_a, x_a)$ and $b = (\lambda_b, x_b)$

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If the energy stored in the system is described by two state variables, λ and x, the *total differential* of stored energy is:

$$dW_m = \frac{\partial W_m}{\partial \lambda}d\lambda + \frac{\partial W_m}{\partial x}dx$$

and it is also:

$$dW_m = id\lambda - f^e dx$$

So that we can make a direct equivalence between the derivatives and:

$$f^e = -\frac{\partial W_m}{\partial x}$$

This generalizes in the case of multiple electrical terminals and/or multiple mechanical terminals. For example, a situation with multiple electrical terminals will have:

$$dW_m = \sum_k i_k d\lambda_k - f_e dx$$

And the case of rotary, as opposed to linear, motion has in place of force f_e and displacement x, torque T_e and angular displacement θ.

In many cases we might consider a system which is electricaly *linear*, in which case inductance is a function only of the mechanical position x.

$$\lambda(x) = L(x)i$$

In this case, assuming that the energy integral is carried out from $\lambda = 0$ (so that the part of the integral carried out over x is zero),

$$W_m = \int_0^\lambda \frac{1}{L(x)}\lambda d\lambda = \frac{1}{2}\frac{\lambda^2}{L(x)}$$

This makes

$$f_e = -\frac{1}{2}\lambda^2 \frac{\partial}{\partial x}\frac{1}{L(x)}$$

Note that this is numerically equivalent to

$$f_e = \frac{1}{2}i^2 \frac{\partial}{\partial x}L(x)$$

This is true *only* in the case of a linear system. Note that substituting $L(x)i = \lambda$ too early in the derivation produces erroneous results: in the case of a linear system it produces a sign error, but in the case of a nonlinear system it is just wrong.

4.1   Example: Junkyard Magnet

This is a practical application of the POVW as applied to magnetic field systems, and the example itself is a practical example of a magnetic actuator. You may have seen one of these things picking up cars in junkyards.

A cartoon view of the device is shown in Figure 10. The shaded volumes are highly permeable material ($\mu \to \infty$). They are uniform in shape in the direction perpendicular to the view with
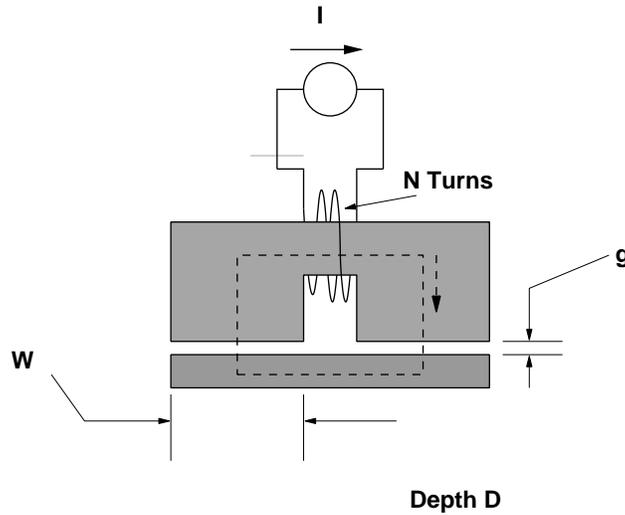
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Figure 10: Cartoon of a junkyard magnet

dimension D. Wrapped arount the upper body is a coil with N turns. That coil is driven with a DC (constant) current of magnitude I. To find flux, consider Ampere's Law around the loop indicated as a dotted line.

$$\oint \vec{H} \cdot d\vec{\ell} = NI$$

If magnetic field $\vec{H}$ is taken to be down in the right-hand gap and up in the left hand gap, and if the gap dimension g is relatively small so the field is uniform over the gap area, and since there is no magnetic field $\vec{H}$ in the magnetic material the field in the gap is:

$$H_g = \frac{NI}{2g}$$

Note that, since magnetic flux has no divergence, the flux leaving the right-hand pole of the upper magnetic body must be matched by flux entering that body in the left-hand pole. Total flux is flux density times area:

$$\Phi = \mu_0 W D \frac{NI}{2g}$$

and since flux linked by the coil is just $N\Phi$, the inductance of the system is:

$$L_g(g) = \mu_0 \frac{N^2 W D}{2g}$$

Magnetic stored energy is:

$$W_m = \int_0^{\lambda_0} \frac{\lambda}{L_g(g)} d\lambda = \frac{1}{2} \frac{\lambda_0^2}{L_g(g)}$$

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Force acting between the two elements would then be:

$$f_e = -\frac{\lambda_0^2}{2}\frac{\partial}{\partial g}\frac{1}{L_g(g)} = -\frac{\mu_0 I^2 N^2 W D}{2g^2}$$

The expression for force is negative as it is in the direction of making the gap smaller.

It should be clear that viewing this problem as linear leads to major limitations: the force cannot, as suggested by this expression, go to infinity as $g \to 0$. Note that it is possible to express the lifting force in terms of the magnetic flux density in the air-gaps: See that, since $\lambda_0 = N\Phi = NWDB_g$, lift force can be written as:

$$f_e = 2WD\frac{B_g^2}{2\mu_0}$$

As it turns out, this is an accurate description of the lift force, this time in terms of the magnetic field in the gap. If the gap is small enough or the excitation current is large enough, the magnetic material will exhibit saturation and the flux density will approach a constant value (in common steels, te saturation flux density is about 1.8 T).

4.2 Example: Linear Actuator

This example is related to common 'solenoid actuators' that are widely used in relays, motor starters, circuit breakers and the like. The actual form of such actuators is not quite what is shown in Figure 11, but the analysis is nearly the same. A flat bar of highly permeable (ferromagnetic) material can slide into a gap in a ferromagnetic core. The permeability of both the bar and the core is large enough that we do not need to consider it ($\mu \to \infty$).
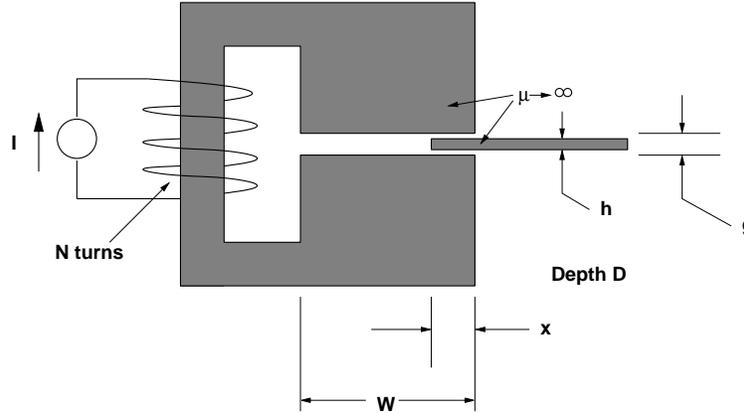

Figure 11: Cartoon of a linear actuator

The inductance of this device is:

$$L(x) = \mu_0 N^2 D \left(\frac{W-x}{g} + \frac{x}{g-h}\right) = \mu_0 N^2 D \left(x\left(\frac{1}{g-h} - \frac{1}{g}\right) + \frac{w}{g}\right)$$

Magnetic Stored Energy is

$$W_m = \int_0^{\lambda_0} \frac{\lambda}{L(x)} d\lambda = \frac{1}{2} \frac{\lambda_0^2}{L(x)}$$

Force developed is then:

$$f_e = -\frac{\partial W_m}{\partial x} = -\frac{1}{2}\lambda_0^2 \frac{\partial}{\partial x} \frac{1}{L(x)} = \frac{1}{2}\left(\frac{\lambda_0}{L(x)}\right)^2 \frac{\partial L(x)}{\partial x}$$

If the system is driven by a current I, this has a simple form:

$$f_e = \frac{I^2}{2} \frac{\partial L(x)}{\partial x} = \mu_0 N^2 D I^2 \left(\frac{1}{g-h} - \frac{1}{g}\right)$$

4.3   Example: Railgun

This seemingly exotic device has practical applications, not only as a weapon but also for some types of materials testing. It has an advantage over gas powered guns (the conventional type that derive the gas from an explosive charge) that means it can drive very high speed projectiles. We introduce it here because it allows us to show a few important concepts.
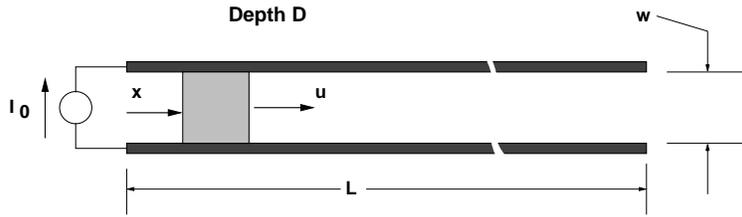

Figure 12: Cartoon of a Rail Gun

A cartoon of the basic rail gun is shown in Figure 12. It consists of two flat 'rails' that are parallel to each other and separated by the distance w, driven at one end by a distributed source of current. The system has depth D in the direction you cannot see. Allowed to slide in the horizontal direction is the projectile, which makes good electrical contact with the two rails.

It is a common assumption that magnetic field is important only within the region between the two rails. This will be close to being right if $D >> w$ or if the magnetic flux is returned by a ferromagnetic barrel (not shown in this cartoon). However, this assumption means that we may estimate the magnetic field in the region to the left of the projectile to be:

$$H_z = -\frac{I_0}{D}$$

Flux to the left of the projectile is $\Phi = \mu_0 \frac{I_0}{D} wx$, and because this is inherently a one-turn inductor, we can estimate the inductance that is a function of projectile position:

$$L(x) = \mu_0 \frac{wx}{D}$$

From the derivation of force, recognizing this thing is linear,

$$f_e = \frac{1}{2}I_0^2 \frac{\partial L(x)}{\partial x} = \frac{1}{2}\mu_0 I_0^2 \frac{w}{D}$$

The advantage over a gas gun should be clear by now: The force on the projectile depends on the square of current. In a gas gun the pressure behind the projectile falls rapidly as the volume behind the projectile increases.

Note that this description was incomplete in an inconsequential way: in calculating inductance we included only the region behind (to the left in this picture) of the projectile. The projectile itself has some dimension and there will be some energy stored and so inductance associated with the projectile itself. However, that inductance does not change with projectile position and so does not affect action of the railgun. That dimension does, however, allow us to compare the force derived by the energy (POVW) method with the Lorentz Force Law. Look at Figure 13.
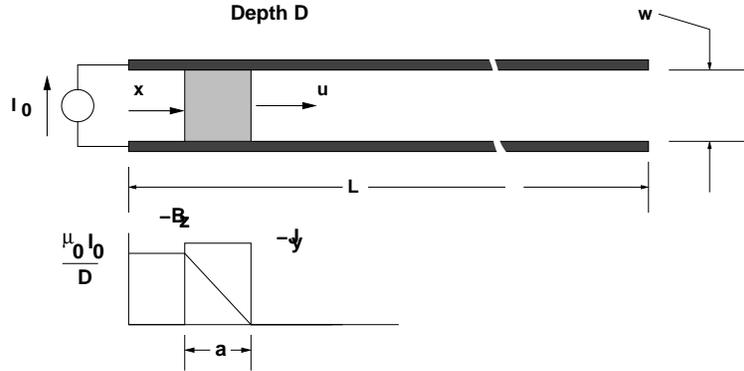

Figure 13: Cartoon of a Rail Gun

In this derivation, assume that the coordinate system to be used has $\hat{x}$ to the right (so that the position of the left-hand edge of the projectile is at x, the $\hat{y}$ direction is up and then the $\hat{z}$ direction is out of the paper. It is reasonable to assume that current in the projectile block will be uniform, and if so it is:

$$\vec{J} = -\hat{y}\frac{I_0}{aD}$$

The flux density in the region of the block must have the form shown in the figure, since the curl of H is current density:

$$\nabla \times \vec{H} = \vec{J}$$

or, in this case,

$$\frac{\partial H_z}{\partial x} = -J_y$$

Force on the block is determined from $F = \vec{J} \times \vec{B}$ and total force is the integral of that over the volume of the block. It is straightforward to see that, if the current is uniform and the consequent flux density has a triangular form as shown,

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$$f_e = \hat{x}\frac{1}{2}\mu_0 J_y B_z = \hat{x}\frac{1}{2}\frac{I_0}{aD}\mu_0\frac{I_0}{D}wDa = \frac{1}{2}\mu_0 I_0^2\frac{w}{D}$$

As with the electric field force example we started with, the dimension of the projectile, a is inconsequential.

6.007 Electromagnetic Energy: From Motors to Lasers
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