18.700 Problem Set 9

Due in class **Wednesday December 4** (changed from syllabus); late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (8 points) Suppose V is a real or complex inner product space. A linear map $S \in \mathcal{L}(V)$ is called *skew-adjoint* if $S^* = -S$. Suppose V is complex and finite-dimensional, and S is skew-adjoint. Show that the eigenvalues of S are all purely imaginary (that is, real multiples of i) and that there is an orthogonal direct sum decomposition

$$V = \bigoplus_{\lambda \in \mathbb{R}} V_{i\lambda}.$$

2. (16 points) Suppose V is an n-dimensional real inner product space, and S is a skew-adjoint linear transformation of V.

a) Show that Sv is orthogonal to v for every $v \in V$.

b) Show that every eigenvalue of S^2 is a real number less than or equal to zero.

c) Suppose (still assuming S is skew-adjoint) that $S^2 = -I$ (the negative of the identity operator on V). Show that we can make V into a *complex* inner product space, by defining scalar multiplication as

$$(a + bi)v = av + bSv$$

and the complex inner product as

$$\langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle - i\langle Sv, w\rangle.$$

What is the dimension of V as a complex vector space?

d) Now *drop* the assumption that $S^2 = -I$, but still assume S is skew-adjoint. Show that there is an orthonormal basis of V in which the matrix of S is

$$\begin{pmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -\lambda_r & & & \\ & & & \lambda_r & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix},$$

with $\lambda_1 \geq \cdots \geq \lambda_r > 0$. That is, the matrix of S in this basis is block diagonal, with r 2×2 blocks of the form

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

with $\lambda > 0$, and $n - 2r$ 1×1 blocks (0). (Hint: first diagonalize S^2.)

3. (6 points) Give an example of a square complex matrix A with the property that A has exactly three distinct eigenvalues, but A is *not* diagonalizable. (For full credit, you should *prove* that your matrix has the two required properties.)