18.310 Exam 2 practice questions

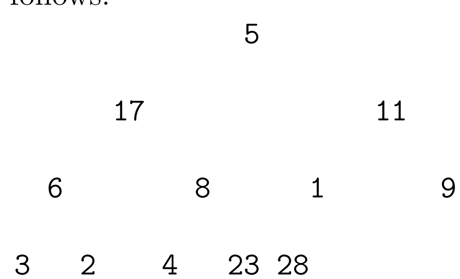
Collection of problems from past quizzes and other sources. It does not necessarily reflect what will be on the exam on Friday.

1. (From quiz, Fall 2012; was one out of 5 questions.) Consider a 2-player zero-sum game with payoff matrix
$$A = \begin{pmatrix} 1 & 5 & 4 \\ 5 & 2 & 2 \end{pmatrix}.$$
   1. Let y_1, y_2 denote a mixed strategy for player 1. Write a linear program that gives the mixed strategy for player 1 that maximizes his expected payoff.
   2. Write the dual of the above linear program.
   3. Is the strategy $y_1 = y_2 = \frac{1}{2}$ optimal for player 1? Explain your reasoning. What is the expected payoff for player 1 if he plays $y_1 = y_2 = \frac{1}{2}$?
   4. What is an optimum strategy for player 2?

2. (Practice problem 2012.) Determine the dual of the following LP:
$$\begin{aligned} \min \quad & 6x - 3y - z \\ \text{s.t.} \quad & 4x - 2y + z = 4 \\ & x + 3y - z \geq 2 \\ & x, y, z \geq 0. \end{aligned}$$

3. (From quiz, Fall 2010; was one of 4 questions.)

   Suppose that we are doing heapsort. At some point, the numbers in our tree are arranged as follows:



   (a) (5 points): How would this heap be stored in an array?

   (b) (5 points). Which numbers have already been processed, and should now be considered inactive?

   (c) (10 points). What will the tree look like after the next step, when it has been turned into a heap again?

4. (From Quiz, Fall 2010; was one out of 4 questions.) For each question below, asnwer TRUE or FALSE and give a one-line justification.

   (a) A heap on n elements can be built with $O(n)$ comparisons, i.e. with a number of comparisons bounded by a constant times n.

   TRUE or FALSE

   (b) The pigeonhole principle implies that the number of comparisons required for merging two sorted arrays of size $n/2$ is at least $\log_2 \binom{n}{n/2}$.

   TRUE or FALSE

   (c) From the construction of Batcher's network described in lecture, one can obtain a non-adaptive algorithm to merge two sorted arrays of size $n/2$ with a *linear* number of comparisons.

   TRUE or FALSE

   (d) A heap with k levels can store n keys where $2^{k-1} \leq n < 2^k - 1$.

   TRUE or FALSE

5. (From a 2010 problem set.) Prove the following: To show that a sorting network on n inputs correctly sorts any input, one only needs to consider all inputs with 0's and 1's (there are 2^n of them). (This is much less than trying all permutations, which would be $n!$.)

6. (Old problem set question, modified.) Suppose you have a nonnegative (i.e. all entries are nonnegative) $m \times n$ matrix A such that all row sums

$$r_i := \sum_{j=1}^{n} a_{ij}$$

   for $i = 1, \cdots, m$ and all column sums

$$c_j := \sum_{i=1}^{m} a_{ij}$$

   for $j = 1, \cdots, n$ are all integers. Then show that there exists a matrix B with

   1. $b_{ij} = 0$ if $a_{ij} = 0$, and
   2. the *same* row sums and column sums, and
   3. with all entries being integers.

7. (From a pset.) Derive an upper bound on the number of comparisons needed to find the median based on partitioning into subarrays of 7 elements. You may use the fact that 7 elements can be sorted with 13 comparisons.

18.310 Principles of Discrete Applied Mathematics
Fall 2013