Fall 2013

18.310 Homework 8 Solutions

Due Monday October 28th at 6PM

Instructions: Remember to submit a separate PDF for each question.Do not forget to include a list of your collaborators or to state that you worked on your own.

1. What is the optimum solution of the following linear program:

$$\begin{aligned} \text{Max} \quad & 5x_1 + 7x_2 + 9x_3 + 11x_4 + 13x_5 \\ \text{subject to:} \quad & \\ & 15x_1 + 28x_2 + 18x_3 + 44x_4 + 65x_5 = 2010 \\ & x_i \geq 0 \qquad\qquad i = 1, \ldots, 5. \end{aligned}$$

Write the dual linear program, and find an optimum solution to the dual. What are the optimum values to the primal and dual linear programs?

Solution: The dual of:

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s. t.} \quad & a^T x = b \\ & x \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{Min}_y \quad & by \\ \text{s.t.} \quad & ya \geq c, \end{aligned}$$

and in this case, it is:

$$\begin{aligned} \text{Min}_y \quad & 2010y \\ \text{s.t.} \quad & 15y \geq 5, \\ & 28y \geq 7 \\ & 18y \geq 9 \\ & 44y \geq 11 \\ & 65y \geq 13 \end{aligned}$$

The dual optimum solution is thus $y^* = \max_i \frac{c_i}{a_i} = \frac{1}{2}$, and the value of the solution is $by^* = 1005$. Note that by strong duality, the primal solution must satisfy $c^T x^* = 1005$. By complementary slackness, since only the inequality $18y \geq 9$ is satisfied at equality by y^*, we have that only x_3^* can be nonzero. This means $9x_3^* = 1005$ and the solution is $x^* = (0, 0, 335/3, 0, 0)$.

2. Solve the following LP using the simplex method, showing your tableau and choice of pivot clearly at each step, and writing the final answer and objective value clearly.

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 \le 4 \\ & 2x_1 + 3x_3 \le 5 \\ & 2x_1 + x_2 + 3x_3 \le 7 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

(You will need to introduce some slack variables to get the LP into the required form.)

Solution: By adding three slack variables, to get the problem into the standard form, we get a basic feasible solution in the new variables and the following tableau:

$$\left[\begin{array}{c|ccccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline -1 & 3 & 2 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 2 & 1 & 0 & 0 & 4 \\ 0 & \boxed{2} & 0 & 3 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 3 & 0 & 0 & 1 & 7 \end{array}\right].$$

The current basis is (s_1, s_2, s_3) with corresponding bfs $(0, 0, 0, 4, 5, 7)$. We choose to have x_1 enter the basis as its reduced cost coefficient is positive $(3 > 0)$. The min-ratio test $\min\{\frac{4}{1}, \frac{5}{2}, \frac{7}{2}\}$ shows that we will pivot on the second row and s_2 will leave the basis. After pivoting, we obtain:

$$\left[\begin{array}{c|ccccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline -1 & 0 & 2 & -1/2 & 0 & -3/2 & 0 & -15/2 \\ \hline 0 & 0 & \boxed{1} & 1/2 & 1 & -1/2 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 & 0 & 5/2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 2 \end{array}\right].$$

Our new basis is (s_1, x_1, s_3) with bfs $(5/2, 0, 0, 3/2, 0, 2)$ and objective value $15/2$. The only variable with positive cost coefficient is x_2, so we will have x_2 enter the basis. The min-ratio test $\min\{\frac{3}{2}, 2\} = \frac{3}{2}$ shows that we will pivot on the first row (see box above for pivot element). We get:

$$\left[\begin{array}{c|ccccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline -1 & 0 & 0 & -3/2 & -2 & -1/2 & 0 & -21/2 \\ \hline 0 & 0 & 1 & 1/2 & 1 & -1/2 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 & 0 & 5/2 \\ 0 & 0 & 0 & 1/2 & -1 & -1/2 & 1 & 1/2 \end{array}\right].$$

Our new basis is (x_2, x_1, s_3) with bfs $(5/2, 3/2, 0, 0, 0, 1/2)$ and objective function value $21/2$. As all reduced cost coefficients are nonpositive, the current bfs is an optimum solution. Forgetting the slack variables, we have the optimum solution $x_1 = \frac{5}{2}$, $x_2 = \frac{3}{2}$ and $x_3 = 0$. As a sanity check, we verify that this solution satisfies all the constraints and has value $21/2$.

3. Using strong duality for linear programming, prove the following.

Theorem. *$Ax = b, x \geq 0$ has no solution if and only if there exists y with $A^T y \geq 0$ and $b^T y < 0$.*

Solution: For the implication to the left, suppose there is such an x and such a y. Then $0 > y^T b = y^T Ax = x^T A^T y \geq 0$, which is a contradiction. For the reverse implication consider the problem:

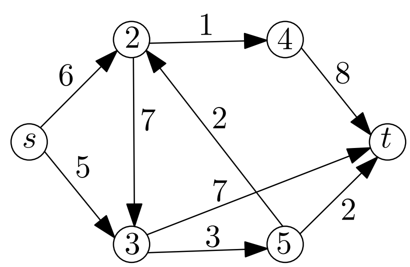
$$\begin{aligned} \text{Max} \quad & 0 \\ \text{subject to:} & \\ & Ax = b \\ & x \geq 0, \end{aligned}$$

and its dual

$$\begin{aligned} \text{Min} \quad & b^T y \\ \text{subject to:} & \\ & A^T y \geq 0. \end{aligned}$$

Observe that the dual is feasible (since $y = 0$ is feasible). If the primal is not feasible, then the dual must be unbounded and in particular, this means that we can find a $y$such that $A^T y \geq 0$ and $b^T y < 0$. This shows the reverse direction. (Observe that by scaling y, we can have $b^T y$ take any negative value while y remains dual feasible.

4. Consider the following flow problem instance, with source s and sink t:



Find a maximum flow (show your augmenting paths), and also exhibit an s-t cut of the same value.

Solution:NOTE :It would be neater to write the solution as a series of diagrams, but we'll skip this. For now, here are the augmenting paths and the cut.
First take the path $s, 2, 3, t$ and push 6 through this path, giving the flow:

	$(s,2)$	$(s,3)$	$(2,3)$	$(2,4)$	$(3,5)$	$(3,t)$	$(4,t)$	$(5,2)$	$(5,t)$
x	6	0	6	0	0	6	0	0	0
u	6	5	7	1	3	7	8	2	2

Next take the path $s, 3, 2, 4, t$ and push 1 units through it, giving the flow:

	$(s,2)$	$(s,3)$	$(2,3)$	$(2,4)$	$(3,5)$	$(3,t)$	$(4,t)$	$(5,2)$	$(5,t)$
x	6	1	5	1	0	6	1	0	0
u	6	5	7	1	3	7	8	2	2

Take the path $s, 3, t$ and push one unit through it.

	$(s,2)$	$(s,3)$	$(2,3)$	$(2,4)$	$(3,5)$	$(3,t)$	$(4,t)$	$(5,2)$	$(5,t)$
x	6	2	5	1	0	7	1	0	0
u	6	5	7	1	3	7	8	2	2

And finally take the path $s, 3, 5, t$ and push 2 units through it.

	$(s,2)$	$(s,3)$	$(2,3)$	$(2,4)$	$(3,5)$	$(3,t)$	$(4,t)$	$(5,2)$	$(5,t)$
x	6	4	5	1	4	7	1	0	4
u	6	5	7	1	3	7	8	2	2

This results in a flow of 10, which matches the minimum cut value corresponding to the cut $S = \{s, 2, 3, 5\}$ (the total capacity of the arcs $(2,4), (3,t), (5,t)$ is indeed 10). S was obtained by looking at the vertices reachable from s in the residual graph.