Fall 2013

# 18.310 Homework 3 Solutions

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**Instructions:** Remember to submit a separate PDF for each question. Do not forget to include a list of your collaborators or to state that you worked on your own.

1. **Writing Assignment:** Following the discussion in recitation, revise Section 3 (and Section 3 **only**) of the lecture notes on Chernoff Bound. You will find a LaTeX version of the notes in the webpage for Homework 3 under the name "chernoff.tex".

   You have complete editorial freedom and should feel free to add, delete and move text within Section 3, following the instructions given at the top of "chernoff.tex". You should pay particular attention to instances of missing guiding or explanatory text, such as those that were pointed out in recitation.

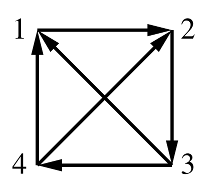
   After revising the notes, write a short supporting text (1-2 paragraphs) explaining your most significant revisions to the notes and the rationale behind them.

   Please submit online the following documents: i-ii) pdf and latex of your revised lecture notes, iii) pdf of your supporting text (as a separate document).

2. A *tournament* on $n$ vertices is an orientation of a complete graph on $n$ vertices, i.e. for any two vertices $u$ and $v$, exactly one of the directed edges $(u, v)$ or $(v, u)$ is present. A Hamiltonian path in a tournament is a directed path passing through all vertices exactly once. For example, in the example below, $1-2-3-4$ is a Hamiltonian path since all edges $(1, 2), (2, 3)$ and $(3, 4)$ are directed in the direction of traversal and it visits every vertex once, while for example $1-2-4-3$ is not a directed path since the second and third edges are directed in the wrong direction.

   Show that, for any $n$, there exists a tournament with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

   For example for $n = 4$, this shows that there exists a tournament on 4 vertices with at least 3 Hamiltonian paths. Here is such a tournament with Hamiltonian paths $1-2-3-4$, $2-3-4-1$, $3-4-1-2$, $4-1-2-3$, $4-2-3-1$.


   (Hint: think probabilistically...)

   **Solution:** For every undirected edge $\{u, v\}$ choose with probability $1/2$ to orient it $(u, v)$, and with probability $1/2$ to orient it $(v, u)$. Do this independently for every edge in the complete graph $K_n$. Clearly the resulting graph $G$ is a tournament, since for every pair of vertices there is in the end exactly one directed edge. Now given a permutation $\sigma$ of $[n]$, the probability that this permutation corresponds to a Hamiltonian path in $G$ is

$$\mathbb{P}(\forall i = 1, \ldots, n-1 \quad (\sigma_i, \sigma_{i+1}) \in E(G)) = \prod_{i=1}^{n-1} \mathbb{P}((\sigma_i, \sigma_{i+1}) \in E(G)) = \frac{1}{2^{n-1}},$$

because every edge is chosen independently and has a 1/2 chance of appearing. Now define $X$ to be the number of Hamiltonian paths in $G$. Clearly $X$ is a random variable and

$$X = \sum_{\sigma \in S_n} X_\sigma,$$

where $X_\sigma$ is the random variable that takes the value 1 when $\sigma$ is a Hamiltonian path and 0 otherwise, and $S_n$ is the set of permutations of $[n]$. Then by linearity of expectation

$$\mathbb{E}(X) = \sum_{\sigma \in S_n} \mathbb{P}(X_\sigma = 1) = \frac{n!}{2^{n-1}},$$

since there are $n!$ permutations of $[n]$. But this implies that there is a choice of directed edges that achieves at least this number of Hamiltonian paths, otherwise

$$E(X) = \sum_{k \geq 0} k\mathbb{P}(X = k) = \sum_{k=0}^{\frac{n!}{2^{n-1}}-1} k\mathbb{P}(X = k) \leq \left(\frac{n!}{2^{n-1}} - 1\right) \sum_{k \geq 0} \mathbb{P}(X = k) < \frac{n!}{2^{n-1}}.$$

3. The examples of Chernoff bounds discussed in class bound the probability of deviating from the expectation by a certain amount. We can also ask the reverse question of finding intervals where we have high confidence that the random variable will fall into.

   Consider flipping a fair coin 100 times. Using the Chernoff bounds shown in class, find $x$ such that with at least 95% probability we obtain at least $x$ heads.

   **Solution:** We will use the lower tail Chernoff bound

$$\mathbb{P}(X < (1-\delta)\mu) \leq e^{-\mu\delta^2/2},$$

   for all $0 < \delta < 1$, valid when $X$ is a sum of independent, identically distributed Bernoulli random variables, and $\mu = \mathbb{E}(X)$. Let $X$ be the number of heads. Obviously it satisfies the hypotheses of the Chernoff bound. Note that,

$$\mathbb{E}(X) = \mathbb{E}(100 - X) = 50,$$

   so applying the Chernoff bound we obtain

$$\mathbb{P}(X < (1-\delta)50) \leq e^{-25\delta^2}$$

   or equivalently

$$\mathbb{P}(X \geq (1-\delta)50) \geq 1 - e^{-25\delta^2}.$$

Now $1 - e^{-25\delta^2} = 0.95$ if and only if

$$\delta = \sqrt{\frac{\ln 20}{25}} \in (0,1).$$

We then conclude that if

$$x = (1-\delta)50 > 32$$

then

$$\mathbb{P}(X \geq 32) \geq \mathbb{P}(X \geq x) \geq 19/20.$$

4. Recall the sequential choice problem discussed in class: $n$ objects are shuffled in random order, and are presented to you one at a time. For each object, you know how it compares against all previous ones that you've seen, However, we're now considering the setting in which, rather than your choice being final, you can alter it at any time. Of course, now the strategy becomes obvious: you will switch to the current object if it is better than everything you have seen so far (or equivalently better than the object you last selected). Find the expected number of times you will select a new object.

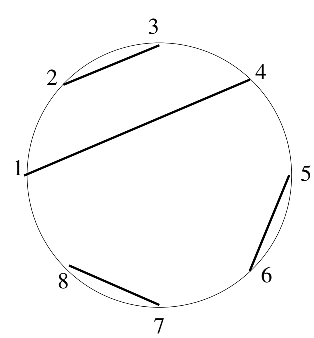
   Formally, given a random permutation of $[n]$, find the expected number of entries smaller than all entries before it, and express the answer in terms of $n$.

   **Solution:** Take $\sigma$ to be a random permutation of $[n]$. Given a position $i \in [n]$ the probability that all entries before position $i$ were smaller than $\sigma_i$ is equal to $1/i$, since each position between 1 and $i$ is equally likely to be the maximum of these $i$ values. Now define the random variable $X$ to be the number of entries that are bigger that its predecessors. Then

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(\sigma_i > \sigma_j : \forall j < i) = \sum_{i=1}^{n} \frac{1}{i} = H_n,$$

   where $H_n$ is the $n$-th Harmonic number.

5. Consider $2n$ points on the plane labelled $1, 2, \cdots, 2n$, all spaced equally on a circle. A *matching* of these points is a collection of $n$ straight line segments, with every point being the endpoint of precisely one of the line segments. A matching is *noncrossing* if no two of its line segments cross. Here is an example of a noncrossing matching on 8 points (so $n = 4$).



Determine (with proof) the number of noncrossing matchings of $2n$ points, as a function of $n$. *(You might want to look for an appropriate bijection.)* Check your result for $n = 1$, 2 and 3.

**Solution:** We will show that the number of noncrossing matchings of $2n$ points is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, by showing a bijection between these drawings and Dyck paths of length $2n$. Define a Dyck path from a drawing by setting the step in the $i$-th position to be an up step if $i$ is connected to $j$ where $j > i$ and a down step otherwise. First we need to prove that the result is a Dyck path. First each edge has one end that is going to correspond to an up step, and one that will correspond to a down step, so the path ends in zero. Furthermore, take $j$ to be a down step position in the path. Take $i < j$ to be the corresponding up step position(i.e. the vertex connected to $j$ in the matching). Since there are no edges crossing $ij$, all steps given by edges between the numbers $\{i + 1, \ldots, j - 1\}$ sum to zero. And then the same is true for the steps given by edges between the numbers $\{i, \ldots, j\}$. If at position $j$ the sum of up steps and down steps resulted in a negative position, the last argument shows that there is a position before $i$ that also has this property. And since this cannot go on forever, and at the beginning we are in zero, there is no such $j$. To see that this is a bijection, given a Dyck path and an up step in position $i$ connect $i$ to $j$ where $j$ is the position of the first down step at the same height as the step in $i$. Clearly there is such a step, since the Dyck path goes back to zero, and also this step occurs before any other up step of the same height, because a Dyck path is continuous, so for every $i$ up step there is a unique $j$. Now paths don't cross since if $i_1 < i_2 < j_1$ the height of $i_2$ must be greater than that of $i_1$ otherwise $j_1$ would have occurred before. This implies $j_2 < j_1$ by the continuity of the Dyck path. So this drawing is a noncrossing matching, whose image is clearly the original Dyck path, so we have found a bijection.