Problem Set 1 Solutions II

Problem 1

Prove that the major index is equidistributed with the number of inversions of $w \in S_n$. Recall that the major index is the sum of the indices of the descents

$$\sum_{w_i > w_{i+1}} i.$$

Solution by Yogeshwar Velingker. We're trying to show that

$$\sum_{w \in S_n} q^{\text{maj}(w)} = \sum_{w \in S_n} q^{\text{inv}(w)}.$$

The right hand side, as proved in class, is

$$1(1+q)(1+q^2)\cdots(1+q+\cdots+q^{n-1}) = [n]_q!.$$

This can be shown by induction, and we'll do a similar thing here. This is obvious when $n=1$: there are no descents and no inversions.

Now let's say we have a permutation $w = (w_1, \cdots, w_n)$. If we insert $n+1$ here, there are several cases.

- Case 1: if we insert $n+1$ at the end, the major index stays the same. Otherwise, we insert $n+1$ somewhere in the middle. Let $w_0 = 0$: then $n+1$ goes after some w_i.
- Case 2: If we insert it between w_i and w_{i+1}, and $w_i < w_{i+1}$, we add a descent at spot $i+1$ and then shift all descents after it by 1 spot. So in this case, the major index increases by $i+1+d(i)$, where $d(i)$ is the number of descents in w at or after i.
- Case 3: If we insert $n+1$ between w_i and w_{i+1}, but $w_i > w_{i+1}$, then the descent goes from spot i to spot $i+1$, and we also shift all later descents by 1. So the major index just increases by $d(i)$ here.

Our goal is to show that all of these increases are the numbers 0 through n, as this would show the result by induction! Notice that $d(i)+i+1$ in the ascent case is at most n, because the descents in $d(i)$ can occur at spots $i+1, \cdots, n-1$, and also $d(i) < n$ in the ascent case. So we just need to show that the values are all different.

We can't have $d(i) = d(j)$ in the two descent insertions, because one descent occurs before the other. If i and j are both ascents, then $d(i)+i+1 = d(j)+j+1$, and this implies $d(i)-d(j) = j-i$. But that would mean we need descents at all spots between i and j, which is not possible since i is an ascent.

Finally, let's say i is a descent and j is an ascent, so

$$d(i) = d(j) + j + 1 \implies d(i) - d(j) = j + 1,$$

but (????), which is bad.

This means that all increases in major index from inserting $n+1$ are all different, so we're done by induction! $\square$

Problem 2

Find the number of permutations $w \in S_n$ that are 231-avoiding and $k(k-1)\cdots 1$-avoiding, where $k = 4$.

Solution by Congyue Deng. Denote this $T(n,k)$. We claim that

$$T(n+1,k) = \sum_{i=0}^{n} T(i,k)T(n-i,k-1).$$

To show this, consider any $w \in S_{n+1} = \sigma(n+1)\tau$, where σ is i elements and τ is the remaining $n-i$ elements. Since w is 231-avoiding, the elements in σ are all smaller than the elements in τ. Note that both σ is a 231 and $k(k-1)\cdots 1$-avoiding S_m and τ is a 231 and $k(k-1)\cdots 1$-avoiding S_{n-m} permutation, which is the exact recurrence relation we want as we sum over m!

Now, let's compute $T(n,3)$. This is the number of 231 and 321-avoiding permutations. If we insert $n+1$ into a permutation $w_n \in S_n$, but we can't put it in slot i, then we also can't put it in slot $i-1$. We know that we can put $n+1$ in the end of w, and there are $k+1$ ways to put $n+2$, or we can put $n+1$ in one of the $k-1$ last spots and place $n+2$ in the end. So by induction, there are 2^{n-1} permutations in S_n that avoid 231 and 321.

So now

$$T(n+1,4) = 2^{n-1}T(0,4) + 2^{n-2}T(1,4) + \cdots + T(n,4),$$

which yields

$$T(n+1,4) = 3T(n,4) - T(n-1,4)$$

which is just the alternating Fibonacci numbers F_{2n-1}. $\square$

Recall the problem from last lecture:

Problem 3

Prove that the number of set-partitions of $[n]$ that have no $i, i+1$ in the same partition is the same as the number of set-partitions of $[n-1]$.

Proof by Wanlin Li. Draw an arc-diagram: for example, $(1,3,5)$ in the same partition would correspond to arcs between 1 and 3 and between 3 and 5. If we have a set-partition of $[n]$ with i and $i+1$ not in the same partition, this is the same as having no arcs of diameter 1. Now shrink every arc by 0.5 in each direction, this is easily reversible and is a bijection! $\square$

Problem 4

For the biased drunk-walk, find the probability that if the man starts at i_0 and moves right with probability p, he falls off after m steps.

Solution by Sarah Wang. The man needs to take k steps to the right and $k + i_0$ steps to the left, so

$$k = \frac{m - i_0}{2}$$

(so we must assume $m \geq i_0$ and m, i_0 have the same parity.)

Draw the path as a Dyck path: we want to count the number of paths from $(0, i_0)$ to $(m, 0)$. Much like with the Catalan numbers, we will instead count the number of bad paths. Those intersect the x-axis: pick the first point where this happens, and reflect the remaining path from there to $(m-1, 1)$ over the y-axis to $(m-1, -1)$. This is a bijection, so the total number of good paths is

$$\binom{m-1}{k} - \binom{m-1}{k-1}.$$

Each path occurs with probability

$$p^k(1-p)^{k+i_0},$$

so our final answer is the number of paths times the probability of each. □

Problem 5

Prove the baby-hook length formula: the number of linear extensions of a poset whose Hasse diagram is a rooted tree is

$$\frac{n!}{\prod_{v \in T} h(v)}.$$

Proof by Sanzeed Anwar. Use strong induction. The base case $n = 1$ is easy to show.

Take a poset whose Hasse diagram is a rooted tree of n vertices. From the vertex, there are some subtrees $T_1, T_2, \cdots, T_k$, with $m_1, \cdots, m_k$ vertices. Note that $m_1 + \cdots + m_k = n - 1$.

By strong induction, the number of linear extensions of T_i is just

$$\frac{m_i!}{\prod_{v \in T_i} h(v)},$$

so if we want a rooted tree on all n vertices, we can do this in

$$\frac{m_1! m_2! \cdots m_k!}{\prod_{v \neq \text{root}} h(v)} \binom{n-1}{m_1, m_2, \cdots, m_k}$$

ways, since we can pick which m_1 nodes to use for T_1, which m_2 nodes to use for T_2, and so on. This simplifies to

$$\frac{(n-1)!}{\prod_{v \neq \text{root}} h(v)} = \frac{n!}{\prod_v h(v)},$$

as desired. □

Problem 6

Show that a permutation of length $mn + 1$ has an increasing subsequence of length $m + 1$ or decreasing subsequence of length $n + 1$.

Proof. For each element in the permutation, assign an ordered pair (i, j) where i is the length of the longest increasing subsequence up to that element and j is the length of the longest decreasing subsequence up to that element. These are all distinct (since any subsequence can be extended), so we can't have $1 \leq i \leq m$ and $1 \leq j \leq n$ for all i, j by the Pigeonhole principle. □

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