

18.212: Algebraic Combinatorics

Andrew Lin

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This class is being taught by **Professor Postnikov**.

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Remember that last lecture, we started talking about plane partitions. As a review, these are $m \times n$ arrays of integers that are weakly decreasing across rows and columns. Let's assume that the numbers that appear are in the set $\{0, 1, 2, \dots, k\}$.

Denote $P(m, n, k)$ to be the number of such plane partitions: we learned that there are alternate ways to interpret this number. First of all, we can represent plane partitions with k weakly noncrossing paths (it's okay to intersect but not cross), and then we can shift them a bit to get noncrossing paths. The Lindstrom lemma then tells us that

$$P(m, n, k) = \det C,$$

where our matrix has size $k \times k$ and entries

$$c_{ij} = \binom{m+n}{m+i-j}.$$

Here, we still have to calculate a determinant, but it's a bit of work.

Proposition 1 (MacMahon, 1895)

It turns out we can write this as

$$P(m, n, k) = \prod_{i=1}^m \prod_{j=1}^n \prod_{\ell=1}^k \frac{i+j+\ell-1}{i+j+\ell-2}.$$

Example 2

Let's set $(m, n, k) = (5, 4, 3)$: then the value should be the determinant

$$\det \begin{pmatrix} \binom{9}{5} & \binom{9}{4} \\ \binom{9}{6} & \binom{9}{5} \end{pmatrix} = \frac{2(9!)^2}{(5!)^2 \cdot 3!4!}$$

which is the same as the product formula answer if we bash.

Notice that this formula is symmetric in m, n, k , which may not be obvious from the determinant formula!

It's good that plane partitions can be thought about in many ways. One thing we did last class was think of a plane partition as a set of cubes, where the numbers on top of the squares represent the height or number of cubes: for example,

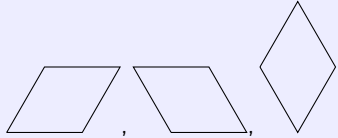
2	2
2	1

is a $2 \times 2 \times 2$ cube with one cube removed.

This is (in some way) a three-dimensional Young diagram! Basically, we want to put a bunch of cubes in our $k \times m \times n$ box so that they're all "justified" towards a specific corner. If we deform the shape a bit, we can also view this as an equiangular hexagon with a **rhombus tiling**:

Proposition 3

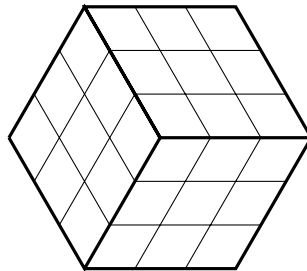
$P(m, n, k)$ is the number of ways to tile an equiangular hexagon with sides of length m, n, k, m, n, k with tiles



Example 4

Take $m = n = k = 3$.

It's important to note that by default, we tile our rhombus like this:

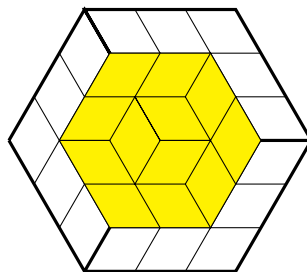


Now the center of the hexagon is our "justified corner" for our cubes. For example, the plane partition

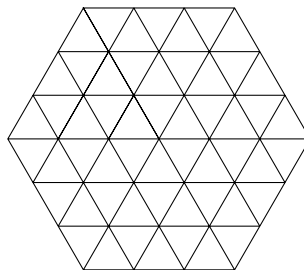
2	2	
2	1	

 will

yield the following rhombus tiling (which can also be interpreted as being inside a cube):



Another way to think of this is to start with an equilateral triangle tiling of our hexagon.



Then rhombi are formed by connecting two faces with an edge, and therefore we can draw the **planar dual graph** of our equilateral triangle grid. We now have a hexagonal tiling, and rhombus tiling is now a perfect matching of the edges of

a hexagon, which is also called a **dimer model** by physicists.

Next, we'll talk about another combinatorial object that looks similar: domino tilings!

Definition 5

Start with an $m \times n$ rectangular grid. A **domino tiling** is a covering of the mn squares with 1×2 dominoes with no holes: define $N(m, n)$ to be the number of such domino tilings.

Is there anything we can say about these?

Lemma 6

If m and n are both odd, then $N(m, n) = 0$.

This is because there are an odd number of squares, and each domino takes up an even number of spots. In general, the best way to approach this is to color our $m \times n$ rectangle in the usual chessboard way: we want a perfect matching of the white and black vertices, which form a bipartite graph.

It turns out this is related to the **permanent** of a matrix, which is like the determinant but without any negative signs. Unfortunately, this is much harder to calculate because we don't have tools like eigenvalues to calculate it! Well, Kasteleyn figured out how to deal with this: instead of using the adjacency matrix as usual, we replace vertical 1s in the matrix with i s.

Theorem 7 (Kasteleyn, 1961)

Take n to be even without loss of generality. Then the number of ways to tile an $m \times n$ rectangle is

$$N(m, n) = \prod_{k=1}^{n/2} \prod_{\ell=1}^{\lfloor m/2 \rfloor} \left(4 \cos^2 \frac{\pi k}{n+1} + 4 \cos^2 \frac{\pi \ell}{m+1} \right)$$

times another $(2 \cos \frac{\pi \lfloor m/2+1 \rfloor}{m+1})$ when m is odd.

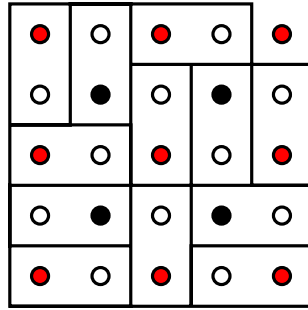
This problem has applications to statistical physics: it's used because in physics, we care about the asymptotic number of matchings as our grid gets larger! We won't prove this unless we have time to come back to it later.

Let's look at another domino-tiling result:

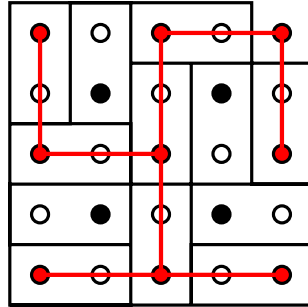
Theorem 8 (Temperley, 1974)

Suppose m, n are both odd. Then the number of ways to tile an $m \times n$ grid without a corner square, if $m = 2k + 1, n = 2\ell + 1$, is the number of spanning trees of a $k \times \ell$ grid.

This is a good way to get approximate asymptotics! We can prove this with **Temperley's bijection**. Given a domino tiling, color the underlying $(2k + 1) \times (2\ell + 1)$ grid black and white so that the corners are black: now take those black entries (i, j) where i, j are both odd, and color them red.



Now we form a spanning tree on the red $k \times \ell$ grid as follows: if there's a domino that is pointed along the line between two adjacent nodes, connect those two, and otherwise don't. As an exercise, show that this always forms a spanning tree!



Fact 9

By the way, if we remove some other box (which is of the same checkerboard color as the corner) along a side, then the number of domino tilings is still the same. Try to find a bijection!

Next class, we'll do presentations on the problem set due today!

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