18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Recall that we've been talking about ranked posets: given a poset P with a rank function $\rho : P \to \mathbb{Z} \geq 0$, we can construct sets $P_i = \{x \in P \mid \rho(x) = i\}$ of a given rank, and we let $r_i = |P_i|$ be the rank numbers.

We defined P to be **rank-symmetric** if $r_i = r_{N-i}$ for all i and **unimodal** if the ranks increase to a point and then decrease. We also defined **Sperner** posets to be those in which the maximal size M of an antichain in P is the maximum among the r_is.

Theorem 1 (Sperner's theorem (1928))

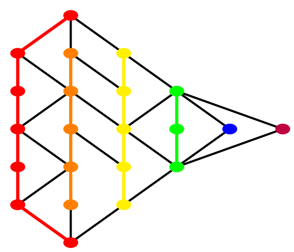
The Boolean lattice B_n is Sperner.

There's a property of posets that implies all three of the ideas above!

Definition 2

A **saturated chain** C has elements $\{x_0 \lessdot x_1 \lessdot x_2 \cdots\}$, so we can't put anything between the elements of the chain. In particular, the rank $\rho(x_i) = \rho(x_0) + i$. A **symmetric chain decomposition** (SCD) is a decomposition of a poset P's elements into a disjoint union of saturated chains C_i such that for all chains $C_i = \{x_0 \lessdot \cdots \lessdot x_\ell$, $\rho(x_\ell) = N - \rho(x_0)$.

For example, here is a poset that has a symmetric chain decomposition:



Lemma 3

If P has a symmetric chain decomposition, then it is rank-symmetric, unimodal, and Sperner.

Proof. Each chain contributes 1 to some set of rank numbers which is symmetric about the middle, so the rank numbers will be symmetric. (The sum of palindromic vectors is palindromic). Unimodality is also obvious: the sum of vectors that are unimodal and symmetric about the same mean is unimodal.

To show that it is Sperner, note that the middle rank is exactly the number of symmetric chains we have in our symmetric chain decomposition: this is because we can write $P = C_1 \cup C_2 \cup \cdots \cup C_m$, and each chain intersects the middle level exactly once (as they are saturated and symmetric about the middle). (If there are two middle levels, it's true for both.) The middle rank is the maximal rank number, and any antichain cannot contain two elements in the same chain C_i. Thus the maximal size of any antichain is m, and we're done! □

Why is this an important lemma at all? Let $[n]$ denote the poset with n elements in a chain.

Fact 4

Then the Boolean lattice $B_n = [2] \times [2] \times \cdots \times [2]$ (n times).

For example, $[2] \times [2]$ is a square, $[2] \times [2] \times [2]$ is the 1-skeleton of a cube, and so on.

Theorem 5 (de Bruijn, 1948 + generalization)

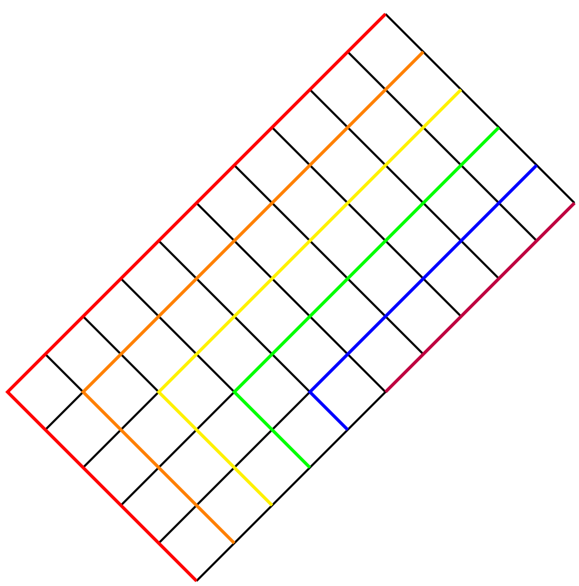
B_n has a symmetric chain decomposition! More generally, $[a] \times [b] \times \cdots \times [c]$ has a symmetric chain decomposition for any product of this form.

To show this, let's use some sublemmas:

Lemma 6

$[a] \times [b]$ has a symmetric chain decomposition.

Here's a proof by picture:



Lemma 7

If posets P and Q have symmetric chain decompositions, then $P \times Q$ has a symmetric chain decomposition.

Proof. Let $P = C_1 \cup C_2 \cup \cdots \cup C_k$, and $Q = C'_1 \cup \cdots \cup C'_\ell$, we can write P as a disjoint union of products of chains

$$P \times J = \bigcup_{(i,j)} C_i \times C'_j.$$

But each saturated chain $C_i \times C'_j$ is of the form $[a] \times [b]$, and pick a symmetric chain decomposition for each $C_i \times C'_j$ as we did in the lemma above! This gives a symmetric chain decomposition for the whole poset P. □

Let's go back to looking at finite posets in general. Given any poset P, remember that we define $M(P)$ to be the maximum number of elements in any antichain of P. Define $m(P)$ to be the minimum number of disjoint chains needed to cover all elements of P.

Theorem 8 (Dilworth, 1950)

For any finite poset P, $M(P) = m(P)$.

There's also a dual version of this theorem:

Theorem 9 (Minsky, 1971)

The statement is also true if you flip the words "chain" and "antichain" in Dilworth's theorem.

In fact, there's a generalization of this duality.

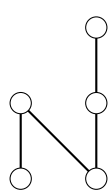
Definition 10

Given a poset P, define ℓ_k to be the maximum size of a union of k (not necessarily disjoint, not necessarily saturated) antichains in P. For example, ℓ_1 is $M(P)$, the maximum number of elements in an anti-chain. Similarly, define m_k to be the maximum size of a union of k-chains in P.

Theorem 11 (Greene, 1976)

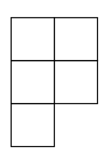
Define $\lambda(P) = (\ell_1, \ell_2 - \ell_1, \cdots)$, and define $\mu(P) = (m_1, m_2 - m_1, \cdots)$. Then λ and μ are both partitions of n that are weakly decreasing, and they are **conjugates**: their Young diagrams are transposes of each other.

For example, consider the following poset:



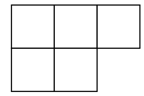
Here, we can cover $2, 4, 5$ elements with $1, 2, 3$ antichains, so

$$\lambda = (2, 4-2, 5-4) = (2, 2, 1) \implies$$



Meanwhile, we can cover $3, 5$ elements with $1, 2$ chains, so

$$\mu = (3, 5 - 3) = (3, 2) \implies$$



So Dilworth's theorem just says that the first row of λ is the same as the first column of μ, and Minsky says the same thing with row and column swapped!

But remember the Schensted correspondence: the shape λ of a Young diagram tells us something about increasing subsequences in permutations. Well, we can make a poset out of permutations such that increasing subsequences are chains, and decreasing subsequences are antichains!