18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

February 11, 2019

As MIT students, we probably know a lot about computer science.

Fact 1

The Bible of computer science is "The Art of Computer Programming" by Knuth, and a lot of the material of this class comes from it.

Definition 2

A **queue** is a data structure that is first in first out, while a **stack** is last in first out.

A queue contains one or several entries, and it's like a line: if you enter the line first, you will exit first. Meanwhile, a stack is like a pile of papers. Main idea: we can use these to sort permutations, and Catalan numbers will appear again!

Proposition 3

The number of queue-sortable permutations of $(1, 2, \cdots, n)$ is equal to C_n.

What does it mean to be **queue-sortable?**

Example 4

For $n = 4$, $(2, 4, 1, 3)$ is queue-sortable. Put 2, then 4 in the queue, then put 1 in our list directly, take 2 out of the queue, put 3 directly, and then take 4 out of the queue.

But an example of something not queue-sortable is $(3, 2, 1)$. We would have to put 3 in the queue and then 2, and that's bad because 3 comes out first.

Proposition 5

The number of stack-sortable permutations of $(1, 2, \cdots, n)$ is also equal to C_n.

For example, $(4, 1, 3, 2)$ is stack-sortable, since we put 4 in the stack, put 1 in the list directly, then put 3 and 2 in the stack and pop everything back out. But $(2, 3, 1)$ is not stack-sortable.

The idea of sortability is related to the concept in combinatorics of **pattern avoidance**.

Definition 6

Given a permutation $w = (w_1, w_2, \cdots, w_n)$ of size n, where $w \in S_n$, the symmetric group, and a permutation $\pi = \pi_1, \pi_2, \cdots, \pi_k$ of size $k \le n$, we say that w **contains** pattern π if there exists a not-necessarily-consecutive set (**subsequence**) of entries $w_{i_1}, w_{i_2}, \cdots, w_{i_k}$, whose entries are in the same relative order as π. Meanwhile, w is **π-avoiding** if it does not contain pattern π.

For example, let $w = (3, 5, 2, 4, 1, 6)$. If $\pi = (2, 1, 3)$, then w does contain π.

Proposition 7

Queue-sortable permutations are exactly those permutations that are 321-avoiding. Meanwhile, stack-sortable permutations are those that are 231-avoiding. Finally, for any pattern π of size 3, the number of π-avoiding permutations is C_n.

This is left as an exercise! By the way, we will not be required to solve all problems in a problem set, so some will be easier and some will be harder.

Time to move to the next topic! We're going to talk about **partitions, Young diagrams, and Young tableaux**.

Fact 8

Since the word "tableau" is French, we add an "x" to the end to make it plural.

Definition 9

A **partition** of n is a list of integers

$$\lambda = (\lambda_1, \cdots, \lambda_e)$$

such that $n = \lambda_1 + \cdots + \lambda_e$, the λ_is are weakly decreasing, and all λ_i are positive integers.

We're talking about partitions, not compositions, so order doesn't matter. It's just by convention that people write them this way!

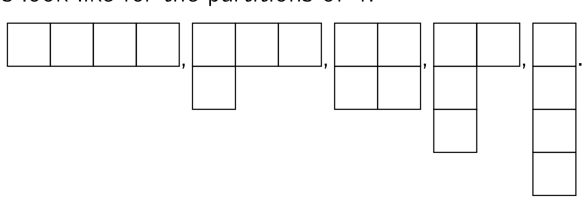
Example 10

Since $4 = 1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4$, there are 5 partitions of 4.

Definition 11

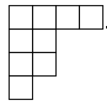
A **Young diagram** is a shape where there are rows of $\lambda_1, \lambda_2, \cdots$ boxes that are left-justified. A similar term, Ferrers shapes, refers to similar diagrams with dots instead of boxes.

Here's what the Young diagrams look like for the partitions of 4:



Definition 12

A **Standard Young Tableau** (SYT) is a way to fill in the boxes of a Young digram with numbers $1, 2, \cdots, n$ (without repetition) such that the numbers are increasing across rows and down the columns.

For example, if $\lambda = (4, 2, 2, 1)$, our tableau looks like . By abuse of notation, we'll use λ for the Young diagrams as well, and here's an example of a Young diagram:

1	3	4	7
2	6		
5	9		
8			

Definition 13

Let f^λ be the number of standard Young tableaux of shape λ.

Lemma 14

For $\lambda = (n, n)$, the number of standard Young tableau is

$$f^{(n,n)} = C_n.$$

Proof. Given a Standard Young Tableaux, construct a sequence $\varepsilon_1, \cdots, \varepsilon_{2n}$ such that

$$\varepsilon_i = \begin{cases} + & \text{if } i \text{ is in the first row} \\ - & \text{if it is in the second row} \end{cases}.$$

These are exactly the sequences that correspond to Dyck paths! For example, the following Young tableau corresponds to the Dyck path $(+, +, -, +, -, -, +, +, -, -)$:

1	2	4	7	8
3	5	6	9	10

Any Young tableau will correspond to a path, since we always have at least as many integers in the top row as the bottom, and any path will correspond to a Young tableau: just write it out!

$\square$

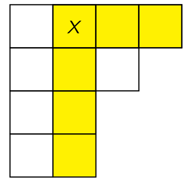
So we have a nice formula for $f^{(n,n)}$: we can think of these Young tableaux as an extension of the Catalan numbers. Is there a nice formula for f in general?

Theorem 15 (Hook Length Formula: Frame, Robinson, Thrall)

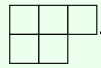
The number of standard Young tableaux for a partition λ is

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

where $h(x)$ are the **hook lengths**; that is, the number of squares in a hook that goes to the right and down from square x. The **arm** is the length of the horizontal component not including x, and the **leg** is the length of the vertical component not including x.

For example, this hook has length 6, arm length 2, and leg length 3: 

Example 16

Take the partition $\lambda = (3, 2)$, so the Young diagram looks like .

The possible Young tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\,.$$

Meanwhile, here are the hook lengths for each square in the tableau:

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$$

By HLF, the number of possible standard Young tableaux is indeed

$$f^{\lambda} = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5$$

as expected. There are many proofs of the Hook Length Formula, and it initially comes from representation theory. However, now there is a probabilistic proof using random walks, which we will cover later!