Practice Quiz 2

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: SOLUTIONS

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

Problem 1. [5+5+5 points]

Let (X, d) be a metric space.

(a) State the definition of a connected subset of X via separated sets, as in Rudin.

$E \subset X$ is connected if

$$\left.\begin{array}{l} E = A \cup B \\ \bar{A} \cap B = \emptyset \\ A \cap \bar{B} = \emptyset \end{array}\right\} \Rightarrow A = \emptyset \text{ or } B = \emptyset$$

(b) Let (X, d) be connected. Show that a subset $A \subset X$ is both open and closed if and only if $A = \emptyset$ or $A = X$. (This was a homework problem, but the task is to reprove this fact.)

- by def$^{\underline{n}}$ of closed, $\emptyset$ (has no limit pts) / X (has all limit pts in X) are closed
- complements of closed sets are open $\Rightarrow$ $\emptyset, X$ open & closed
- $A \subset X$ open & closed $\Rightarrow$ $B := A^c \subset X$ closed & open

$$\left.\begin{array}{l} \bar{A} \cap B \underset{A \text{ closed}}{=} A \cap B = A \cap A^c = \emptyset \\ \\ A \cap \bar{B} \underset{\substack{A \text{ open} \\ \Rightarrow B \text{ closed}}}{=} A \cap B = A \cap A^c = \emptyset \\ \\ X = A \cup A^c = A \cup B \end{array}\right\} \underset{X \text{ connected}}{\Longrightarrow} A = \emptyset \text{ or } \underset{\substack{\| \\ A^c}}{B} = \emptyset \Leftrightarrow A = X$$

(c) Suppose that (X, d) is a metric space with the following property: A subset $A \subset X$ is both open and closed if and only if $A = \emptyset$ or $A = X$. Then show that (X, d) is connected.

- $X = A \cup B$, $A \cap B = \emptyset \Rightarrow B = A^c$
- $\bar{A} \cap B = \emptyset \Rightarrow \bar{A} \subset B^c = (A^c)^c = A \Rightarrow \bar{A} = A \Rightarrow A$ closed
- $A \cap \bar{B} = \emptyset \Rightarrow \bar{B} \subset A^c = B \Rightarrow \bar{B} = B \Rightarrow B$ closed $\Rightarrow A$ open

$$\xrightarrow[\text{by assumption}]{} A = \emptyset \text{ or } A = X$$

$$A = X \Leftrightarrow B = A^c = \emptyset$$

Problem 2. [10+10 points]

(a) Find $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ for each of the following sequences.

Are these sequences bounded and/or convergent?

$$a_n = \sin\left(\frac{n\pi}{4}\right), \qquad b_n = \frac{(-1)^n}{n^{3/2}} = \left(-1, \frac{1}{2^{3/2}}, \frac{-1}{3^{3/2}}, \frac{1}{4^{3/2}}, \dots\right)$$

$$a_n = \left(\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, \dots\right)$$

$\{a_n \mid n \in \mathbb{N}\} = \{0, \frac{\sqrt{2}}{2}, 1, -\frac{\sqrt{2}}{2}, -1\} = \{\text{subsequential limits of } (a_n)\}$

all assumed infinitely often

$\limsup a_n = \max\{\dots\} = 1$

$\liminf a_n = \min\{\dots\} = -1$

so (a_n) is bounded but not convergent.

$$|b_n| = \frac{1}{n^{3/2}} \xrightarrow[n\to\infty]{} 0 \Rightarrow b_n \to 0$$

$$\Rightarrow \liminf_{n\to\infty} b_n = \limsup_{n\to\infty} b_n = \lim_{n\to\infty} b_n = 0$$

So (b_n) is bounded and convergent.

(b) Let (a_n), (b_n) and (c_n) be sequences in $\mathbb{R}$ such that for all $n \geq N$ we have $a_n \leq b_n \leq c_n$. Assume also that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ for some real number L.

Prove that $\lim\limits_{n\to\infty} b_n = L$.

Given $r>0$, we have $N_a(r) \in \mathbb{N} : \forall n \geq N_a(r) \quad |a_n - L| < r$,

$N_c(r) \in \mathbb{N} : \forall n \geq N_c(r) \quad |c_n - L| < r$.

So let $N_b(r) := \max\{N_a(r), N_c(r)\}$, then $\forall n \geq N_b(r)$

$$-r < a_n - L \leq b_n - L \leq c_n - L < r$$

$$\Rightarrow \quad |b_n - L| < r$$

Q.E.D.

Problem 3. [10 points] Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series and that $a_n \geq 0$ for all $n \geq N$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{|a_n|}$ converges. (Hint: You can use the general inequality $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$.)

- $0 \leq \frac{1}{n}\sqrt{|a_n|} \leq \frac{1}{2}\left(\frac{1}{n^2} + |a_n|\right) = \frac{1}{2}\left(\frac{1}{n^2} + a_n\right)$

- $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n^2} + a_n\right) = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} a_n\right)$ converges

OK because right hand side converges

$\Rightarrow$ by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{|a_n|}$ converges

Problem 4. [20 points: +4 for each correct, −4 for each incorrect; no proofs required.] (Hint: Note the penalty – it may be wise to leave some questions unanswered.)

a) Let (X, d) be a metric space, and let $E \subset X$. Then the closure of E is equal to the set $L(E)$ of all limits of sequences in E:

$$L(E) = \{x \in X \mid \exists (x_n)_{n\in\mathbb{N}} \subset E : \lim_{n\to\infty} x_n = x\}.$$

TRUE   FALSE

(• $x \in E \rightsquigarrow x_n = x$
• $x \notin E$ limit pt $\Rightarrow \exists\, x_n \to x$ shown in Rudin
• $x_n \to x,\ x_n \in E \Rightarrow x \in \bar{E}$ shown in Rudin)

b) If $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \geq 0$ then $a_n \to 0$.

TRUE   FALSE

(follows from partial sum convergence criterion
$\sum_{k=n}^{n+1} a_n = a_n \underset{n\to\infty}{\longrightarrow} 0$)

c) The subset $\{z \in \mathbb{Q} \mid |z| < 1\}$ of $\mathbb{Q}$ is connected.

TRUE   FALSE

($\mathbb{Q} \cap (-1,1) = A \cup B$ $\quad A = \mathbb{Q} \cap (-1, \frac{\sqrt{2}}{2})$ $\quad \bar{A} = \mathbb{Q} \cap [-1, \frac{\sqrt{2}}{2})$
$B = \mathbb{Q} \cap (\frac{\sqrt{2}}{2}, 1)$ $\quad \bar{B} = \mathbb{Q} \cap (\frac{\sqrt{2}}{2}, 1]$ $\quad \bar{A} \cap \bar{B} = \emptyset$)

d) Let (x_n) be a sequence in the metric space (X, d) such that $d(x_n, x_{n+1}) \leq \frac{1}{n}$. Then (x_n) is a Cauchy sequence.

TRUE   FALSE

(e.g. $x_n = \sum_{k=1}^{n-1} \frac{1}{k}$ diverges $\Rightarrow$ not Cauchy
$\mathbb{R}$ complete
but $|x_{n+1} - x_n| = \frac{1}{n}$)

e) Suppose $\sum_{n=1}^{\infty} c_n z^n$ is a power series with convergence radius $R = 2$ and such that it converges for $z = 2$. Then it converges for all other $z \in \mathbb{C}$ with $|z| = 2$.

TRUE   FALSE

(this would be true if $c_n \geq 0$, but otherwise e.g. $c_n = \frac{(-1)^n}{n \cdot 2^n}$
has $R^{-1} = \limsup \sqrt[n]{|c_n|} = \limsup \frac{1}{2\sqrt[n]{n}} = \frac{1}{2}$, $\sum_{n=1}^{\infty} c_n \cdot 2^n = \sum \frac{(-1)^n}{n}$ converges
but $\sum c_n (-2)^n = \sum \frac{1}{n}$ diverges)