18.100B : Fall 2010 : Section R2
Homework 8

Due Tuesday, November 2, 1pm

Reading: Tue Oct.26 : continuity and compactness, connectedness, Rudin 4.13-24
Thu Oct.28 : discontinuities, monotone functions, Rudin 4.25-34

1 . (a) Problem 4, page 98 in Rudin

(b) Problem 14, page 100 in Rudin (Hint: Rephrase the problem as $g(x) = 0$ and use the fact that $[0, 1]$ is connected.)

2 . Let $f\colon \mathbb{R} \to Y$ be a map to a metric space Y. Show that, for $a \in \mathbb{R}$ and $y \in Y$, the statement $f(a+) = y$ is equivalent to the following:

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in (a, a + \delta) \;:\; d(f(x), y) < \epsilon.$$

Formulate the analogous statement in the case of the left limit $f(a-) = y$.

3 . (a) Let $f\colon X \to Y$ be a uniformly continuous function between metric spaces. Show that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X, then $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in Y. Show, therefore, that the function $f(x) = 1/x^2$ defined on $(0, \infty)$ is not uniformly continuous.

(b) Problem 6, page 99 in Rudin (You may assume that f is a real valued function on $E \subset \mathbb{R}$. This result does hold in general also with the metric $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$ on the product $X \times Y$ of two metric spaces (X, d_X) and (Y, d_Y). Hint: One direction is a little subtle. Try e.g. a proof by contradiction to the ϵ, δ-definition of continuity, and use sequential compactness.)

4 . Let P denote the space of power series with radius of convergence $R > 1$:

$$P = \left\{ \sum_{n=0}^{\infty} a_n z^n \,;\, a_n \in \mathbb{C}, \limsup_{n\to\infty} |a_n|^{1/n} < 1 \right\}.$$

(a) Define $d\colon P \times P \to \mathbb{R}$ as follows: If $p(z) = \sum_n a_n z^n$ and $q(z) = \sum_n b_n z^n$, then

$$d(p, q) = \sum_{n=0}^{\infty} |a_n - b_n|.$$

Show that d is a metric on P. [*Hint*: You can use your knowledge of ℓ^1.]

(b) Fix $z_0 \in \mathbb{C}$ with $|z_0| \le 1$. Show that the *evaluation map* $\mathrm{ev}_{z_0}(p) = p(z_0)$ for $p \in P$ is a *uniformly continuous* function $P \to \mathbb{C}$ (in terms of the metric d from part **(a)** on P, and the usual Euclidean metric on $\mathbb{C}$). [*Hint*: Try first with $z_0 = 1$.]

5. Let (X, d_X) and (Y, d_Y) be metric spaces. For any continuous map $f\colon X \to Y$ define a function $\delta_f\colon X \times (0,\infty) \to (0,\infty) \cup \{\infty\}$ as follows:

$$\delta_f(x,\epsilon) = \sup\{\delta > 0 \,|\, \forall t \in X \; d_X(x,t) < \delta \Rightarrow d_Y(f(x), f(t)) < \epsilon\}.$$

Note that this supremum may be ∞ if the continuity condition holds for all $\delta > 0$; e.g. for f constant. In the following, we use the definition of order $<$ and infimum in the extended reals.

(a) Show that the statement "f is continuous at x" is equivalent to "$\delta_f(x,\epsilon) > 0$ for each $\epsilon > 0$".

(b) Show that f is uniformly continuous on X iff $\inf_{x\in X} \delta_f(x,\epsilon) > 0$.

(c) Consider the function $f(x) = x^2$ defined on the metric space $X = [0,\infty)$. Show that for each $x \in X$

$$\sup_{t\in X, d_X(x,t)<\delta} |f(x) - f(t)| = 2x\delta + \delta^2.$$

(d) Use part **(c)** to show that $\delta_f(x,\epsilon) = \sqrt{x^2+\epsilon} - x$. Show that, for fixed $\epsilon > 0$, $\lim_{x\to\infty}[\sqrt{x^2+\epsilon} - x] = 0$. Conclude that f is *not* uniformly continuous on X. [*Hint*: you can use Calculus here, but you needn't. Set $\varphi(x) = \sqrt{x^2+\epsilon} - x$. Show that $\varphi(x) \geq 0$, and that $\varphi(x)^2 + 2x\,\varphi(x) = \epsilon$ for every x. Hence $0 \leq \varphi(x) \leq \epsilon/2x$.]

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