18.100B Fall 2010
Practice Quiz 2 Solutions

1.(a) $E \subset X$ is connected if

$$\left.\begin{array}{l} E = A \cup B \\ \bar{A} \cap B = \emptyset \\ A \cap \bar{B} = \emptyset \end{array}\right\} \Rightarrow A = \emptyset \text{ or } B = \emptyset$$

(b)

- By definition of closed, $\emptyset$ (has no limit points) and X (has all limit points in X) are closed.
- Complements of closed sets are open $\Rightarrow \emptyset, X$ are open and closed.
- $A \subset X$ open and closed $\Rightarrow B := A^C \subset X$ closed and open

$$\left.\begin{array}{l} \bar{A} \cap B \underset{A \text{ closed}}{=} A \cap B = A \cap A^C = \emptyset \\ \\ A \cap \bar{B} \underset{A \text{ open } \Rightarrow B \text{ closed}}{=} A \cap B = A \cap A^C = \emptyset \\ \\ X = A \cup A^C = A \cup B \end{array}\right\} \underset{X \text{connected}}{\Rightarrow} A = \emptyset \text{ or } B (= A^C) = \emptyset \Leftrightarrow A = X$$

(c)

- $X = A \cup B, A \cap B = \emptyset \Rightarrow B = A^C$
- $\bar{A} \cap B = \emptyset \Rightarrow \bar{A} \subset B^C = (A^C)^C = A \Rightarrow \bar{A} = A \Rightarrow A$ closed
- $A \cap \bar{B} = \emptyset \Rightarrow \bar{B} \subset A^C = B \Rightarrow \bar{B} = B \Rightarrow B$ closed $\Rightarrow A$ open

$\underset{\text{By assumption}}{\Longrightarrow} A = \emptyset$ or $A = X (\Leftrightarrow B = A^C = \emptyset)$

2.(a) $a_n = \sin(\frac{n\pi}{4}) = (\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}, -1, \frac{-\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, \dots)$

$b_n = \frac{(-1)^n}{n^{3/2}} = (-1, \frac{1}{2^{3/2}}, \frac{-1}{3^{3/2}}, \frac{1}{4^{3/2}}, \dots)$

$\{a_n | n \in \mathbb{N}\} = \{0, \frac{\sqrt{2}}{2}, 1, \frac{-\sqrt{2}}{2}, -1\} = \{$subsequential limits of $(a_n)\}$; subsequential limits all assumed infinitely often.

$$\left.\begin{array}{l} \limsup a_n = \max\{\dots\} = 1 \\ \liminf a_n = \min\{\dots\} = -1 \end{array}\right\} \text{so } (a_n) \text{ is bounded but not convergent.}$$

$|b_n| = \frac{1}{n^{3/2}} \underset{n \to \infty}{\longrightarrow} 0 \Rightarrow b_n \to 0$

$\Rightarrow \lim_{n \to \infty} \inf b_n = \lim_{n \to \infty} \sup b_n = \lim_{n \to \infty} b_n = 0$

So (b_n) is bounded and convergent.

(b) Given $r > 0$, we have $\begin{matrix} N_a(r) \in \mathbb{N} : \forall n \geq N_a(r) & |a_n - L| < r, \\ N_c(r) \in \mathbb{N} : \forall n \geq N_c(r) & |c_n - L| < r \end{matrix}$

So let $N_b(r) := \max\{N_a(r), N_c(r)\}$, then $\forall n \geq N_b(r)$

$-r < a_n - L \leq b_n - L \leq c_n - L < r$

$\Rightarrow |b_n - L| < r$

Q.E.D.

3.

$\bullet 0 \leq \frac{1}{n}\sqrt{|a_n|} \leq \frac{1}{2}\left(\frac{1}{n^2} + |a_n|\right) = \frac{1}{2}\left(\frac{1}{n^2} + a_n\right)$

$$\bullet \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n^2} + a_n\right) \underset{\text{OK because right hand side converges}}{=} \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty} a_n\right) \text{ converges}$$

$\Rightarrow$ by comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{|a_n|}$ converges.

4.(a) TRUE

$\bullet x \in E \rightsquigarrow x_n = x$

$\bullet x \notin E$ limit point $\Rightarrow \exists x_n \to x$ shown in Rudin

$\bullet x_n \to x, x_n \in E \Rightarrow x \in \bar{E}$ shown in Rudin

(b) TRUE

Follows from partial sum convergence criterion $\displaystyle\sum_{k=n}^{n+1} a_n = a_n \underset{n\to\infty}{\longrightarrow} 0$

(c) FALSE

$$\mathbb{Q} \cap (-1, 1) = A \cup B \quad \left.\begin{matrix} A = \mathbb{Q} \cap (-1, \frac{\sqrt{2}}{2}) & \bar{A} = \mathbb{Q} \cap [-1, \frac{\sqrt{2}}{2}) \\ B = \mathbb{Q} \cap (\frac{\sqrt{2}}{2}, 1) & \bar{B} = \mathbb{Q} \cap (\frac{\sqrt{2}}{2}, 1] \end{matrix}\right\} \bar{A} \cap \bar{B} = \emptyset$$

(d) FALSE

e.g. $x_n = \displaystyle\sum_{k=1}^{n-1} \frac{1}{k}$ diverges $\underset{\mathbb{R}\text{ complete}}{\Longrightarrow}$ not Cauchy

but $|x_{n+1} - x_n| = \frac{1}{n}$

(e) FALSE

This would be true if $c_n \geq 0$, but otherwise e.g. $c_n = \frac{(-1)^n}{n \cdot 2^n}$ has

$R^{-1} = \limsup \sqrt[n]{|c_n|} = \limsup \frac{1}{2\sqrt[n]{n}} = \frac{1}{2}, \displaystyle\sum_{n=1}^{\infty} c_n \cdot 2^n = \sum \frac{(-1)^n}{n}$ converges

but $\sum c_n(-2)^n = \sum \frac{1}{n}$ diverges.