PROFESSOR STRANG: Alright. So this is Lecture 15. It's the last topic, today and Friday, like just 15 and 16. Trusses within Chapter 2. The last topic we'll do for discrete systems. Then it's a lot of fun. I wanted to say a few words first about last night's exam. Several words first. Overall, I'm sure it's going to be fine. Ramis is grading the first two problems, he'll pass them to Peter for the next two. And I'll get them back. I'm pretty sure it'll be next week. I felt it was a fair exam, except I should have done a better job in helping you with the matrix A. Especially in problem one. I'm glad that hint was there, the matrix A_0, that goes with free-free, to sort of say what kind of matrix to be looking for. And I thought I'd just repeat, make the connections that I should have made earlier. So we all see the point about these. These A's and the A transpose A's. So if I take, for that A_0, free-free one. Everybody sees that this is-- And this connects of course with our graphs.

Our graph is just this simple graph with well actually is that how many, is it five nodes? I guess there are five. Because as it stands I have one, two, three, four, five columns, I've got five u's, u_0 down to u_4. And if I take A_0 transpose A_0, that would be the free-free matrix. What size would it be? And what matrix would it be? Just if we do that multiplication, this is a first difference matrix. When I do A_0 transpose A_0 I'll get one of our second difference matrices. So it'll be one of our special ones. Which special one would it be? B. It'll be the matrix B that has both ends free. And what size will it be? I guess it'll be five by five. That's right; that would be five by four times A_0, which is four by five. So it'll be the five by five matrix B. Can I call it B_5? OK. So that was there as a hint. That isn't the correct matrix for problem one because problem one was fixed-fixed. Let's get there in two steps. Suppose it's fixed-free. So suppose I make u_0 equal 0. So I ground the top node, I support the top node-- Oh no, shall I do u_0? Yeah. I'll do u_0. So that would knock out this one. If I say fix u_0, say, at zero or whatever.

Now I've got, the next A, I won't call it A_0 anymore. So now four by four, now if I do A transpose A, which of our special matrices am I going to get? T. It'll be T. It'll be the one that has the first, the (1, 1) entry will only be a one. So that'll be the fixed-free matrix. It'll be of what size? Four. Now I've only got four unknowns. u_1 to u_4. OK, that still is not what problem one is. Problem one was fixed-fixed. So as I did in the review, that would knock out both of these columns. So this now is the matrix that I was looking for in problem one, and I wish I had emphasized these steps in advance. I apologize. OK, so what fixed-fixed if, without the C part in it. Just focusing on the A, what fixed-fixed matrix would I get? Which one of our guys would it be? K of size three. And while we're at it, what would be the story, how would I get one of these circular ones, which is sort of on our special list. For Fourier it's the really special guy. So a circular one I'm going to connect u_4 back to u_0.

So I'm going to put these guys back in. And what else would change, if u_4 was connected back to u_0, now I'm aiming for this circulant. What matrix A is going to give me the circulant? So again, these guys are the A transpose A's. This over here was the A, and over here is the A transpose A. And now I want to fix A, and then I want to see that A transpose A. So suppose I give you that graph, then. Oops, I should have, well. Just connect the whole guy. So fifth node coming back to the first. So that's my circle of nodes. That's a simple graph. What's the A_circulant now? So this would be the A for the circulant case. So it's got that back in. That shouldn't be erased, that shouldn't be erased. And what else has it got? If I ask you for the incidence matrix, now I'm in Section 2.4, like I've given you a graph, or you can think of masses and springs in a circle. So I've got five masses, five springs. What's my matrix A missing? It needs another row. We just put in another edge, it needs another row. That edge went from the last node back to the first node. So we've got a fifth row.

So you see, now it really is circulant. I would call this one also a circulant matrix. The diagonals are constant. That's what I and MATLAB and everybody else would call a Toeplitz matrix, and the command toeplitz could create this. That diagonal is constant. That diagonal is constant. The other diagonals are constant. But more than that, what's additional here in the circulant, which is the thing that makes Fourier happy? The diagonal circles around. That diagonal has only got four entries in it, but it circles around sort of periodically to its fifth entry. So that's more than Toeplitz. It's circulant because it's coming around again. This we'll see in the discrete Fourier transform. It's really all good stuff. And now there is my a circulant. And what would be my A transpose circulant A_circulant? What would be A transpose A if I take that five by five matrix? C. Finally I've created, so I've already got B, I've got T, I've got K, all those three special guys. And now the A transpose A for this circulant, so that's a first difference matrix for a periodic problem. And A transpose A will be a second difference matrix for a periodic problem, C_5, I guess. It'll be five.

OK. I hope that brings together what I, if I was on the ball, I would have brought it together before the quiz. Can I just say a few words about the quiz and grades? They come out fine. Really they do. I've been doing this a long time. And just, enjoy October. I'm sorry to give you any exam at all, but it's a chance for you yeah, I'm working on this stuff. I'm learning it. Everybody didn't learn it first time. I don't learn it first time, every time I teach the course I learn something more. And if you're learning from this course then I'm totally happy. And I believe that's the case. So I am entirely happy. And I hope the quiz, some points of it I wish I'd prepared better. But I feel pretty good about it. I feel good about it, let me just say. So, and I'm happy to have any comments, email or in person. But allow me to go forward with trusses. However, I'm ready always for a comment. Yeah. OK. Anyway, enjoy trusses. Enjoy life. Yeah.

And this should have been in the book, this page. So if it wasn't too late I would paste it in. Because this connects A transpose A to special matrices, in the way I had in my mind, but I didn't put it on the board until just now. OK, so I'll cover that up and ready to go with trusses. OK. So trusses, we want to know what's up. We want to get the setup right. Once we get the setup we'll know we're looking for. OK, so a truss is a bunch of elastic bars with pin joints connecting them. Now, what do I mean by a pin joint? I mean that stretching the bars takes force. Turning around the pin joint doesn't take force. So the pin just lets them turn, so we'll have forces in those bars. So it's like masses and springs. Exactly like masses and springs. But yet we have a 2-D problem. So it's a two dimensional problem with masses and springs. And we could certainly have a 3-D truss, but 2-D makes all the important points. And then I can count the bars. One, two, three, four, five. And I can count the nodes. There happen to be five here.

But now comes the moment. I have to tell you, what are the unknowns? What are the u's. Because of course, you know that I'm going to go from u's to e's to w's, to forces, f. And you know that a matrix A is going to do that. A matrix C is going to do that. A matrix A transpose is going to do that. You're all ready, we need to know what's the setup. What are these matrices. OK, and how many-- So let me explain the setup. Typical node, node one. we have forces in these bars, so that node one could have a force. We're in the plane. So we have a horizontal force and a vertical force. Together, that would give, produces a force in any direction whatever. So this is the key point. That there is a horizontal force. f horizontal one. The one being the force on node one. But there's also a vertical force. And let me take horizontal to the right, positive to the right. Vertical positive upwards. Just to have a convention. So how many f's have I got? Well, the point is I now have two per node. That's the difference. I have two per node, two forces. And I have two displacements per node. Because that point under, there will be more forces. Some maybe pulling this way, whatever. Maybe let's look at node two.

So node two could have a couple of forces on it. f h two, and f v two. And it moves like the other nodes. So now I'm introducing the unknowns. u is the movement. u, again horizontal, and again, now we're talking about node two. And it moves up. Or doesn't, or moves down. But that's an unknown. u, a displacement, a vertical displacement of node two. Do you see the setup? Two forces per node. Two displacements per node. So that's like, the number of unknown is like doubled. Like, doubled, and that produces an interesting situation. I've marked supports here. So let's just speak about supports. So what's happening at the supports? At the support there's no movement. The whole-- That point is pinned. So this is telling me that u horizontal five is zero and u vertical, sorry that was four and it'll be the same for five. u vertical five is zero. It's like grounding a node in the electrical case. We just see this pattern over and over. And we want to see OK, what does it look like for trusses? So here's a support that fixes those. So those are not unknowns. And similarly, they're not unknowns there. Still saying five when I mean four.

So those are boundary conditions, n conditions, whatever. And similarly here. So how many unknowns are there? Now look at this picture, how many unknown displacements are there in this truss? Six. Six, right? Two here, two here, two here and none there. So the number of actual unknowns is six. My idea would be that it's twice the number of nodes minus the number of fixed things, reaction, whatever. That R would be four in this case. I've got two fixed here and two fixed here, so this would be two times five. Ten possible displacements but R counts the number of fixed displacements, four, and leaves us with six. OK. So my matrix A will now be, it's always m by n. My matrix A will be five by six. OK. Now you're going to ask what is that matrix. But let me hold that off for a little moment. I want to just see its shape first. So you could now do this for a large truss, right? You count the bars, and you could count the nodes. And then you could count the unknown displacements, u. So there are six u's here. And there are five e's. And there are five bar forces. And there are six equilibrium, balance, force balances. Six, six for the node count, for the unknowns count, five, five for the bars count.

OK, now here's a point about this particular truss. It's not safe to get on it. Right? And I want to say why is it not safe. So this is a feature that comes into the truss question that makes it a little new and more interesting. A little twist compared to the previous examples. That bar, that truss, I wouldn't stand on it. Now, why not? Well, purely for linear algebra reasons. Of course.

The matrix A is five by six. So now what do we know about a matrix that's five by six? So A is five by six. Five rows, as always the m; six columns, because now we have six unknowns. And what do I know about any five by six matrix? I want to ask about the equation Au=0. So I want to ask about it in linear algebra language and then I want to ask about it in physical language. And the beauty is the thing that makes trusses sort of fun is, these matrices, A, get pretty big fast. Because when I put a few more nodes on, the book has a picture of a sort of treehouse. Then A is growing. And I don't, all the time, write down the matrix A. I haven't written it down here. What I've written down is just its size, because that's enough to tell us something about this set of equations Au=0.

What's the story on Au=0? Well of course it has the solution u=0. Nothing moving. If I have no displacement, if the u's are all zero, then I have no stretching. The e's are stretching. Elongation, as before. How far does the bar stretch? OK. So if I have zero u's and zero e's. But, what other possibility am I going to have here? I'm going to have probably one solution to this system that isn't zero. I'm probably going to have one set of displacements u, look what's happening here. This is Au is the e. So I'm going to have at least one and probably in this case it will be one, there will be one, the neat word for it is a mechanism. And what does that mean? A mechanism is a solution to Au=0. So that, a mechanism is a movement of the bar. So it's going to be non-zero. The bars are going to move a little. Sorry, the nodes are going to move a little. The nodes will move a little bit. In this u, because it isn't zero. But the bars won't stretch. So that tells us we've got instability here. If there's a solution to that, that's always telling us that A transpose A is singular. So let me just put that A transpose A -- or A transpose C A, C couldn't save it -- will be singular. It's just like our free-free thing in being singular, but the picture doesn't look free-free, does it? It's got supports in here, just not good enough. And I believe that if you look at this truss, you could describe, you could tell me, and you could draw, a movement of that truss in which there is displacement but no stretching.

Let me ask you how to draw that. And I believe-- Everybody understood, why was there a solution? It was because we have six unknowns and we only have five equations. So this was five equations. Any time you have five equations with a zero on the right-hand side, so five homogeneous equations, whatever you want to say when that's zero on the right and six unknowns. Six u's. Then you're going to have solution. You can't help it. You've got that many degrees of freedom, you've only got that many constraints, there's going to be solution. OK, tell me how to draw that. Let me put in the truss now. What's the solution? So this is the fun part in a particular example at the start. How could that move without stretching bars? Let me see. What could happen? What do you mean now, who's going to move where? What's the movement here? And I want to draw it over there. So you give the answer by drawing it as well as by telling me the six unknown u's. So what can happen at this thing? So you're going to say the truss could, these bars could, turn a little? And notice that word a little. We're talking small displacement, small stretches all the time here. I'll show you why we're always making that linearity assumption, or small assumption. OK, those move a little. And what happens to that triangle at the top?

It sort of just moves along, right? So the picture you would draw would be that you started there. And it moved along a little, I'll make it a larger displacement than I really have in mind. These guys of course are here. So they come out and the rest of the truss, the top of the truss, just kind of goes with it. Goes with the flow. That would be the answer that I would be looking for, to draw the mechanism. That would show it. And if I wanted to write down the u that goes with it, what would it be? Let me again number these guys; this is one, two, and this is three. So what are the displacements of nodes one, two, and three? I'll always write u_1 horizontal before vertical. Can we make an agreement? So I want to know about the horizontal movement then the vertical movement of node one, then node two, then node three. So I'll have six numbers there. And what could I put in for those six numbers?

So the horizontal, let me suppose that that first guy, I'll put a one. Really that's a bigger number than I should put, but it's a convenient number. So I'll just take it to be one even though I really have in mind-- Let's say that's one angstrom, or one tiny little person. OK, so what about the rest? What's the vertical-- Oh yeah, this is a key point here, what's the vertical movement? This movement to me is horizontal. I'm going to say that the vertical moment is zero. Of node one, just moves over. And node two does the same. And node three does the same. So that's my solution. [1, 0, 1, 0, 1, 0]. That's a simple movement, a simple set of displacements, think most to the right. I have not written down the matrix A, but probably won't even do it until next time. But you will see that when we do the matrix A for this particular truss will have this particular u as a mechanism. In linear algebra, u is in the null space of A. Au=0, that's all that means. OK, do you see more or less what's up? But now there's one little thing that may be bothering you. Which is what? If I come back to the zero, zero, zero there, you could correctly say wait a minute, if those bars didn't stretch, if they just rotated as you told me to do, then this was mostly across but a little bit down, right? And I'm saying no. I'm saying zero. OK, how do I get away with that? So I'm saying in 18.085 it's a zero. And why?

So this is like a little time-out just to focus in on, let me focus in on node two. So here's the bottom node four, so it used to be vertical up to two. This was node two and this was number four. And then it rotated a little, to there. To this position. So it went, if this angle was, let's say, theta, then what is that actual position? So let's say this was, let's say the bar had length one. This is the origin, (0, 0). This is the point (0, 1) above it, OK? And now, that's before it moved. Then it moved a little bit. It moved to an angle theta. What's the position of that bar? Of that node? What's the new position of the node, and then we'll look at the difference and we'll see the movement u, the displacement. So how far did it move? What's the x-coordinate? How far did it go across? If I put in that line you'll know. So the movement across was, sin(theta)? Good. sin(theta). And the movement down, well, yes, so let's find its position and then we'll take the difference. So what's the vertical new position for that guy? It moved by, it moved across by sin(theta). And down by 1-cos(theta). Are you agreed with that? Because here is cosine theta, right there. And here's the little bit it moved down. OK. So these are exactly correct. Yeah this is in the position of sin(theta), cos(theta), and the difference was the 1-cos(theta). OK, so now here comes the key point. Approximately, sin(theta) is approximately, if theta is small and now here comes the smallness, sin(theta) is approximately theta. sin(theta)'s approximately theta. And 1-cos(theta) is approximately what? Now, this is the important point. So theta is like the first term. If I expand, I mean, the exact term would be theta minus theta cubed over six, dot dot dot. But I'm only keeping that term. And 1-cos(theta), now what's the formula for cos(theta)? This is like worth, just should-- It's a one, because of course cos(0) is one, and then you subtract what? Theta squared over two. And so on. And then plus theta fourth over 24 or whatever. OK, so the ones cancel as we expect. And I'm getting theta squared over two. And this, here was theta to the first power. Theta cubed was, we didn't care. And we don't care about theta squared. So that's why it's zero. Because it's a second order movement. If theta is small, as I'm going to assume, small displacement, theta squared would, if I allowed theta squared and cos(theta) in here, I'd have a non-linear problem. And I don't want that. And I don't need it. I mean, finite elements, structures, bridges, whatever. Your first hope and expectation and calculation is small theta, linear problem. So to a linear person theta squared is zero. That's why those guys are zero.

OK, so that's an assumption we'll often see, so it kind of was. There are two kinds of non-linearities in structures and elasticity. One would be to allow this geometric non-linearity. Thetas, large displacements, theta large enough so that you can't neglect theta squared. That's a tough one. If you allow geometric non-linearity in, as finite element codes have to do. If you-- ABAQUS is a code that does major finite element calculations, nonlinear ones. I mean they, at the beginning they were studying what happens, what are the stresses on cables under the Atlantic. I mean, those are fascinating problems. Or I mention car crashes. I mean, car crashes, the geometry changes, you have big displacements. But we're talking here about linear small displacement cases. OK, so don't forget that part. That when the truss gets more complicated, the principle stays the same. That we distinguish between the thetas that matter and the theta squareds that don't.

OK, so now what? Now I guess I'm ready to complete this picture a little more. OK. So let me, so we've understood what's the idea of a mechanism. Oh, how could I prevent a mechanism? In other words, if I stood on this truss, the slightest bit of wind would crash it down, right? So that's unstable. That's an unstable truss. How could I make it stable? I mean if you were designing this thing, what would you do? Add another edge. You'd stick in another bar. Maybe stick in a bar there. What would happen now? Would it now be stable? You'd have to answer that question. You couldn't just put in bars, whatever. You want to put bars that do the job. OK, now how many bars? We've now got six bars. So m is now up to six. The matrix A is six by six now, whatever that matrix may be. We have six bars, six displacements. We can hope that we now have a six by six, well, we do have a six by six matrix, whatever it looks like. And we can hope that it's not singular. We can hope it's invertible, we can hope that that mechanism is killed. And you see it is killed. The six by six, that truss is now stable. No mechanism there, right? Again I haven't written down the matrix, but I'm really calling for engineering intuition here. That this truss is now stable, and of course I can make it even more stable by adding a seventh edge. A seventh bar. So when it was six I had square matrices, A transpose and then C and then A would have been square.

Now I've got seven bars and, so now I've put in a seventh guy. m is now up to seven. My matrix a would now be seven by six. Mechanism will be gone because I've now got, what, seven equations. Same six u's. So we begin to get a feel of, are there solutions or not? What I'm saying is, I can't tell just from the count that A is not singular. I could have a lot of bars and still be unstable. Invent a truss for me. Just because, how could you invent a truss that had, maybe it has seven bars. With those seven bars, those diagonal guys, that did it. That made it stable. Our eye tells us that before we do any linear algebra. Tell me a seven by-- A thing. Well, yeah. OK, so here would be, shall we support both of these? I'll start out the same, OK. Now, yeah, how could I, let's see, I've haven't prepared. How could I get a whole lot of bars. I might not get seven by six exactly, but how could I have plenty of bars and still unstable? Well, suppose I do this. Oh yeah, that's a good example. That's not stable, right? OK. Every let's practice with that one. That's just my idea, and problems in the book just ask you to practice with things like that. Tell me the count, first. What is m, the number of bars? Six. What is n, the number of unknowns, little n, the number of unknowns? What's the shape of my matrix here? A is, it's got six bars and how many unknowns? Eight. Eight, right? Two here, two, two, two. None here. Six by eight.

OK. And how many mechanisms am I now expecting? Probably two. Probably two, there would be two independent mechanisms here. Can you tell me what they look like? Draw them. What would they look like? What would be two different things that could happen, could go wrong with that truss? You see it, right? This could turn. As in our example with the top part moving with it. Or, a second one possibility would be the top part goes. And the bottom part stays. Or any combination. So the whole thing could go like that. That would be one. But that wouldn't be the only one, of course. So in other words, we have a two-dimensional space of mechanisms and you could give me two different, and there are not just two guys, all their combinations are there. So this would have two mechanisms. Two mechanisms. And I could put in bars, of course, that would try to save it. Well, how many bars, what's the minimum number of bars I absolutely need to make this thing stable again? Two. Well, now suppose I put in these two bars. Right? I've got enough bars, I've got an eight by eight matrix, but I haven't saved it. Right? Because it still has that mechanism. So you can't assume that because the count is right you've avoided mechanisms because in that example you haven't.

OK, so that would be a case of square eight by eight, but not good. So as soon as I say there's a solution to Au=0, I know that A transpose A will be singular. And unstable. OK, before I go to the framework let's just do one more thing. Suppose I take away the supports. All right, let me put in some bars, though. I'll put in some bars. OK, plenty of bars. Want another one? OK, how many bars have I got? Lots, right? OK, now the matrix A, what do you think about this? Are there solutions? You haven't even seen the matrix A, of course, but you've seen the truss, that's what matters. How many solutions, are there solutions to Au=0? Are there ways that this truss could move without stretching? Are there ways that this truss could move without stretching? And what are they, and how many are there? And what name should we use? OK, what are they? How could that move without stretching? Well, it's got no supports at all. It's just free out there in space. So it could move. How many ways could it move? Three. It could move, everybody could move this way. All ones on the horizontal guys. Everybody could move this way, all six ones on the vertical guys, it'll be 12 unknowns here. And it could also rotate, what would be the rotation? I'm not talking about this rotation. This could not happen. What could happen? What rotation could happen here? For this, there's a third rigid motion. Translation, translation, and rotation around, well take this one as an example. The whole thing could swing around this. That would be a motion.

Well now you're going to say well, why didn't I swing it around that one? And of course it could. But what would be the deal? It would have to be, there are only three rigid motions, right, up and around. So if you give me another one, like, around this one, then somehow it had to be a combination of those. I don't even want to think what combination it is. But there are three rigid motions. So I sort of distinguish mechanisms, this word mechanism. So that's where the truss deforms. In these-- And rigid motions, so I'll say plus, possibly. Plus rigid motions, and rigid motions would be, you know, it doesn't deform internally, the whole thing moves. And this is of course what we get in the case of not enough supports. And this is what we get in the case of not enough bars. Yeah. So maybe it's worth separating those two. In the examples we do, we'll usually put in enough supports to kill the rigid motions. And then the question would be are there some mechanisms.

OK. Now, I have to start on what this is. Well it'll be just a very quick start. So what I'll do at the beginning of Friday, so Friday's the other lecture on this topic. And then the homework will ask you to do some trusses in this section. It's probably Section 2 point something. 2.7, maybe. What's the matrix C? Last second question, what's the matrix C, what size is it? What size is the matrix C for our original problem? Or no. What size is C, is C involving, if I know these numbers, what size is C? Five by five. m by m, right? C is the diagonal matrix, one entry for each bar, C is just C. It has a c_1, c_2, c_3, and this $w=Ce$, it's just Hooke's Law on each bar. So, simple. It gets there in the middle, just the way that C in the first exam problem popped in, and other C's. That gets there in the middle. But it's very, extremely, simple. OK, so the real attention is on A, as usual. And that will come Friday morning.