Part I Problems and Solutions

Problem 1: *Find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \leq \pi$ by*

$$f(t) = \begin{cases} 0, & -\pi < t \leq 0 \\ 1, & 0 < t \leq \pi \end{cases}$$

as in the same problem in the previous session – but this time use the known Fourier series for $sq(t)$ = the standard square wave.

Solution:

$$sq(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

$f(t) = \frac{1}{2}(1 + sq(t))$. Known Fourier series for $sq(t)$ is

$$sq(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt$$

so

$$f(t) = \frac{1}{2}(1 + sq(t)) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt$$

This is the same as was shown in the same problem in the previous session.

Problem 2: *Find the Fourier series of the function $f(t)$ with period 2π given by $f(t) = |t|$ on $(-\pi, \pi)$ by integrating the Fourier series of the derivative $f'(t)$.*

Solution: Note that $f(t) = \begin{cases} -t & -\pi < t \leq 0 \\ t & 0 \leq t < \pi \end{cases}$ so that

$$f'(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 \leq t < \pi \end{cases} = sq(t)$$

the standard square wave. Also note that $f(t)$ is an even function, so that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$$

For $n = 0$, $a_0 = \frac{1}{4} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{4} \int_{-\pi}^{\pi} |t| dt = \frac{2}{\pi} \int_0^{\pi} t dt = \frac{2}{\pi} \frac{t^2}{2} \Big|_0^{\pi} = \pi$

For $n \geq 1$, we can integrate the Fourier series for $sq(t)$ term-by-term.
$n \geq 1$, odd: nth term of $sq(t)$ is $\frac{4}{\pi}\frac{1}{n}\sin nt \to \frac{4}{\pi n}\int \sin(nt)dt = -\frac{4}{\pi n}\frac{1}{n}\cos nt = -\frac{4}{\pi n^2}\cos nt \to$ $a_n = -\frac{4}{\pi n^2}$ for n odd.
We thus now have the Fourier series for $f(t)$:

$$f(t) = \pi - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos(nt)$$