18.03SC Practice Problems 32

First order linear systems

Solution suggestions

Vocabulary/Concepts: system of differential equations; linear, time-independent, homogeneous; matrix, matrix multiplication; solution, trajectory, phase portrait; companion matrix.

1. *Practice in matrix multiplication: Compute the following products.*

(a) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x + 2y]$

(b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} x + 2y & u + 2v \\ 3x + 4y & 3u + 4v \end{bmatrix}$

2. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be any* 2×2 *matrix.*

Multiplying by the matrix A *sends any vector* $\begin{bmatrix} x \\ y \end{bmatrix}$ *to another vector* $A \begin{bmatrix} x \\ y \end{bmatrix}$*. This operation can be visualized by thinking about where it sends the square with corners*

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{i} + \mathbf{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For each of the following matrices A*, draw segments connecting the dots* $\mathbf{0}$, $A\mathbf{i}$, $A(\mathbf{i} + \mathbf{j})$, $A\mathbf{j}$, $\mathbf{0}$, *and come up with a verbal description of the operation.*

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$: hold the x-direction unchanged, but stretch the y-direction by a factor of 2.

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: hold the bottom two vertices fixed on the x-axis, but move the upper two vertices horizontally to the right by one unit.

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$: reflect about the x-axis.

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: reflect about the line $y = x$.

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$: holding the vertex at the origin fixed, first rotate the square 45° clockwise, then reflect it about the x-axis, and finally stretch the four sides by a factor of $\sqrt{2}$.

3. *Examine the equation*

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

(a) *What is the companion matrix A of this second order equation?*

By definition, the companion matrix of a higher order differential equation is the matrix of the linear system obtained by introducing enough new variables to denote the intermediate derivatives of x.

Since this is a second order equation, we will only need to introduce one new variable, and A will be a 2×2 matrix.

So let $y = \dot{x}$. Then $\dot{y} = \ddot{x}$, and our second order equation becomes $\dot{y} + 2y + 2x = 0$. With the introduction of this new variable, our single-variable second order equation translates into the following first order system in two variables:

$$\begin{cases} \dot{x} = & y \\ \dot{y} = & -2x - 2y. \end{cases}$$

We can now read off that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

(b) *Find two independent real solutions of this equation.*

The characteristic polynomial of the original equation is $p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$. This polynomial has complex conjugate roots $s = -1 \pm i$, so two independent complex solutions are $e^{(-1+i)t}$ and $e^{(-1-i)t}$, and two independent real solutions are

$$\{e^{-t} \cos t,\ e^{-t} \sin t\}.$$

(c)* *Now let $x_1(t)$ denote the solution with initial condition $x_1(0) = 0$, $\dot{x}_1(0) = 1$. Find it, and then write down the corresponding solution $\mathbf{u_1}(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix}$ of the equation $\dot{\mathbf{u}} = A\mathbf{u}$. What is $\mathbf{u_1}(0)$?*

Sketch the graphs of $x_1(t)$ and of $\dot{x}_1(t)$, and sketch the trajectory of the solution $\mathbf{u_1}(t)$. Compare these pictures.

The solution that satisfies the given initial conditions is $x_1(t) = e^{-t}\sin t$, so the corresponding solution of the companion system is

$$\mathbf{u_1}(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}(\cos t - \sin t) \end{bmatrix}.$$

At time $t = 0$ this solution has value $\mathbf{u_1}(t) = (0, 1)$.

The function x_1 has envelope $\pm e^{-t}$, which decays exponentially. The graph of x_1 oscillates inside this envelope, touching the envelope at odd multiples of $\pi/2$. Its derivative $\dot{x}_1$ has envelope $\pm\sqrt{2}e^t$, and the graph of $\dot{x}_1$ touches the envelope when t has the form $\frac{4k-1}{4}\pi$.

As a result, the trajectory given by $\mathbf{u_1}(t)$ in the $(x, y = \dot{x})$-plane is an inward spiral, elongated in the northwest-southeast direction.

(d) Sketch a few more trajectories to fill out the phase portrait. In particular, sketch the trajectory of $\mathbf{u_2}(t)$ with $\mathbf{u_2}(0) = \mathbf{i}$.*

When trajectories of this companion equation cross the x axis, at what angle do they cross it?

When trajectories cross the x axis, they cross at an angle of $\pi/2$.

4. *Let $a + bi$ be a complex number. There is a matrix A such that if $(a + bi)(x + yi) = (v + wi)$ then*

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}.$$

***(a)** Find this matrix A for general $a + bi$.*

Multiplying out, we get that $(a + bi)(x + yi) = ax + ayi + bxi - by = (ax - by) + i(ay + bx)$, so we must have $v = ax - by$, $w = bx + ay$.

This defines the rows of our transformation matrix, so

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

***(b)** What is the matrix for $a + bi = 2$? For $a + bi = i$? For $a + bi = 1 + i$? Draw the parallelograms discussed in **(2)** for these matrices.*

For $a + bi = 2$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and the parallelogram is a square of length 2.

For $a + bi = i$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and the parallelogram is a square of length 1, rotated by 90 degrees counterclockwise around the origin.

For $a + bi = 1 + i$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and the parallelogram is a square of length $\sqrt{2}$, rotated by 45 degrees counterclockwise around the origin.