18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #4

Due by Friday, March 13, 2009, by NOON.

1. (a) Show that if u_1 is a solution of the third-order variable-coefficient linear differential equation

$$u''' + p_1(x)u'' + p_2(x)u' + p_3(x)u = 0,$$

then the substitution $u(x) = u_1(x)v(x)$ leads to the second-order differential equation for v':

$$u_1 v''' + (3u_1' + p_1 u_1)v'' + (3u_1'' + 2p_1 u_1' + p_2 u_1)v' = 0.$$

(b) Verify that $u_1(x) = e^x$ is a solution of the differential equation $(2-x)u''' + (2x-3)u'' - xu' + u = 0$. Use the method in part (a) to find the general solution of the differential equation.

2. Find a particular solution of the differential equation

$$x^2 u'' + (1 - \alpha - \beta)xu' + \alpha\beta u = x^2 f(x)$$

(a) when $\alpha \neq \beta$ and (b) when $\alpha = \beta$.

3. Birkhoff-Rota, pp. 62, #2.

4. (a) Find annihilators of $x^m e^{\alpha x}$, $x^m \sin \beta x$, and $e^{\alpha x} \cos \beta x$.

(b) Find the general solution of $(D^2 + 1)(D - 3)^2 u = 12e^{3x}(10x + 1)$.

5. Consider the nth order linear homogeneous differential equation

$$u^{(n)} + a_1 u^{(n-1)} + \cdots + a_{n-1}u' + a_n u = 0$$

with real constant coefficients.

(a) Show that if the differential equation is asymptotically stable then $a_1, a_2, \cdots, a_n > 0$.

(b) Show that the converse to part (a) is true if all roots of its associated characteristic polynomial are real. (Hint. Assume it is false, and seek for a contradiction.)

(c) Show by a counter example that the converse to part (a) is in general false.

6. (a) Show that the function y^2 is not Lipschitzian on $-\infty < y < \infty$. Discuss how this failure of the Lipschitz condition is reflected in the behavior of the initial value problem

$$y' = y^2, \qquad y(0) = y_0 > 0.$$

(b) Show that the function $y^{2/3}$ is not Lipschitzian in any strip $|y| < h$ containing the origin. Discuss how this failure of the Lipschitz condition is reflected in the behavior of the initial value problem

$$y' = y^{2/3}, \qquad y(0) = 0.$$