18.03 Recitation 23, May 4, 2010

Linear phase portraits

The matrices I want you to study all have the form $A = \begin{bmatrix} a & 2 \\ -2 & -1 \end{bmatrix}$.

1. Compute the trace, determinant, characteristic polynomial, and eigenvalues, in terms of a.

2. For these matrices, express the determinant as a function of the trace. Sketch the $(\mathrm{tr}\, A, \det A)$ plane, along with the critical parabola $\det A = (\mathrm{tr}\, A)^2/4$, and plot the curve representing the relationship you found for this family of matrices. On this curve, plot the points corresponding to the following values of a: $a = -6, -5, -2, 1, 2, 3, 4, 5$.

3. Make a table showing for each a in this list (1) the eigenvalues; (2) information about the phase portrait derived from the eigenvalues (saddle, node, spiral) and the stability type (stable if all real parts are negative; unstable if at least one real part is positive; undesignated if neither); (3) further information beyond what the eigenvalues alone tell you: if a spiral, the direction (clockwise or counterclockwise) of motion; if the eigenvalues are repeated, whether the matrix is defective or complete.

In each case, make a small sketch of the phase portrait which conveys this information, but does not try to get the eigendirections right.

4. Make sure you know how to find the general solution to $\dot{\mathbf{u}} = A\mathbf{u}$ for each of these cases. Special attention is required in the defective node case.