20. Laplace transform technique: coverup

I want to show you some practical tricks which will help you to find the inverse Laplace transform of a rational function. These are refinements on the method of partial fractions which you studied when you learned how to integrate rational functions. Some of this will use complex numbers.

20.1. **Simple case.** First, let's do an easy case:

$$F(s) = \frac{1}{s^2 - 4}.$$

To begin, factor the denominator, and write

$$F(s) = \frac{1}{(s-2)(s+2)} = \frac{a}{s-2} + \frac{b}{s+2} \tag{1}$$

for as yet unknown constants a and b. One way to proceed is to cross multiply and collect terms in the numerator. That is fine but the following is more fun.

To find a, *first multiply through by the corresponding denominator*, $(s-2)$ in this case. You get

$$\frac{1}{s+2} = a + (s-2)(\text{other terms}),$$

in which the "other terms" (namely, $\dfrac{b}{s+2}$) don't have a factor of $(s-2)$ in the denominator. *Then set* $s = 2$:

$$\frac{1}{2+2} = a + (2-2)(\text{other terms}) = a$$

since the second term vanishes. So $a = 1/4$. In exactly the same way, you can multiply (1) through by $s+2$ and then set $s = -2$, to find

$$\frac{1}{-2-2} = b$$

or $b = -1/4$. Thus

$$F(s) = \frac{1/4}{s-2} - \frac{1/4}{s+2}.$$

The tables then show that

$$f(t) = (1/4)(e^{2t} - e^{-2t}).$$

This approach to partial fractions has been called the "cover-up method"; you cover up the denominators of the terms you wish to compute. You can do it without writing anything down; just cover

up the denominators and write down the answer. It works well but does not completely handle the case in which the denominator has a repeated root.

20.2. **Repeated roots.** For example look at

$$F(s) = \frac{s}{s^2 + 2s + 1}.$$

The denominator factors as $(s+1)^2$. We want to find a and b such that

$$F(s) = \frac{s}{(s+1)^2} = \frac{a}{s+1} + \frac{b}{(s+1)^2}.$$

We can begin to use the coverup method: *multiply through by* $(s+1)^2$ *and set* $s = -1$: The left hand side is just -1; the first term vanishes; and the second term is b: so $b = -1$. We can't get a this way, though. One way to find a is to *set* s *to some* other *value*. Any other value will do, and we might as well make our arithmetic as simple as possible. Let's take $s = 0$: then we have

$$0 = \frac{a}{1} + \frac{-1}{1}$$

so $a = 1$:

$$F(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Now the tables show

$$f(t) = e^{-t} - te^{-t}.$$

20.3. **Completing the square.** Suppose

$$F(s) = \frac{1}{s^2 + 2s + 2}.$$

The first part of the method here is to *complete the square in the denominator,* and *rewrite the numerator in the same terms*:

$$\frac{s}{s^2 + 2s + 2} = \frac{a(s+1) + b}{(s+1)^2 + 1}.$$

This works with $a = 1$ and $b = -1$:

$$F(s) = \frac{(s+1) - 1}{(s+1)^2 + 1}.$$

Now the s-shift rule applies, since $F(s)$ is written in terms of $s - a$ (where here $a = -1$). The second part of this method gives you a way to use the s-shift rule without getting too confused. You should *invent*

a new function name—say $G(s)$—and use it to denote the function such that that $F(s) = G(s-a)$. Thus, here,

$$G(s) = \frac{s-1}{s^2+1}.$$

Now the s-shift rule says that if $g(t) \rightsquigarrow G(s)$ then $e^{-t}g(t) \rightsquigarrow G(s+1) = F(s)$, which is to say that $f(t) = e^{-t}g(t)$. The tables give

$$g(t) = \cos t - \sin t$$

so

$$f(t) = e^{-t}(\cos t - \sin t).$$

20.4. **Complex coverup.** Now let's take an example in which the quadratic factor does not occur alone in the denominator: say

$$F(s) = \frac{1}{s^3+s^2-2}.$$

The denominator factors as $s^3+s^2-2 = (s-1)(s^2+2s+2)$. In the example above we learned that the factor s^2+2s+2 should be handled by completing the square and grouping the $(s+1)$ in the numerator:

$$F(s) = \frac{1}{(s-1)((s+1)^2+1)} = \frac{a}{s-1} + \frac{b(s+1)+c}{(s+1)^2+1}.$$

Find a just as before: multiply through by $s-1$ and then set $s=1$, to get $a = 1/5$. To find b and c, *multiply through by the quadratic factor* $(s+1)^2+1$ *and then set* s *equal to a root of that factor.* Having already completed the square, it's easy to find a root: $(s+1)^2 = -1$, so $s+1 = i$ for example, so $s = -1+i$. We get:

$$\frac{1}{(-1+i)-1} = b((-1+i)+1)+c$$

or, rationalizing the denominator,

$$\frac{-2-i}{5} = c+bi$$

Since we want b and c real, we must have $c = -2/5$ and $b = -1/5$:

$$F(s) = \frac{1}{5}\left(\frac{1}{s-1} - \frac{(s+1)+2}{(s+1)^2+1}\right).$$

We're in position to appeal to the s-shift rule, using the tricks described in 20.3, and find

$$f(t) = \frac{1}{5}\left(e^t - e^{-t}(\sin t + 2\cos t)\right).$$

20.5. **Complete partial fractions.** There is another way to deal with quadratic factors: just factor them over the complex numbers and use the coverup method in its original form as in Section 20.1. I don't recommend using this in practice, but it's interesting to see how it works out, and we will use these ideas in Section 22. Using the example

$$F(s) = \frac{1}{s^3 + s^2 - 2}$$

again, we can find *complex* constants a, b, c such that

$$F(s) = \frac{a}{s-1} + \frac{b}{s-(-1+i)} + \frac{c}{s-(-1-i)} \tag{2}$$

Expect that a will be real, and that b and c will be complex conjugates of each other.

Find a just as before; $a = 1/5$. To find b, do the same: multiply through by $s-(-1+i)$ to get

$$\frac{1}{(s-1)(s-(-1-i))} = b + (s-(-1+i))(\text{other terms})$$

and then set $s = -1+i$ to see

$$\frac{1}{(-2+i)(2i)} = b$$

or $b = 1/(-2-4i) = (-1+2i)/10$. The coefficient c can be computed similarly. Alternatively, you can use the fact that the two last terms in (2) must be complex conjugate to each other (in order for the whole expression to come out real) and so discover that $c = \bar{b} = (-1-2i)/10$:

$$F(s) = \frac{1/5}{s-1} + \frac{(-1+2i)/10}{s-(-1+i)} + \frac{(-1-2i)/10}{s-(-1-i)}.$$

The numerators a, b, and c, in this expression are called the **residues** of the poles of $F(s)$; see 22.1 below.

It's perfectly simple to find the inverse Laplace transforms of the terms here:

$$f(t) = \frac{1}{5}\,e^t + \frac{-1+2i}{10}\,e^{(-1+i)t} + \frac{-1-2i}{10}\,e^{(-1-i)t}.$$

The last two terms are complex conjugates of each other, so their sum is twice the real part of each, namely,

$$2\,\frac{e^{-t}}{10}\,\text{Re}\,((-1+2i)(\cos t + i\sin t)) = \frac{e^{-t}}{5}\,(-\cos t - 2\sin t).$$

We wind up with the same function $f(t)$ as before.

List of properties of the Laplace transform

1. $\mathcal{L}$ is linear: $af(t) + bg(t) \rightsquigarrow aF(s) + bG(s)$.

2. $F(s)$ essentially determines $f(t)$.

3. s-shift theorem: If $f(t) \rightsquigarrow F(s)$, then $e^{at}f(t) \rightsquigarrow F(s-a)$.

4. t-shift theorem: If $f(t) \rightsquigarrow F(s)$, then $f_a(t) \rightsquigarrow e^{-as}F(s)$, where

$$f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}.$$

5. s-derivative theorem: If $f(t) \rightsquigarrow F(s)$, then $tf(t) \rightsquigarrow -F'(s)$.

6. t-derivative theorem: If $f(t) \rightsquigarrow F(s)$, then $f'(t) \rightsquigarrow sF(s) - f(0+)$ where $f(t)$ is continuous for $t > 0$ and the notation $f'(t)$ indicates the ordinary derivative of $f(t)$.

7. If $f(t) \rightsquigarrow F(s)$ and $g(t) \rightsquigarrow G(s)$, then $f(t) * g(t) \rightsquigarrow F(s)G(s)$.

8. $\delta(t) \rightsquigarrow 1$.

18.03 Differential Equations

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