

18.03 Class 15, March 8, 2010

Operators, polynomial signals, resonance

- [1] Operators
- [2] Resonance
- [3] Polynomial signals

Several different topics today, and a respite from the gain game.

[1] Operators

Just as $\begin{array}{ccc} & \text{function} & \\ \text{number} & \text{-----} & \text{number} \end{array}$

$\begin{array}{ccc} & \text{operator} & \\ \text{function} & \text{-----} & \text{function} \end{array}$

The *differentiation operator D takes x to x' : $Dx = x'$.

For example, $D \sin(t) = \cos(t)$, $D x^n = n x^{n-1}$, $D 8 = 0$.

We can iterate: $D^2 x = x''$.

There's also the "identity operator": $Ix = x$

And we can take linear combinations of operators:

$$(D^2 + 2D + 2I) x = x'' + 2x' + 2x .$$

The characteristic polynomial here is $p(s) = s^2 + 2s + 2$, and it's irresistible to write

$$D^2 + 2D + 2I = p(D)$$

so $x'' + 2x' + 2x = p(D) x$

This formalism lets us discuss linear equations of higher order with no extra work. Such an equation has the form

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = q(t) \quad (*)$$

It has a characteristic polynomial

$$p(s) = a_n s^n + \dots + a_1 s + a_0$$

and so it can be written

$$p(D) x = q(t)$$

Now we can say that the *operator*

$$p(D) = a_n D^n + \dots + a_1 D + a_0 I$$

represents the system. This is a "linear, time-invariant differential

operator." In the systems and signals yoga, an input signal determines a function $q(t)$, and the system response x satisfies $Lx = q$. We could write

$$x = L^{-1} q$$

and indeed most of this course is about finding ways to "invert" these operators.

The Exponential Response Formula lets us find a solution of

$$p(D) x = B e^{rt}$$

very efficiently, as long as

- (1) r is not a root of $p(s)$ and
- (2) B is constant

Today we'll see how to deal with the first limitation. The second will come on Wednesday.

[2] For example suppose we want to solve $x'' - 4x = e^{-2t}$. $p(s) = s^2 - 4$ vanishes at $r = -2$. We can't apply ERF. To see what we CAN do, I want to recall where ERF came from: [Slide]

$$p(s) = a_n s^n + \dots + a_1 s + a_0$$

$$a_0] \quad e^{rt} = e^{rt}$$

$$a_1] \quad D e^{rt} = r e^{rt}$$

$$a_2] \quad D^2 e^{rt} = r^2 e^{rt}$$

... ..

$$a_n] \quad D^n e^{rt} = r^n e^{rt}$$

$$\frac{D^n e^{rt}}{p(D) e^{rt}} = \frac{r^n e^{rt}}{p(r) e^{rt}} \quad (*)$$

To get the ERF, you divide by $p(r)$ to see that $e^{rt}/p(r)$ is a solution to $p(D) x = e^{rt}$.

This formula (*) is true even if $p(r) = 0$. I will exploit that fact by thinking of t as a constant and r as a variable, and differentiate with respect to r . There are two independent variables now. D is differentiation with respect to t . In 18.02 you learn that at least when applied to nice functions (eg e^{rt}), mixed partials are equal:

$$\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial D}\right) = \left(\frac{\partial}{\partial D}\right)\left(\frac{\partial}{\partial r}\right)$$

So $\left(\frac{\partial}{\partial r}\right) D = D \left(\frac{\partial}{\partial r}\right)$ and

$$\begin{aligned} \left(\frac{\partial}{\partial r}\right) p(D) e^{rt} &= p(D) \left(\frac{\partial}{\partial r}\right) e^{rt} \\ &= p(D) (t e^{rt}) \end{aligned}$$

By (*), the LHS equals

$$\left(\frac{\partial}{\partial r}\right) (p(r) e^{rt}) = p'(r) e^{rt} + p(r) t e^{rt}$$

so

$$p(D) (t e^{rt}) = p'(r) e^{rt} + p(r) t e^{rt}$$

Now the hypothesis that $p(r) = 0$ becomes a virtue! - the second term goes away. As long as $p'(r)$ is not zero we can divide by it:

Resonant Response Formula: If $p(r) = 0$ and $p'(r)$ is not zero, then

$$x_p = B t e^{rt} / p'(r)$$

is a solution to $p(D) x = B e^{rt}$.

Example: If $p(s) = s^2 - 4$, $p'(s) = 2s$, so a solution to $x'' - 4x = e^{-2t}$ is given by

$$x_p = t e^{-2t} / 4$$

When r is a root of $p(s)$ we say that the system is "in resonance" with the input signal. Here's why:

Example: $x'' + 4x = \cos(2t)$

$$z'' + 4z = e^{2it}$$

$p(s) = s^2 + 4$, $p(2i) = 0$. $p'(s) = 2s$, $p'(2i) = 4i$.

$$z_p = t e^{2it} / 4i$$

$$x_p = \operatorname{Re} z_p = (1/4) t \sin(2t) .$$

So it's like a swing - you push in synch with the natural frequency and the amplitude increases, linearly. You're in resonance. Our solution in the first example doesn't increase in size - it decays to zero, but slower than the e^{-2t} decay that ERF would lead us to expect.

Speaking of swings, with the economy the way it is, the CIA is about the only group hiring academics these days. One day a biologist, a physicist, and a mathematician were staking out an empty house. Two people went in, and a little while later three people came out. The biologist started talking about reproduction, the physicist about tunneling, and the mathematician said, "If one more person goes into that house, it will be empty again."

[3] Polynomial signals are next.

Theorem (Undetermined coefficients) Suppose that

$$q(t) = b_k t^k + \dots + b_1 t + b_0 .$$

Then $p(D)x = q(t)$ has exactly one solution which is polynomial of degree less than or equal to k , provided that $p(0)$ is not zero.

Notice that if $p(s)$ is the polynomial

$$a_n s^n + a_{(n-1)} s^{n-1} + \dots + a_1 s + a_0$$

then $p(0) = a_0$.

Proof by example:

$$3x'' + 2x' + x = t^2 + 1$$

The theorem applies since 3 is not 0 : there is exactly one solution of the form

$$x = at^2 + bt + c$$

To find a, b, c , plug in:

$$\begin{array}{l} 1] \ x = at^2 + bt + c \\ 2] \ x' = 2at + b \\ 3] \ x'' = 2a \end{array}$$

$$t^2 + 1 = at^2 + (b+4a)t + (c+2b+6a)$$

The coefficients must be equal. $a = 1$. Then $b+4a = 0$ implies $b = -4$. Finally $c+2b+6a = 1$ reads $c = 1 - 2b - 6a = 1 - 2(-4) - 6 = 3$.

$$x_p = t^2 - 4t + 3$$

If $a_0 = 0$, the theorem doesn't apply, but we can use "reduction of order":

Eg $x'' + x' = t$

Substitute $u = x'$ so $u' + u = t$

$$\begin{array}{l} u = at + b \\ u' = a \\ \hline t = at + (a+b) \end{array}$$

$a = 1, b = -1, u_p = t - 1$. Now integrate: $x_p = t^2/2 - t$.

For the general solution, the roots of $p(s) = s(s+1)$ are 0 and -1, so $x_h = c_1 + c_2 e^{-t}$.

Q15.1. The differential equation $3x^{(4)} + 2x^{(3)} + x^{(2)} = t^2 + 1$

(1) has no polynomial solutions.

- (2) has exactly one polynomial solution.
- (3) has a polynomial solution of degree at most 3 .
- (4) has exactly one polynomial solution of the form $at^5 + bt^4 + ct^3$.
- (5) has only polynomial solutions.
- (6) None of the above
- (7) Several of the above
- (Blank) Don't know

Ans: First of all, if I plug 1 or t into the left hand side, I get zero. These are solutions to the associated homogeneous equation, so I can add them to any solution and get new solutions. So if there is one polynomial solution then there are many.

To find one, set $u = x''$, so the equation is $3u'' + 2u' + u = t^2 + 1$.

We solved this above, but in any case the Undetermined Coefficients Theorem shows that it has a solution $u = at^2 + bt + c$. Integrating twice to get x gives a polynomial involving t^4, t^3, t^2 (and, if you like, t and 1 as well, by different choice of constant of integration).

So I think the answer is (6).

Incidentally using the calculation we did, you get

$$x_p = (1/12) t^4 - (2/3) t^3 + (3/2) t^2 .$$

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