18.03 Class 14, March 5, 2010

Complex gain

1. Recap
2. Phase lag
3. Driving via the dashpot
4. Complex gain

[1] The story so far: We're studying solutions of linear constant coefficient equations

$$a_n x^{(n)} + \dots + a_1 x + a_0 = q(t) \qquad (*)$$

A key is the characteristic polynomial

$$p(s) = a_n s^n + \dots + a_1 s + a_0$$

For the homogeneous case,

$$a_n x^{(n)} + \dots + a_1 x + a_0 = 0 \qquad (*)_h$$

we found that the roots of $p(s)$ give the exponents in exponential solutions, and that the general solution is a linear combination of these or (these times a power of t in case there are repeated roots).
Euler's formula shows that

$$|e^{(a+bi)t}| = e^{at}$$

so: [Slide]

> Transience Theorem:
> All homogeneous solutions of $(*)_h$ decay to zero provided that all the roots of $p(s)$ have negative real parts.

In this case the solutions to $(*)_h$ are called "transients,"
By superposition, all solutions to $(*)$ converge together as t gets large, and we say that the equation is "stable."

If we have a system modeled by a stable equation, and we are only interested in what it looks like after the transients have died down, we can eliminate the initial condition:

```
                   ____________
   input          |            |       steady state
----------------->|   System   |------------------------->
   signal         |____________|       output signal
                                            x_p
```

So we look for a particular solution x_p . Sinusoidal input signals are of particular importance. Experiments indicate that sinusoidal in gives sinusoidal out. We decide to set our clock so that the input signal is

$$\text{input} = A \cos(\omega t)$$

Experiments indicate that the steady state output signal is again sinusoidal, of the same circular frequency:

$$\text{output} = x = B \cos(\omega t - \phi)$$

A consequence of linearity of the system is that B is proportional to A:

$$x = g A \cos(\omega t - \phi)$$

So there are just two numbers I need to know, in understanding this kind of system:

the "gain" g and
the phase lag ϕ.

Both of them will depend upon the input circular frequency ω .

[2] Polar treatment of example from Wednesday:

$$x'' + x' + 2x = \cos(t) \qquad p(s) = s^2 + s + 2$$

The default is to regard the right hand side as the input signal, and x as the output. We are looking for the steady state solution. Make the complex replacement:

$$z'' + z' + 2z = e^{it} \qquad p(i) = - 1 + i + 2 = 1 + i$$

$$z_p = e^{it}/(1+i) \qquad \text{from ERF, [Slide]}$$

Rectangular solution: $1/(1+i) = (1-i)/2$

$$x_p = \mathrm{Re}(z_p) = (1/2) \cos(t) + (1/2) \sin(t)$$

We used the triangle to rewrite this in polar form:

$$x_p = (\sqrt{2}/2) \cos(t - \pi/4)$$

This expression gives more insight: the amplitude is $\sqrt{2}/2 \sim 0.707$ times the amplitude of the input signal - the gain is $\sqrt{2}/2$ - and the steady state system response lags $\pi/4$ radians or $1/8$ cycle behind the input signal.

I want to show you how you can get to this information directly, by passing to polar coordinates earlier. So we start from

$$z_p = e^{it}/p(i)$$

and calculate

$$p(i) = 1+i = \sqrt{2}\, e^{i \pi/4}$$

$$z_p = e^{it}/p(i) = (1/\sqrt{2})\, e^{-i \pi/4} e^{it}$$

$$= (\sqrt{2}/2)\, e^{i(t - \pi/4)}$$

$$x_p = \mathrm{Re}(z_p) = (\sqrt{2}/2) \cos(t - \pi/4)$$

-- much more efficient.

Question 14.1. In this equation, if m and k are left alone
and the damping constant b is increased from 1, the phase lag

1. increases and I can see why from the mathematics
2. increases but I only see this from physical reasoning
3. decreases and I can see why from the mathematics
4. decreases but I only see this from physical reasoning
5. stays the same and I can see why from the mathematics
6. stays the same but I only see this from physical reasoning
7. don't know

Ans: the only effect of b is to produce the imaginary part of p(i) .
If it increases, then the argument of the complex number p(i) increases,
The argument of p(i) is the phase lag in this example, and that increases.

[The class was on board with this one.]

Question 14.2. In this equation, if m and k are left alone and
the damping constant b is increased from 1, the amplitude of the solution

1. increases and I can see why from the mathematics
2. increases but I only see this from physical reasoning
3. decreases and I can see why from the mathematics
4. decreases but I only see this from physical reasoning
5. stays the same and I can see why from the mathematics
6. stays the same but I only see this from physical reasoning
7. don't know

The amplitude of the solution is 1/|p(i)| . p(i) increases if b increases,
so 1/|p(i)| decreases.

[This was harder. Both classes discussed it. I think the mistake was
forgetting that you *divide* by p(i omega).]

[3] Another way to drive the spring system: though the dashpot:

```
     |                    ______
     |                   |      |     ----------------|
     |---\/\/\/\/\/------|      |---------=======     |-------
     |                   |______|     ----------------|   |
     |                      |                             |
     |                      |---------->                  |--------->
                            |     x                       |     y
```

Now the force on the mass exerted by the dashpot is b(y-x)' :

$$m x'' + bx' + kx = by' \qquad (*)$$

Input signal: y
System response: x

Notice! the right hand side is not the input signal; it's not even a multiple of the input signal.

Again let's think about driving this system sinusoidally;

$$y = A \cos(\omega t) .$$

We know we will analyze this by making a complex replacement.
Let's take the next step, push the complex replacement back even farther, and replace the input signal itself with a complex exponential signal:

$$y_{cx} = A e^{i \omega t}$$

Now solve (*) with y_{cx} in place of y :

$$m z'' + b z' + k z = b y_{cx}' = b A i \omega e^{i \omega t}$$

ERF $$z_p = b A i \omega e^{i \omega t} / p(i \omega)$$

where $$p(i \omega) = (k - m \omega^2) + b i \omega$$

[4] Define the *complex gain* as the complex number you multiply the complex exponential input by in order to get the complex exponential system response:

$$z_p = H(\omega) y_{cx}$$

In this case it is

$$H(\omega) = b i \omega / p(i \omega)$$

Now, to return to original equation we pass to real parts:

$$x_p = \mathrm{Re} (H(\omega) e^{i \omega t})$$

Let's compute the real part using the polar approach as in [1].
The following calculation works in general, not just for this particular case.

$$H(\omega) = |H(\omega)| e^{- i \phi}$$

so $-\phi$ is the argument of $H(\omega)$. Then

$$z_p = A |H(\omega)| e^{- i \phi} e^{i \omega t}$$

$$= A |H(\omega)| e^{i(\omega t - \phi)}$$

Now when I take real parts,

$$x_p = A |H(\omega)| \cos(\omega t - \phi)$$

So: $|H(\omega)|$ is the gain of the system

$-\mathrm{Arg}(H(\omega))$ is the phase lag of the system.

(and that accounts for my choice to write - phi for Arg(H(omega)) .)

This last conclusion is not special to this particular system; it is a general fact.

I demonstrated the Mathlet Amplitude and Phase: Second Order II.