18.03 Class 38, May 10, 2010

Linearization: The nonlinear pendulum and phugoid oscillation

[1] Nonlinear pendulum
[2] Phugoid oscillation
[3] Buckling bridge

[1] The bob of a pendulum is attached to a rod, so it can swing clear around the pivot. This system is determined by three parameters:

L length of pendulum
m mass of bob
g acceleration of gravity

We will assume that the motion is restricted to a plane.

To describe it we need a dynamical variable. We could use a horizontal displacement, but it turns out to be easier to write down the equation controlling it if you use the angle of displacement from straight down. Write theta for that angle, measured to the right.

Here is a force diagram:

```
      *
      |\
      |theta
      |  \
      |   \
   mg |   /\   (this is supposed to be a right angle!)
      |   \/
      |   /
      |  / mg sin(theta)
      | /
      |/
      *
```

Write s for arc length along the circle, with s = 0 straight down. Of course,

s = L theta

Newton's law says

ms" = F

The force has the - mg sin(theta) component of the force of gravity (and notice the sign!), and also a frictional force which depends upon

s' = L theta'

Make the simplest model for friction, - cs' = -cL theta'. So:

m L theta" = - mg sin(theta) - cL theta'

Divide through by mL and we get

theta" + b theta' + k sin(theta) = 0

where k = g/L and b = c/m .

This is a nonlinear second order equation. It still has a "companion first order system," obtained by setting

x = theta , y = x'

so y' = theta" = - k sin(theta) - b theta' or

x' = y
y' = - k sin(x) - by

This is an autonmous system. Let's study its phase portrait.

Equilibria: y = 0 , sin(x) = 0 ; that is,

x = 0 , +-pi , +-2pi , ...

Let's compute the Jacobian:

J(x,y) = [0 , 1 ; - k cos(x) , -b]

When x = 0 , +-2pi , +-4pi , ..., cos(x) = 1 and

J(x,y) = [0 1 ; -k -b]

det = k , tr = -b . Suppose b is small, so we get spirals.

When x = +-pi , +-3pi , ..., cos(x) = -1 and

J(x,y) = [0 1 ; k -b]

det = -k , tr = -b : saddles.

Then I revealed the entire phase portrait. It shows another feature common to all companion systems:

The trajectories above the x axis move right, those below move left.

Trajectories coming down from the left represent the pendulum swinging around in counterclockwise complete circles. Trajectories coming up from the right represent the pendulum swinging around in clockwise circles.

With a student I animated the pendulum swinging around. The successive dips represent passing through the vertical position. In very exceptional cases, the trajectory heads straight at the saddle equilibria; they converge to it exponentially, but most likely miss and move away from it exponentially. The saddles represent the unstable equilibria which are straight up. Eventually, the trajectory gets caught in a basin (actually it was always in that basin) and spiral in towards the attractor of that basin, which is straight down.

[2] Phugoid equation.

The forces on an airplane are like this: suppose the plane is moving to the right. It is tipped up a bit, which is hard to draw using ascii.

Along the directon of motion we have the thrust, created by the engines
Directed downwards we have the force of gravity, AKA the weight of the plane.
There is a frictional force, which we decompose into two componets:
Drag is directed against the thrust,
Lift is directed perpendicularly to the direction of motion.

One can analyze the effect of these forces near equilibrium.
Write F for the thrust
g for the acceleration of gravity
m for the mass of the plane
v_0 for equilibrium speed.

At equilibrium the airplane is moving horizontally.
We study what happens when you are jarred off of equilibruium, by a downdraft for example. Then the horizontal component of the velocity is $v_0 + u$ and the vertical component is w, where u and w are small relative to v_0.

$$\frac{d}{dt}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -2F/mv_0 & -g/v_0 \\ 2g/v_0 & 0 \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix}$$

The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 + (2F/mv_0)\lambda + (2g^2/v_0^2)$$

Let's use the language of natural frequency and damping ratio:

$$p_A(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

so

$$\omega_n = (\sqrt{2})\, g/v_0$$

$$2F / mv_0 = 2\zeta\omega_n = 2\zeta(\sqrt{2})\, g / v_0$$

$$\zeta = (1/\sqrt{2})\, F/mg$$

Lessons to be learned from this linearization:

(1) The period of this "phugoid oscillation" depends *only on the equilibrium speed* and not on the thrust or mass of the airplane. In units of meters and seconds, it comes to about $0.45 v_0$.

Boeing 747	260 m/sec	118 seconds
F15	838 m/sec	380 seconds

(2) The damping ratio depends only on *thrust/weight*. Both ζ and thrust/weight are dimensionless, the same in all units.

The system is underdamped as long as $\zeta < 1$, ie $F/mg < \sqrt{2}$

Even an F15 doesn't come close to having F/mg > sqrt(2) ! - so the phugoid system is always underdamped:

747 :   F/mg ~ .27 , zeta ~ .19
F15:   F/mg ~ .67 , zeta ~ .47

[3] I want to show you this movie. In the summer of 1940, a bridge was built ove the Tacoma Narrows in Washington State. It was a mile long and just 39 feet across - an elegant suspension bridge. Very early it acquired the sobriquet "Galloping Gerdie," because it would oscillate vertcially in a very exciting way. The structural engineers studied this and concluded that it wasn't a problem. But in November, the first winter storm picked up. The bridge showed a different behavior now - it wasn't oscillating vertically now, it was twisting. Here's a video -

http://www.youtube.com/watch?v=P0Fi1VcbpAI

What happened here? Resonance? The wind was gusting back and forth at just the natural frequency of the bridge? Nah. What happens is this:

Suppose theta is the angle off of horizontal, at some point along the bridge.

    theta" + b theta' + omega^2 theta = F

F depends on the wind velocity, but look: When the edge of the bridge tips up, the torisonal force from the wind changes. So perhaps F looks like

    F = a(v) theta + c(v) theta'

Actually, it turns out that the dominant contribution is from the theta' term, so let's simplify by just dropping the theta term.

So: theta" + (b-c(v)) theta' + omega_n^2 theta = 0

Now the characteristic polynomial is

    s^2 + (b-c(v)) s + omega_n^2

        = (s + (b-c(v))/2)^2 + omega_d^2

with roots

    r = (c(v)-b)/2 +- omega_d

(The experiment shows the roots are not real!)

It turns out that the shape the graph of c(v) is like a W . For v near zero, c(v) < 0 : the effect increases the damping and causes the bridge to be MORE stable. At some point c(v) = 0, which is still OK. But eventually it equals and then surpasses the damping constant b : then you get effective anti-damping, the real part of the root becomes positive, and we get solutions oscillating under increasing exponentials: disaster!