18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 16. – **Thu, Oct 18, 2007**

Handouts: PS6 solutions, PS7.

Double integrals.

Recall integral in 1-variable calculus: $\int_a^b f(x)\,dx$ = area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\iint_R f(x, y)\,dA$ = volume below graph $z = f(x, y)$ over plane region R.

Cut R into small pieces $\Delta A \Rightarrow$ the volume is approximately $\sum f(x_i, y_i)\,\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint_R f(x, y)\,dA$. (picture shown)

How to compute the integral? By taking slices: $S(x)$ = area of the slice by a plane parallel to yz-plane (picture shown): then

$$\text{volume} = \int_{x_{min}}^{x_{max}} S(x)\,dx, \quad \text{and for given } x,\ S(x) = \int f(x, y)\,dy.$$

In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*

Example 1: $z = 1 - x^2 - y^2$, region $0 \le x \le 1,\ 0 \le y \le 1$ (picture shown):

$$\int_0^1 \int_0^1 (1 - x^2 - y^2)\,dy\,dx.$$

(note: $dA = dy\,dx$, limit of $\Delta A = \Delta y\,\Delta x$ for small rectangles).

How to evaluate:

1) inner integral (x is constant): $\displaystyle\int_0^1 (1 - x^2 - y^2)\,dy = \left[(1 - x^2)y - \frac{1}{3}y^3\right]_0^1 = (1 - x^2) - \frac{1}{3} = \frac{2}{3} - x^2.$

2) outer integral: $\displaystyle\int_0^1 (\frac{2}{3} - x^2)\,dx = \left[\frac{2}{3}x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Example 2: same function over the quarter disc $R : x^2 + y^2 < 1$ in the first quadrant.

How to find the bounds of integration? Fix x constant: what is a slice parallel to y-axis? bounds for y = from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is $x = 0$, last slice is $x = 1$. So we get:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx.$$

(note the inner bounds depend on the outer variable x; the outer bounds are constants!)

Inner: $\displaystyle\left[(1 - x^2)y - y^3/3\right]_0^{\sqrt{1-x^2}} = \frac{2}{3}(1 - x^2)^{3/2}.$

Outer: $\displaystyle\int_0^1 \frac{2}{3}(1 - x^2)^{3/2}\,dx = \cdots = \frac{\pi}{8}.$

($\ldots$ = trig. substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $(1 - x^2)^{3/2} = \cos^3\theta$. Then use double angle formulas... complicated! I carried out part of the calculation to show how it would be done but then stopped before the end to save time; students may be confused about what happened exactly.)

Exchanging order of integration.

$\int_0^1 \int_0^2 dx\,dy = \int_0^2 \int_0^1 dy\,dx$, since region is a rectangle (shown). In general, more complicated!

Example 3: $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y}\, dy\, dx$: inner integral has no formula. To exchange order:

1) draw the region (here: $x < y < \sqrt{x}$ for $0 \le x \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\, dx\, dy = \int_0^1 \frac{e^y}{y}(y - y^2)\, dy = \int_0^1 e^y - ye^y\, dy = [-ye^y + 2e^y]_0^1 = e - 2.$$

(the last integration can be done either by parts, or by starting from the guess $-ye^y$ and adjusting;).

18.02 Lecture 17. – **Fri, Oct 19, 2007**

Integration in polar coordinates. ($x = r\cos\theta$, $y = r\sin\theta$): useful if either integrand or region have a simpler expression in polar coordinates.

Area element: $\Delta A \simeq (r\Delta\theta)\,\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $\Delta\theta, \Delta r \to 0$, we get $dA = r\, dr\, d\theta$.

Example (same as last time): $\iint_{x^2+y^2\le 1,\ x\ge 0,\ y\ge 0} (1 - x^2 - y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^1 (1 - r^2)\, r\, dr\, d\theta.$

Inner: $\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{1}{4}$. Outer: $\int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{2}\frac{1}{4} = \frac{\pi}{8}$.

In general: when setting up $\iint f\, r\, dr\, d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Applications.

1) The area of the region R is $\iint_R 1\, dA$. Also, the total mass of a planar object with density $\delta = \lim_{\Delta A = 0} \Delta m/\Delta A$ (mass per unit area, $\delta = \delta(x, y)$ – if uniform material, constant) is given by:

$$M = \iint_R \delta\, dA.$$

2) recall the average value of f over R is $\bar{f} = \frac{1}{Area} \iint_R f\, dA$. The *center of mass*, or *centroid*, of a plate with density δ is given by weighted average

$$\bar{x} = \frac{1}{mass} \iint_R x\, \delta\, dA, \qquad \bar{y} = \frac{1}{mass} \iint_R y\, \delta\, dA$$

3) **moment of inertia:** physical equivalent of mass for rotational motion. (mass = how hard it is to impart translation motion; moment of inertia about some axis = same for rotation motion around that axis)

Idea: kinetic energy for a single mass m at distance r rotating at angular speed $\omega = d\theta/dt$ (so velocity $v = r\omega$) is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$; $I_0 = mr^2$ is the moment of inertia.

For a solid with density δ, $I_0 = \iint_R r^2 \delta\, dA$ (moment of inertia / origin). (the rotational energy is $\frac{1}{2}I_0\omega^2$).

Moment of inertia about an axis: $I = \iint_R (\text{distance to axis})^2 \delta \, dA$. E.g. about x-axis, distance is $|y|$, so

$$I_x = \iint_R y^2 \delta \, dA.$$

Examples: 1) disk of radius a around its center ($\delta = 1$):

$$I_0 = \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^a = \frac{\pi a^4}{2}.$$

2) same disk, about a point on the circumference?

Setup: place origin at point so integrand is easier; diameter along x-axis; then polar equation of circle is $r = 2a \cos\theta$ (explained on a picture). Thus

$$I_0 = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2 \, r \, dr \, d\theta = ... = \frac{3}{2}\pi a^4.$$