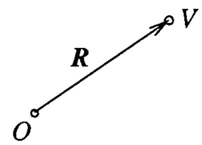
18.02 Multivariable Calculus
Fall 2007

G. Gravitational Attraction

We use triple integration to calculate the gravitational attraction that a solid body V of mass M exerts on a unit point mass placed at the origin.

If the solid V is also a point mass, then according to Newton's law of gravitation, the force it exerts is given by

(1) $$\mathbf{F} = \frac{GM}{|\mathbf{R}|^2}\mathbf{r},$$



where $\mathbf{R}$ is the position vector from the origin $\mathbf{0}$ to the point V, and the unit vector $\mathbf{r} = \mathbf{R}/|\mathbf{R}|$ is its direction.

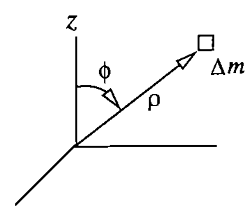
If however the solid body V is not a point mass, we have to use integration. We concentrate on finding just the $\mathbf{k}$ component of the gravitational attraction — all our examples will have the solid body V placed symmetrically so that its pull is all in the $\mathbf{k}$ direction anyway.

To calculate this force, we divide up the solid V into small pieces having volume ΔV and mass Δm. If the density function is $\delta(x, y, z)$, we have for the piece containing the point (x, y, z)

(2) $$\Delta m \approx \delta(x, y, z)\Delta V,$$

Thinking of this small piece as being essentially a point mass at (x, y, z), the force $\Delta\mathbf{F}$ it exerts on the unit mass at the origin is given by (1), and its $\mathbf{k}$ component ΔF_z is therefore

$$\Delta F_z = G\frac{\Delta m}{|\mathbf{R}|^2}\mathbf{r}\cdot\mathbf{k}\ ,$$



which in spherical coordinates becomes, using (2), and the picture,

$$\Delta F_z = G\frac{\cos\phi}{\rho^2}\delta\,\Delta V = G\frac{\delta\,\Delta V}{\rho^2}\cos\phi\ .$$

If we sum all the contributions to the force from each of the mass elements Δm and pass to the limit, we get for the $\mathbf{k}$-component of the gravitational force

(3) $$F_z = G\iiint_V \frac{\cos\phi}{\rho^2}\,\delta\,dV\ .$$

If the integral is in spherical coordinates, then $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and the integral becomes

(4) $$F_z = G\iiint_V \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta\ .$$

Example 1. Find the gravitational attraction of the upper half of a solid sphere of radius a centered at the origin, if its density is given by $\delta = \sqrt{x^2 + y^2}$.

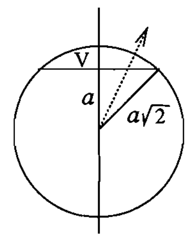
Solution. Since the solid and its density are symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction, and we can use (3) or (4). Since

$$\sqrt{x^2 + y^2} = r = \rho \sin\phi \ ,$$

the integral is

$$F_z = G \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \sin^2\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

which evaluates easily to $\pi G a^2/3$.



Example 2. Let V be the solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center of the sphere. Find the gravitational attraction of V on a unit point mass at the center of the sphere. (Take the density to be 1.)

Solution. To take advantage of the symmetry, place the origin at the center of the sphere, and align the axis of the cap along the z-axis (so the flat side of the cap is parallel to the xy-plane).

We use spherical coordinates; the main problem is determining the limits of integration. If we fix ϕ and θ and let ρ vary, we get a ray which enters V at its flat side

$$z = a, \quad \text{or} \quad \rho \cos\phi = a,$$

and leaves V on its spherical side, $\rho = a\sqrt{2}$. The rays which intersect V in this way are those for which $0 \le \phi \le \pi/4$, as one sees from the picture. Thus by (4),

$$F_z \ = \ G \int_0^{2\pi} \int_0^{\pi/4} \int_{a/\cos\phi}^{a\sqrt{2}} \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta,$$

which after integrating with respect to ρ (and θ) becomes

$$= \ 2\pi G \int_0^{\pi/4} a\Big(\sqrt{2} - \frac{1}{\cos\phi}\Big) \sin\phi \cos\phi \, d\phi$$
$$= \ 2\pi G a \Big(\frac{3\sqrt{2}}{4} - 1\Big) \ .$$

Remark. Newton proved that a solid sphere of uniform density and mass M exerts the same force on an external point mass as would a point mass M placed at the center of the sphere. (See Problem 6a).

This does *not* however generalize to other uniform solids of mass M — it is not true that the gravitational force they exert is the same as that of a point mass M at their center of mass. For if this were so, a unit test mass placed on the axis between two equal point masses M and M' ought to be pulled toward the midposition, whereas actually it will be pulled toward the closer of the two masses.

Exercises: Section 5C