18.01 Single Variable Calculus
Fall 2006

Lecture 28: Integration by Inverse Substitution; Completing the Square

Trigonometric Substitutions, continued

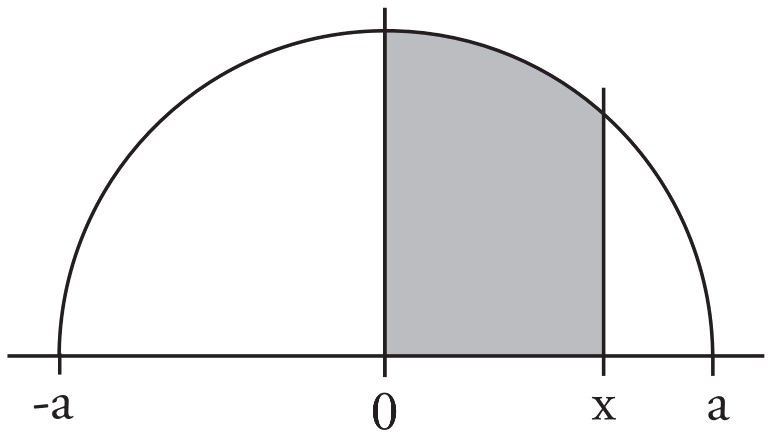

Figure 1: Find area of shaded portion of semicircle.

$$\int_0^x \sqrt{a^2 - t^2}\,dt$$

$$t = a\sin u; \quad dt = a\cos u\,du$$

$$a^2 - t^2 = a^2 - a^2\sin^2 u = a^2\cos^2 u \implies \sqrt{a^2 - t^2} = a\cos u \quad \text{(No more square root!)}$$

Start: $x = -a \Leftrightarrow u = -\pi/2$; Finish: $x = a \Leftrightarrow u = \pi/2$

$$\int \sqrt{a^2 - t^2}\,dt = \int a^2\cos^2 u\,du = a^2 \int \frac{1+\cos(2u)}{2} du = a^2\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right] + c$$

$$\text{(Recall, } \cos^2 u = \frac{1+\cos(2u)}{2}\text{).}$$

We want to express this in terms of x, not u. When $t = 0$, $a\sin u = 0$, and therefore $u = 0$. When $t = x$, $a\sin u = x$, and therefore $u = \sin^{-1}(x/a)$.

$$\frac{\sin(2u)}{4} = \frac{2\sin u\cos u}{4} = \frac{1}{2}\sin u\cos u$$

$$\sin u = \sin\left(\sin^{-1}(x/a)\right) = \frac{x}{a}$$

How can we find $\cos u = \cos\left(\sin^{-1}(x/a)\right)$? Answer: use a right triangle (Figure 2).

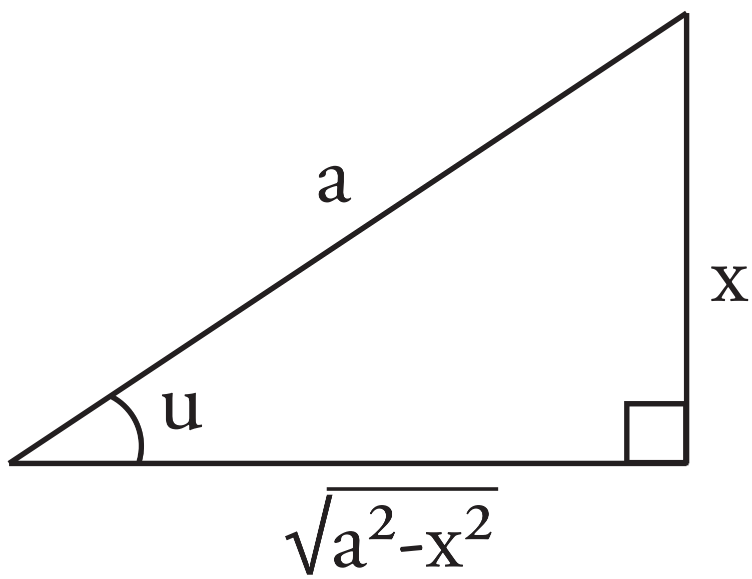

Figure 2: $\sin u = x/a;\ \cos u = \sqrt{a^2 - x^2}/a$.

From the diagram, we see

$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$

And finally,

$$\int_0^x \sqrt{a^2 - t^2}\, dt = a^2 \left[\frac{u}{4} + \frac{1}{2}\sin u \cos u\right] - 0 = a^2 \left[\frac{\sin^{-1}(x/a)}{2} + \frac{1}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a}\right]$$

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2}$$

When the *answer* is this complicated, the route to getting there has to be rather complicated. There's no way to avoid the complexity.

Let's double-check this answer. The area of the upper shaded sector in Figure 3 is $\frac{1}{2}a^2 u$. The area of the lower shaded region, which is a triangle of height $\sqrt{a^2 - x^2}$ and base x, is $\frac{1}{2}x\sqrt{a^2 - x^2}$.

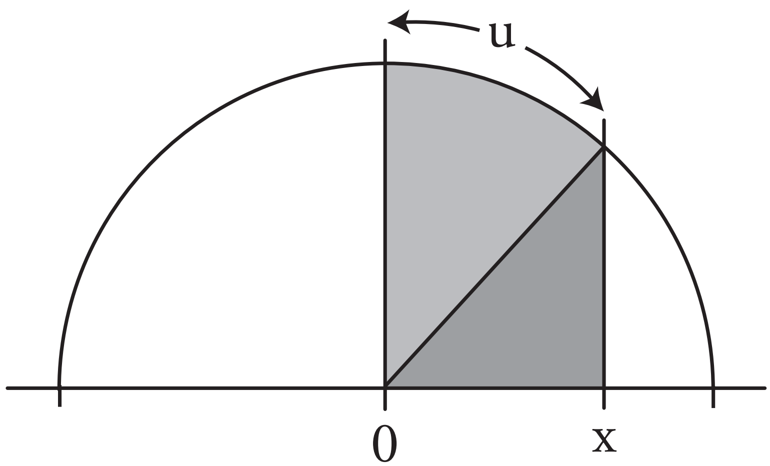

Figure 3: Area divided into a sector and a triangle.

Here is a list of integrals that can be computed using a trig substitution and a trig identity.

integral	substitution	trig identity
$\int \frac{dx}{\sqrt{x^2+1}}$	$x = \tan u$	$\tan^2 u + 1 = \sec^2 u$
$\int \frac{dx}{\sqrt{x^2-1}}$	$x = \sec u$	$\sec^2 u - 1 = \tan^2 u$
$\int \frac{dx}{\sqrt{1-x^2}}$	$x = \sin u$	$1 - \sin^2 u = \cos^2 u$

Let's extend this further. How can we evaluate an integral like this?

$$\int \frac{dx}{\sqrt{x^2+4x}}$$

When you have a linear and a quadratic term under the square root, complete the square.

$$x^2 + 4x = (\text{something})^2 \pm \text{constant}$$

In this case,

$$(x+2)^2 = x^2 + 4x + 4 \implies x^2 + 4x = (x+2)^2 - 4$$

Now, we make a substitution.

$$v = x + 2 \quad \text{and} \quad dv = dx$$

Plugging these in gives us

$$\int \frac{dx}{\sqrt{(x+2)^2 - 4}} = \int \frac{dv}{\sqrt{v^2 - 4}}$$

Now, let

$$v = 2\sec u \quad \text{and} \quad dv = 2\sec u \tan u$$

$$\int \frac{dv}{\sqrt{v^2-4}} = \int \frac{2\sec u\, \tan u\, du}{2\tan u} = \int \sec u\, du$$

Remember that

$$\int \sec u \, du = \ln(\sec u + \tan u) + c$$

Finally, rewrite everything in terms of x.

$$v = 2\sec u \Leftrightarrow \cos u = \frac{2}{v}$$

Set up a right triangle as in Figure 4. Express $\tan u$ in terms of v.

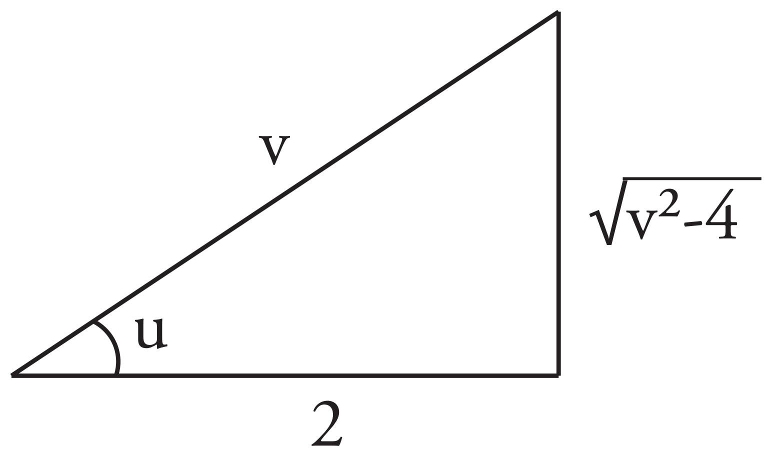

Figure 4: $\sec u = v/2 \quad \text{or} \quad \cos u = 2/v.$

Just from looking at the triangle, we can read off

$$\sec u = \frac{v}{2} \quad \text{and} \quad \tan u = \frac{\sqrt{v^2-4}}{2}$$

$$\int 2\sec u \, du = \ln\left(\frac{v}{2} + \frac{\sqrt{v^2-4}}{2}\right) + c$$

$$= \ln(v + \sqrt{v^2-4}) - \ln 2 + c$$

We can combine those last two terms into another constant, $\tilde{c}$.

$$\int \frac{dx}{\sqrt{x^2+4x}} = \ln(x + 2 + \sqrt{x^2+4x}) + \tilde{c}$$

Here's a teaser for next time. In the next lecture, we'll integrate <u>all</u> rational functions. By "rational functions," we mean functions that are the ratios of polynomials:

$$\frac{P(x)}{Q(x)}$$

It's easy to evaluate an expression like this:

$$\int \left(\frac{1}{x-1} + \frac{3}{x+2}\right) dx = \ln|x-1| + 3\ln|x+2| + c$$

If we write it a bit differently, however, it becomes much harder to integrate:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{x+2+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

$$\int \frac{4x-1}{x^2+x-2} = ???$$

How can we reorganize what to do starting from $(4x-1)/(x^2+x-2)$? Next time, we'll see how. It involves some algebra.