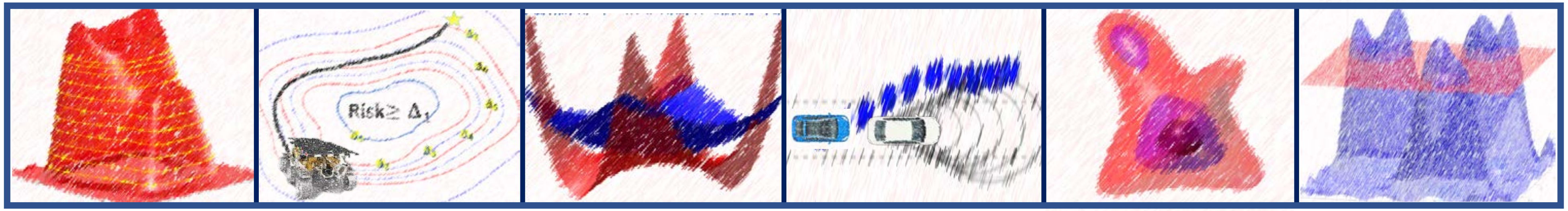
Lecture 7

Nonlinear Chance Constrained and Chance Optimization

Moment-SOS Based SDP Approach

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

Ashkan Jasour



Topics

- Formulation of Chance Optimization and Chance Constrained Optimization
- Geometrical Interpretation
- Challenges
- Moment Based SDP for Chance Optimization
- Dual of Moment-SDP (Sum-of-Squares Program)
- SOS Based SDP for Chance Constrained Optimization
- Outer and Inner approximations of Chance Constrained Sets

Risk Aware Optimization

Chance Optimization

$$\begin{array}{ll} \underset{\text{design parameters}}{\text{maximize}} & \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \\ \text{subject to} & \text{Constraints}(\text{design parameters}) \end{array}$$

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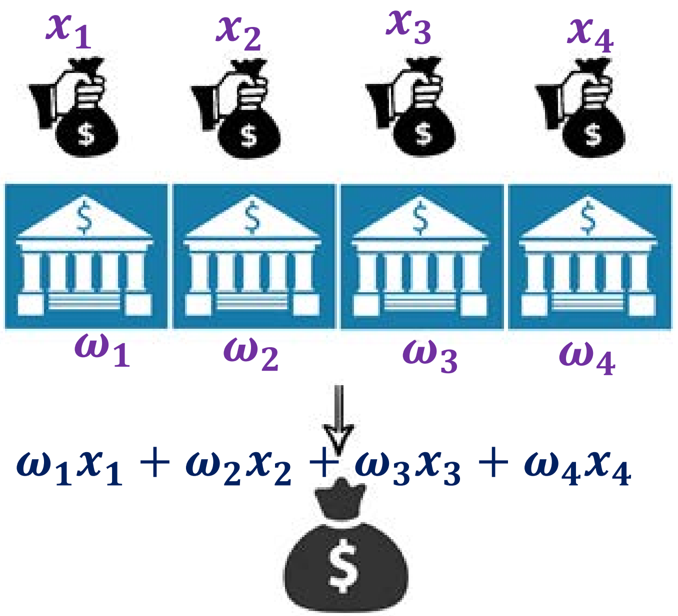
Chance Constrained Optimization

$$\underset{\text{design parameters}}{\text{minimize}} \quad \text{Objective Function}(\text{design parameters})$$
$$\text{subject to} \quad \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta$$

Acceptable risk level

Example: Portfolio Selection Problem



- Assets with uncertain rate of return $\omega_i \sim pr_i(\omega), i = 1, \ldots, 4$
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  $= \{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*\}$

Chance Optimization

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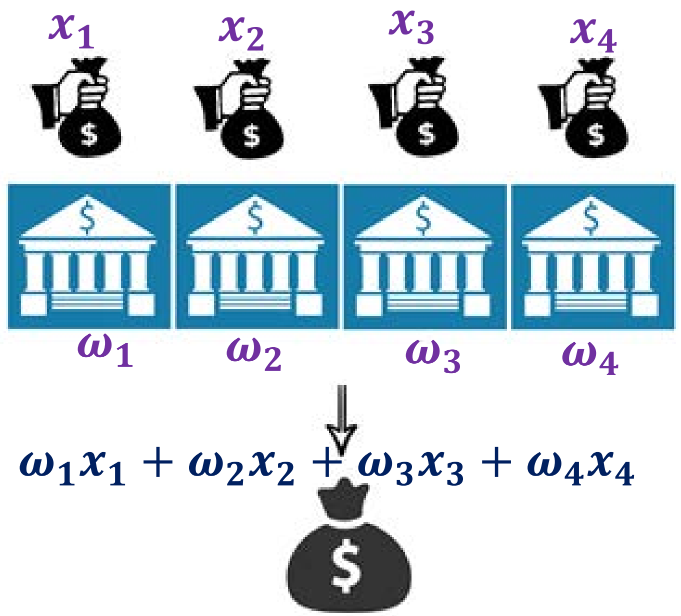
level set of ploynomial in terms of design and uncertain parameters



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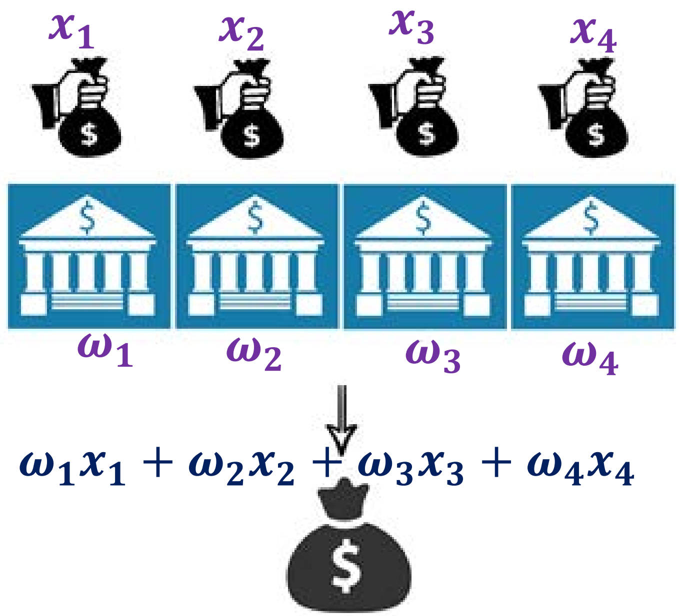

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Example: Obstacle Avoidance

- Probabilistic Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states　inputs　Probabilistic Uncertainty

~ probability distribution

- Unsafe Sets

$$\chi_{obs_i} = \{x : p_{obs_i}(x) < 0\},\ \ i = 1, ..., \ell$$

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Success: level set of ploynomials in terms of design and uncertain parameters

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uncertain parameters

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$$\begin{array}{ll} \underset{u_k}{\text{minimize}} & ||u_k||_2^2 \\ \text{subject to} & \\ & \text{Prob}(f(x_k, u_k, \omega_k) \in \chi_{safe}) \geq 1 - \Delta \\ & u_k \in \mathcal{U} \end{array}$$

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Examples in pages 94, 140, 143

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Risk Aware Optimization

Success: Described with level sets of polynomials in terms of *design parameters* and *uncertain parameters*

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Risk Aware Optimization

Mathematical Formulation:

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}\big(\, \overbrace{p_i(x,\omega) \geq 0,\ i = 1, \ldots, n_p}^{\text{Success Set}} \,\big) \\ \text{subject to} \quad & g_i(x) \geq 0,\ i = 1, \ldots, n_g \end{aligned}$$

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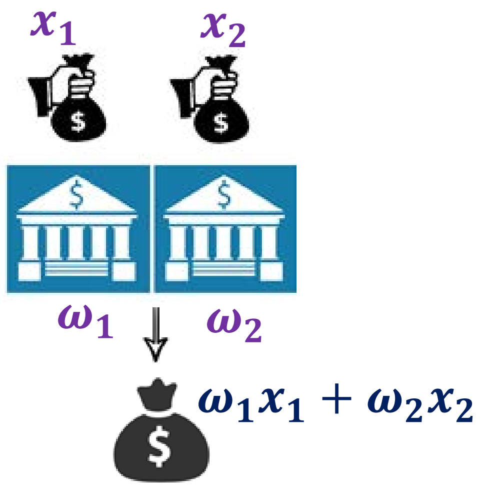
Chance Constrained / Chance Optimization

- Geometrical Interpretation

Example: Portfolio Selection Problem



- Assets with uncertain rate of return

$$\omega_1 \in [l_1 \;\; u_1] \sim pr_1(\omega_1)$$
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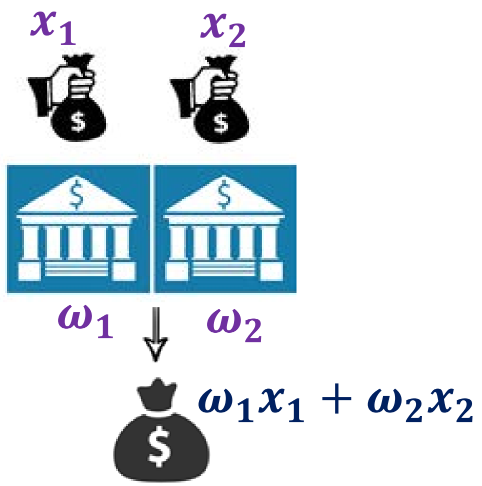

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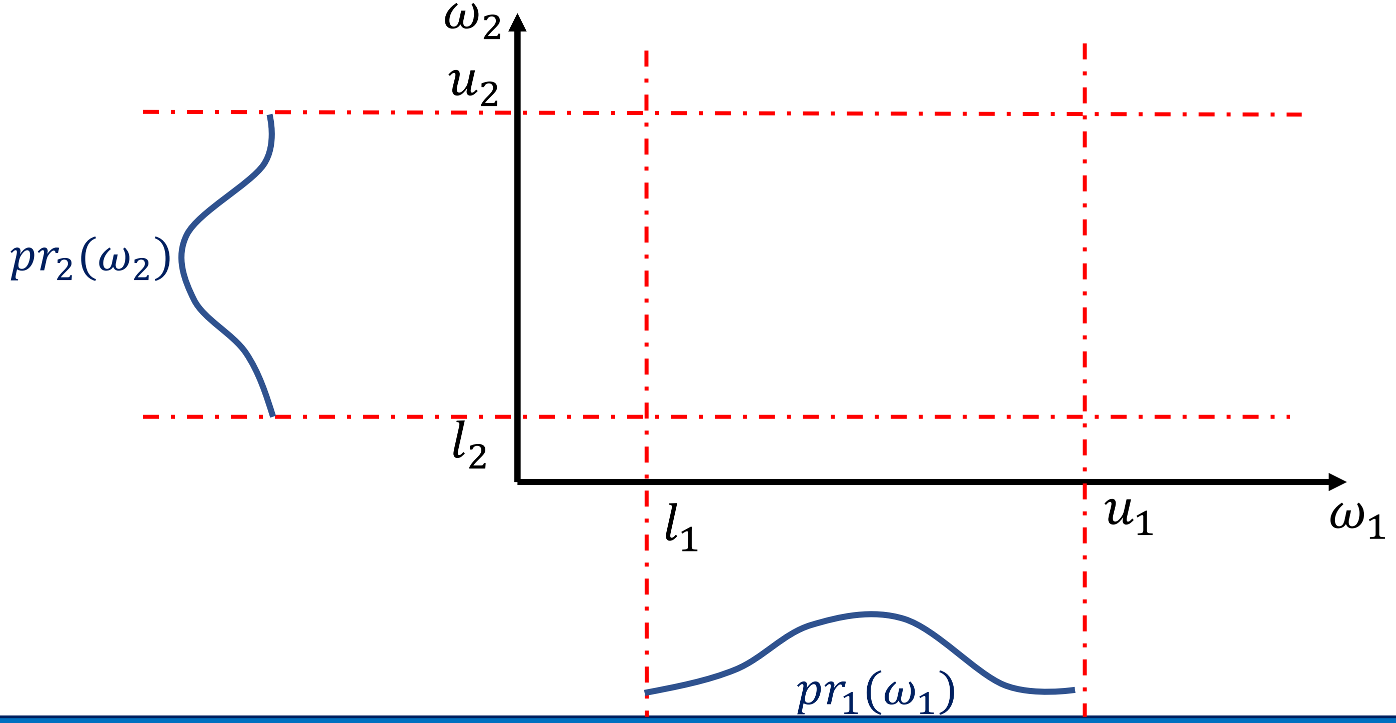

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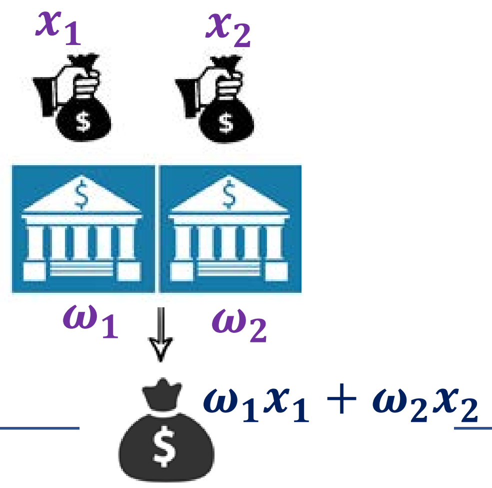
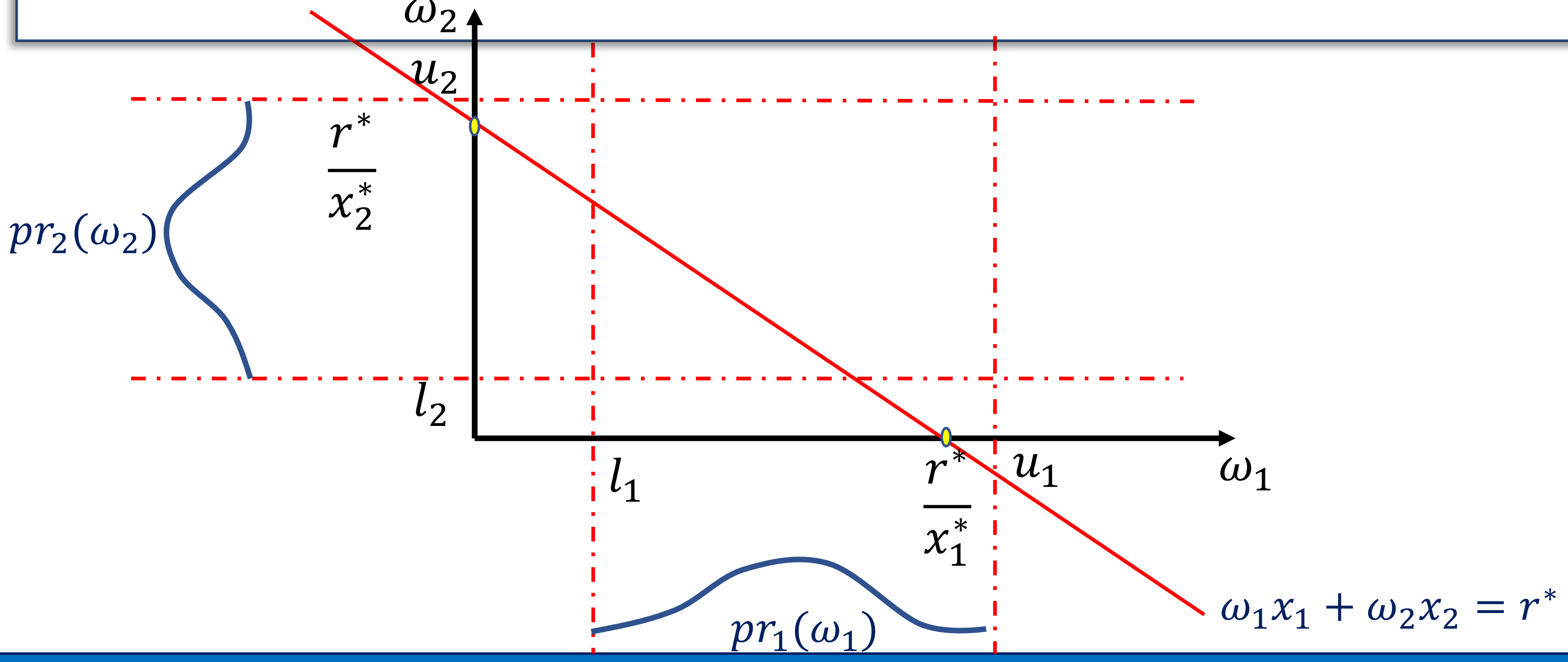


➢ For given design parameters x_1^* and x_2^* :



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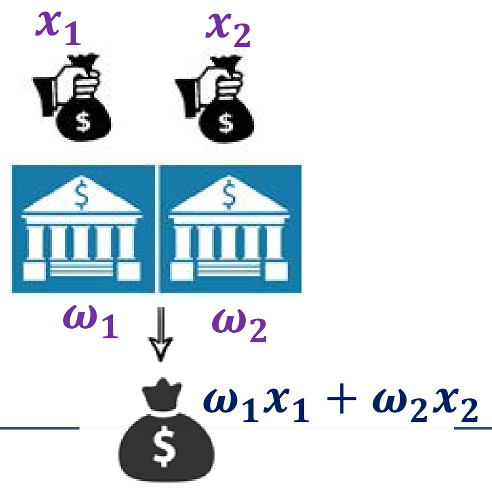

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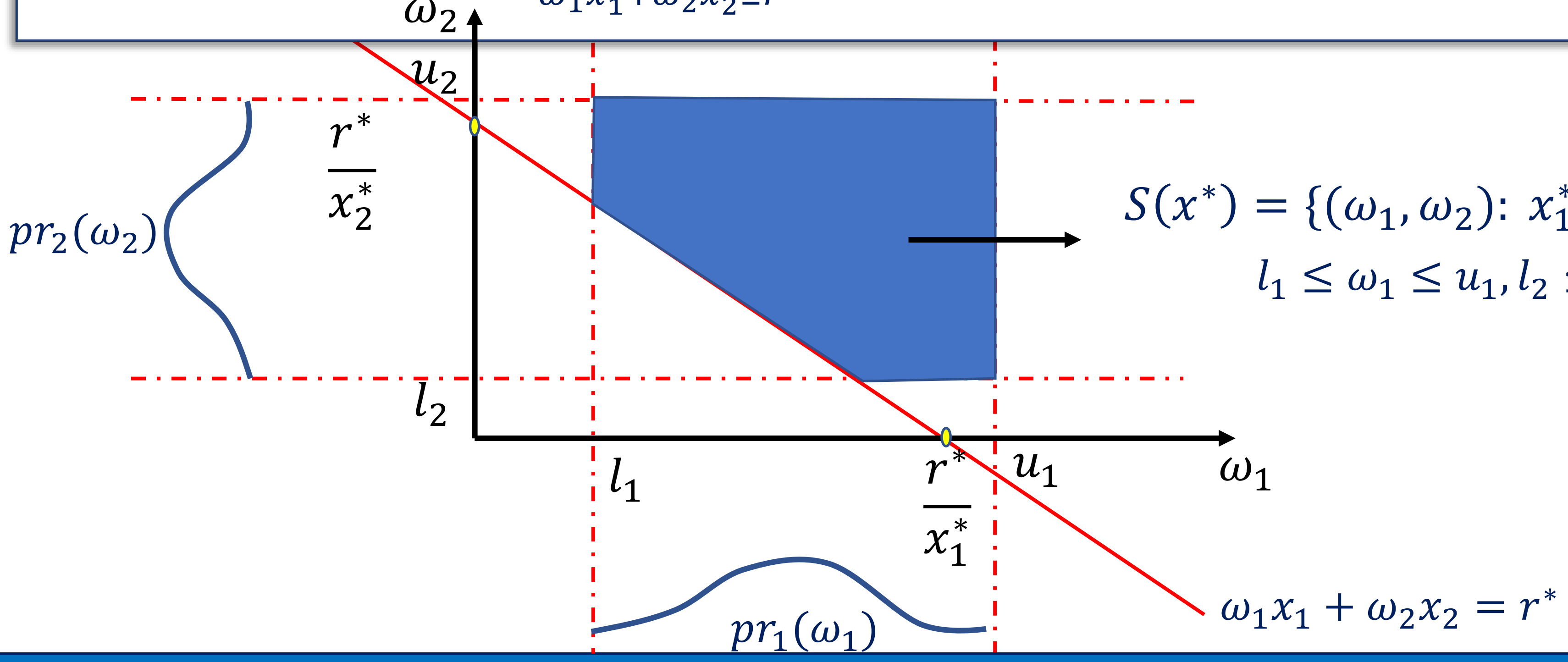
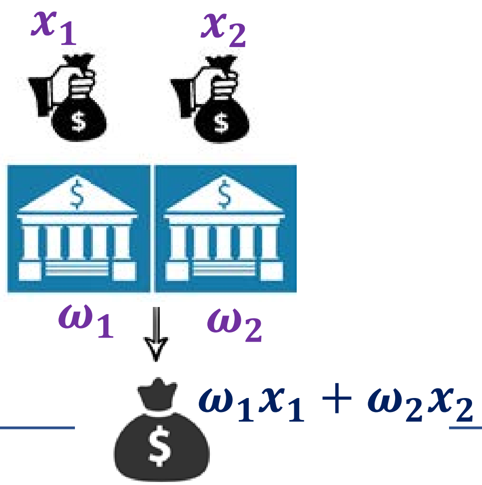


$S(x^*) = \{(\omega_1, \omega_2)\colon \; x_1^* \, \omega_1 + x_2^* \, \omega_2 \geq r^*\}$

$l_1 \leq \omega_1 \leq u_1, l_2 \leq \omega_2 \leq u_2$

Example: Portfolio Selection Problem



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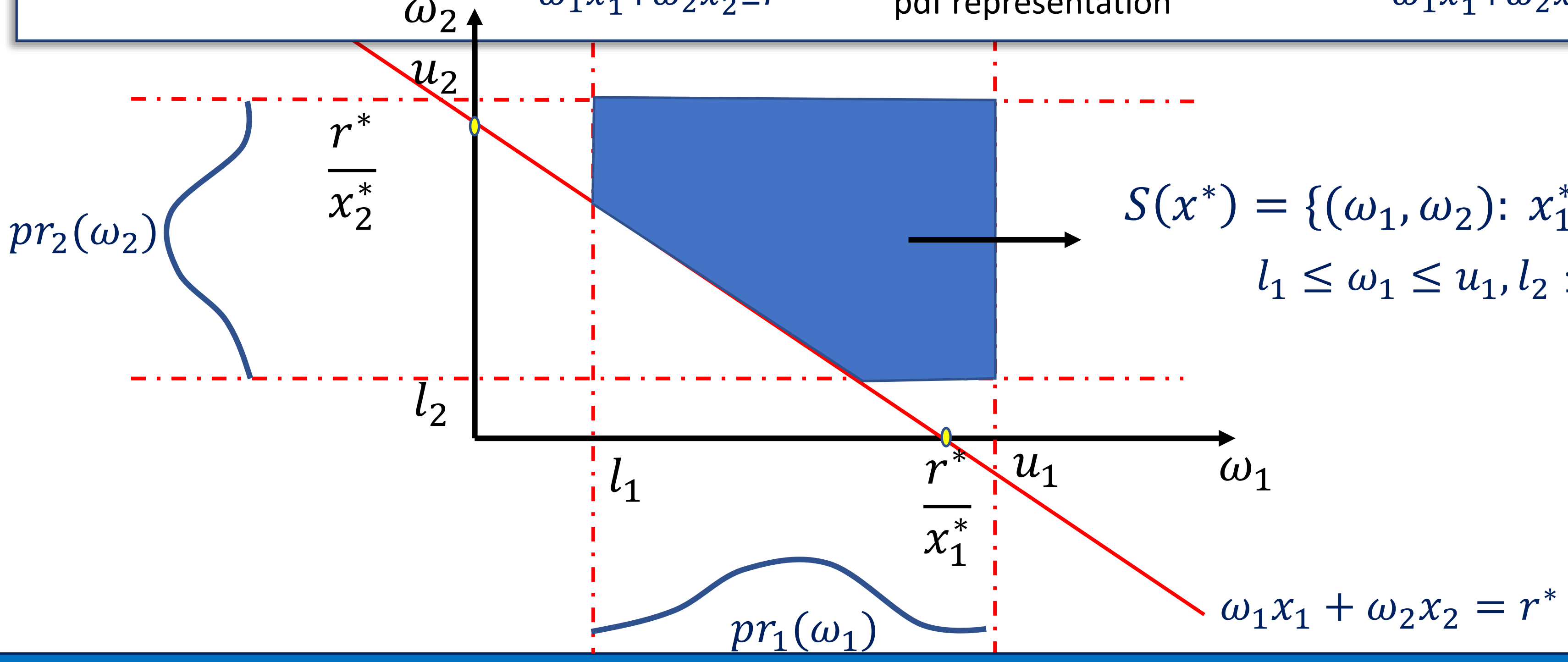
pdf representation — Measure representation



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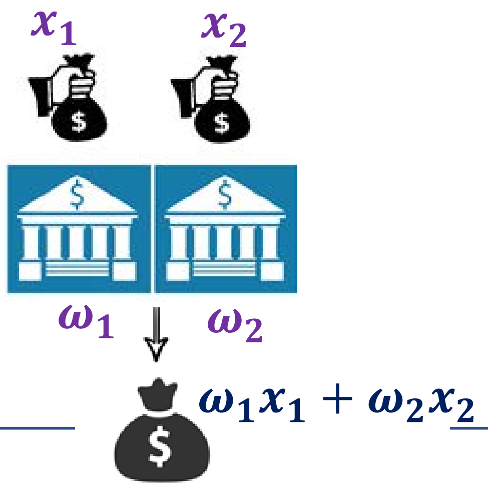

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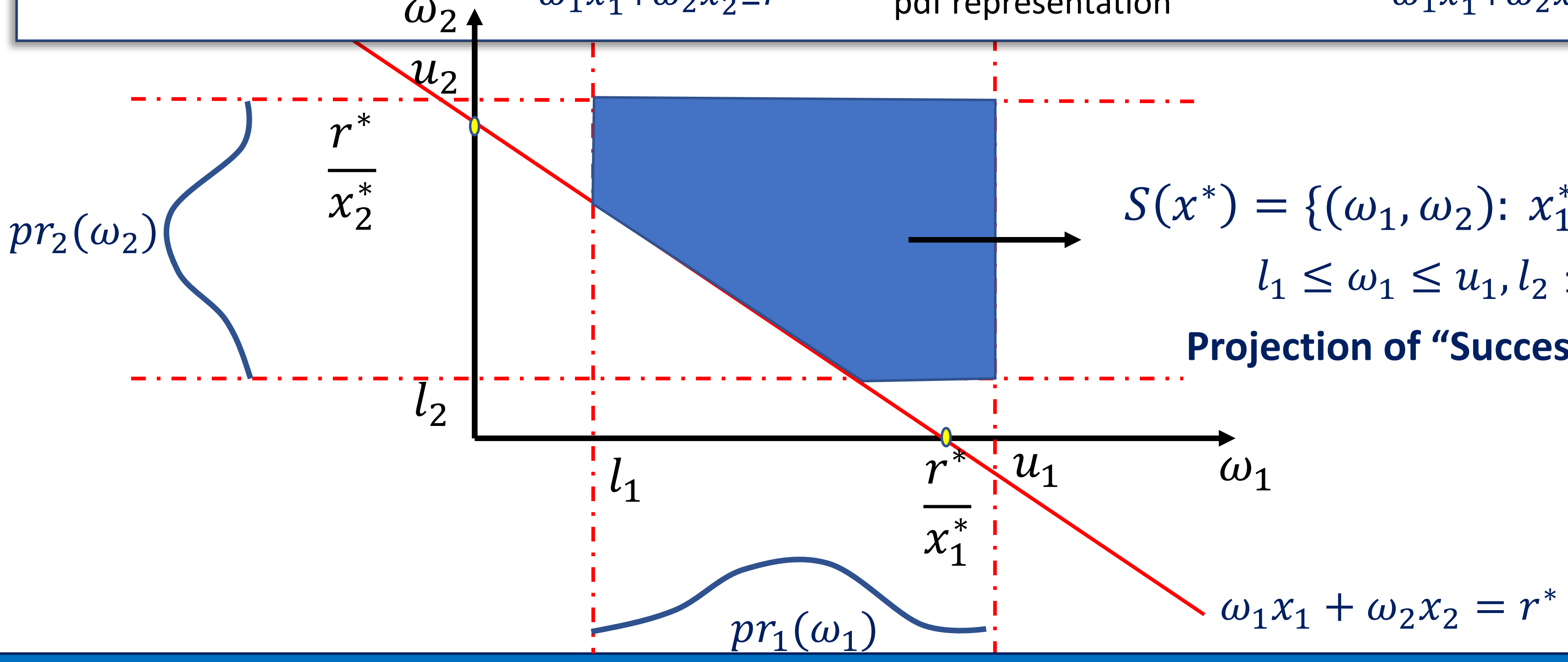

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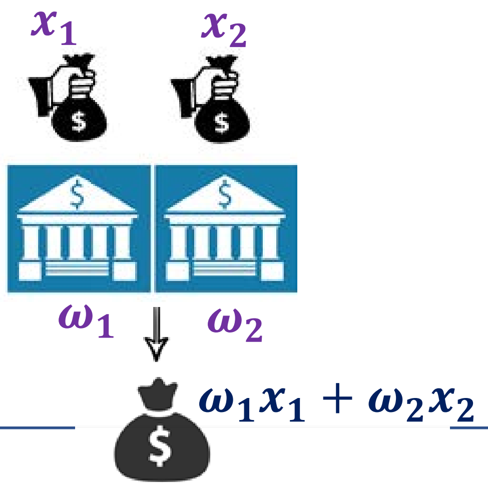
Projection of "Success Set" onto uncertainty space

Example: Portfolio Selection Problem



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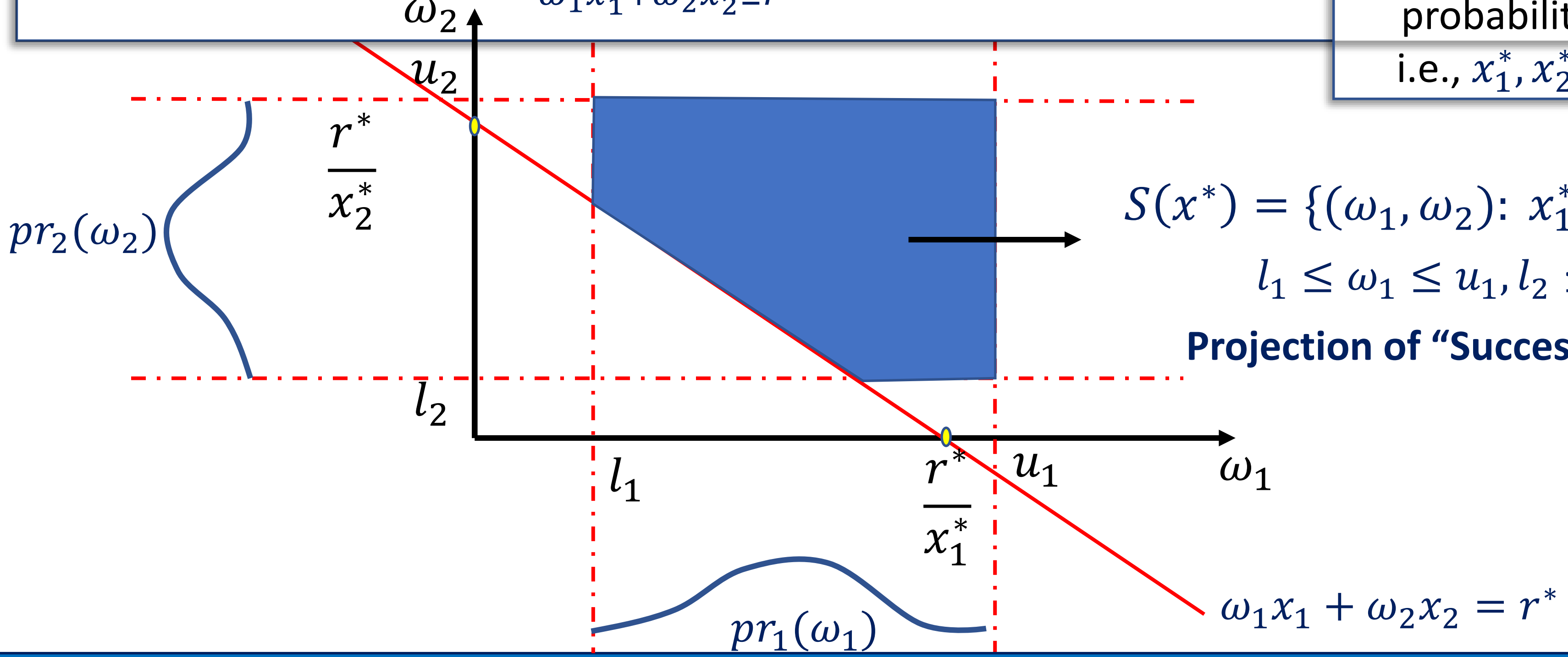
➢ To increase the probability of success, we need to increase the size of set $S(x^*)$ with respect to the probability distribution of uncertainties $pr_i(\omega_i)$ i.e., $x_1^*, x_2^* \uparrow$



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$$p\,(x,\omega) = 0.5\ \omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

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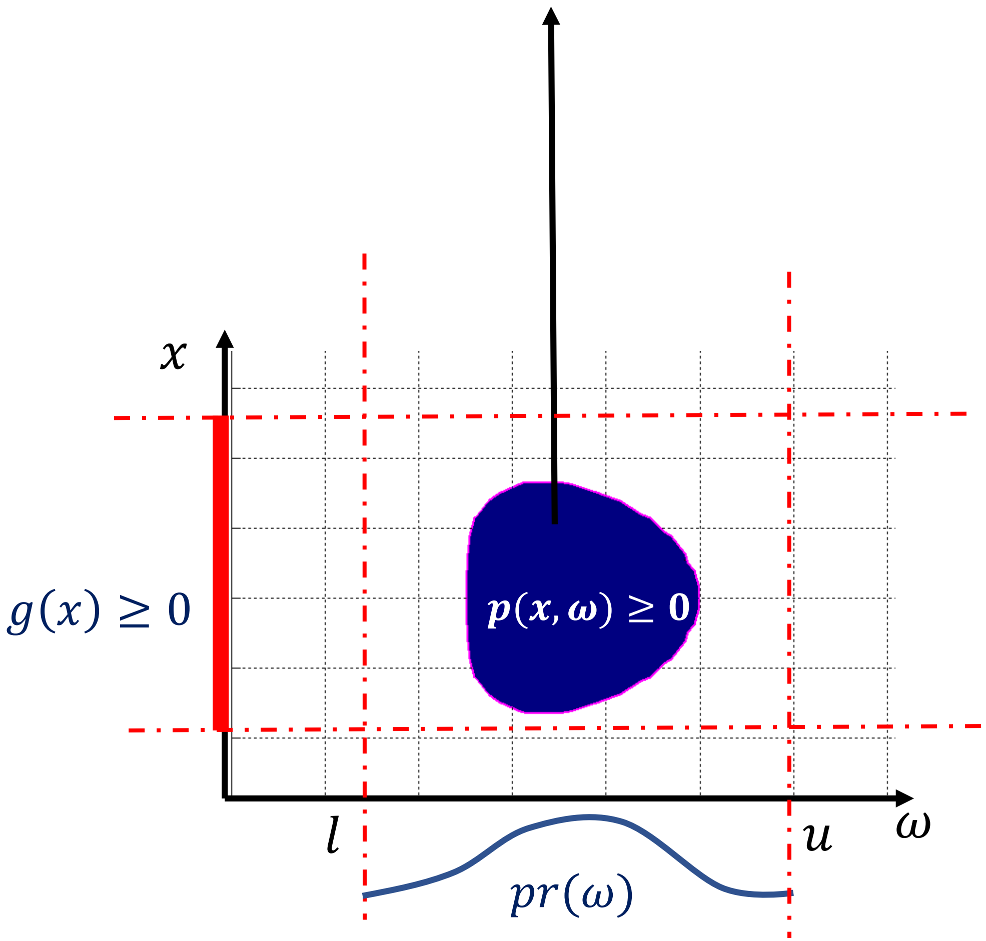

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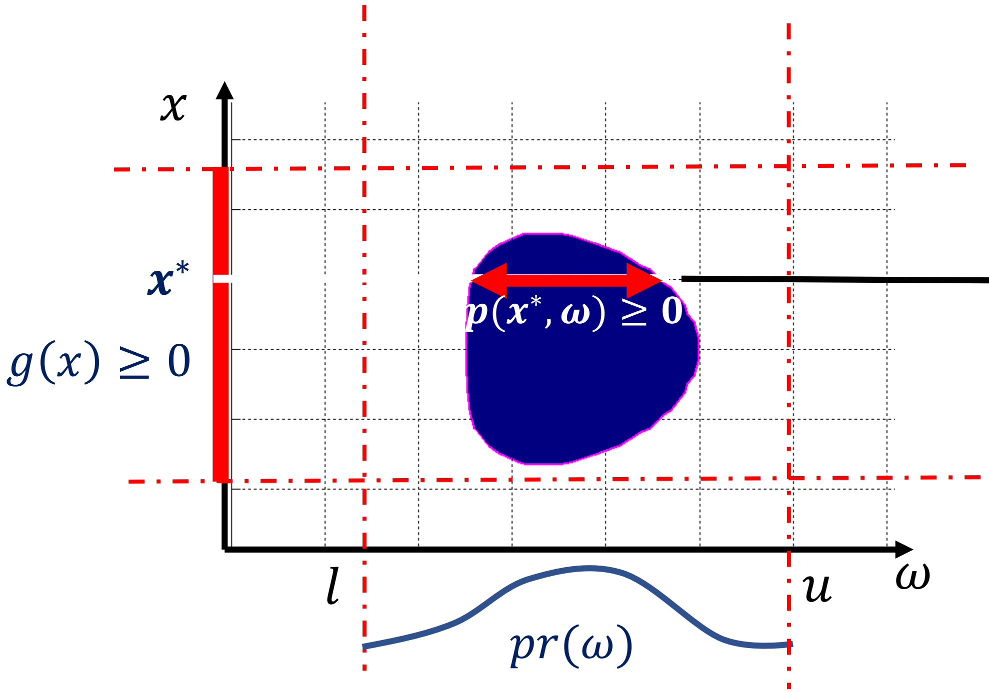

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$\boldsymbol{S(x^*)} = \{p(x^*,\omega) \geq 0\}$ $\quad$ $Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega$

Projection of “Success Set” onto uncertainty space

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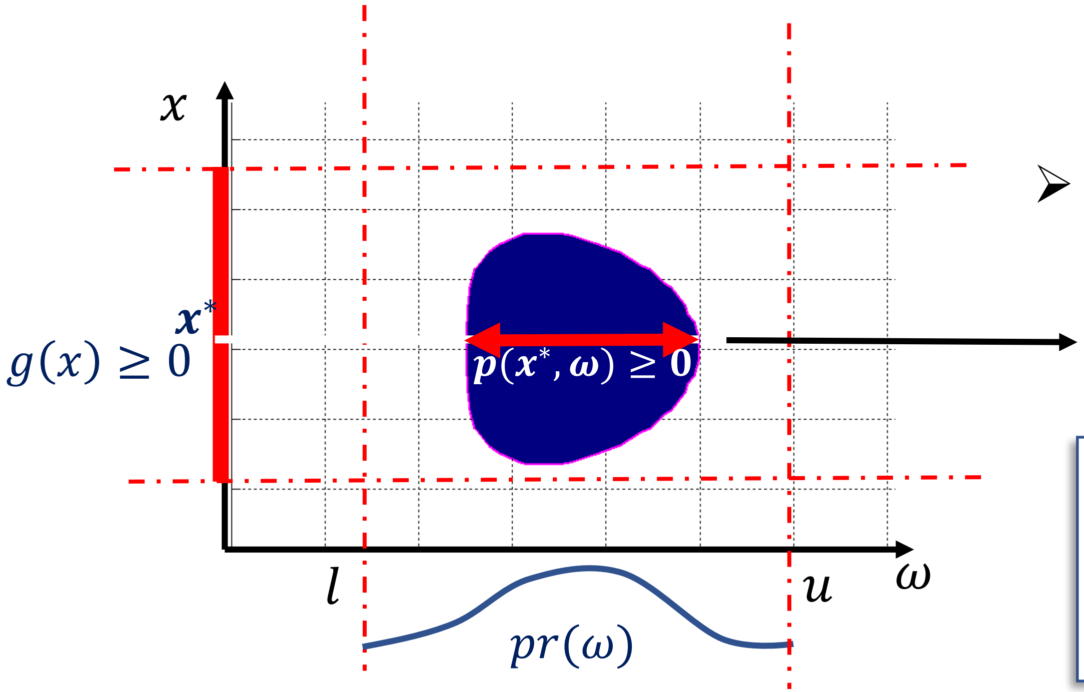


➢ For given design parameters x^*

$\boldsymbol{S(x^*)} = \{p(x^*,\omega) \geq 0\}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega$

Projection of "Success Set" onto uncertainty space

Risk Aware Optimization

Chance Optimization

$$\underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(p_i(x, \omega) \geq 0, \; i = 1, ..., n_p \right)$$

$$\text{subject to} \quad g_i(x) \geq 0, \; i = 1, ..., n_g$$

- Find $x \in \{g_i(x) \geq 0, i = 1, \dots, n_g\}$ to maximize $Prob\{Success\} = \int_{S(x)} pr(\omega) d\omega$

where $\boldsymbol{S}(\boldsymbol{x}) = \{\omega: \; p_i(x^*, \omega) \geq 0, i = 1, \dots, n_p\}$

Projection of "Success Set" onto uncertainty space

Risk Aware Optimization

Chance Constrained Optimization

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & \text{Probability}_{\text{pr}(\omega)}\left(g_i(x,\omega) \geq 0,\ i = 1, \ldots, n_g \right) \geq 1 - \Delta \end{array}$$

- Find set of design parameters χ_{cc} such that

For any $x^* \in \chi_{cc}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega) d\omega \geq 1 - \Delta$

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g_i(x^*, \omega) \geq 0, i = 1, \ldots, n_g\}$

Risk Aware Optimization

Chance Constrained Optimization

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}\left(g_i(x, \omega) \geq 0,\ i = 1, ..., n_g \right) \geq 1 - \Delta$$

- Find set of design parameters χ_{cc} such that

For any $x^* \in \chi_{cc}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega \geq 1 - \Delta$

Chance Constrained Set

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g_i(x^*, \omega) \geq 0, i = 1, \dots, n_g\}$

Risk Aware Optimization

Chance Constrained Optimization

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}\left(\ g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g\ \right) \geq 1 - \Delta$$

Deterministic optimization:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad x \in \chi_{cc}$$

- Find set of design parameters χ_{cc} such that

For any $x^* \in \chi_{cc}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega \geq 1 - \Delta$

Chance Constrained Set

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g_i(x^*, \omega) \geq 0, i = 1, \ldots, n_g\}$

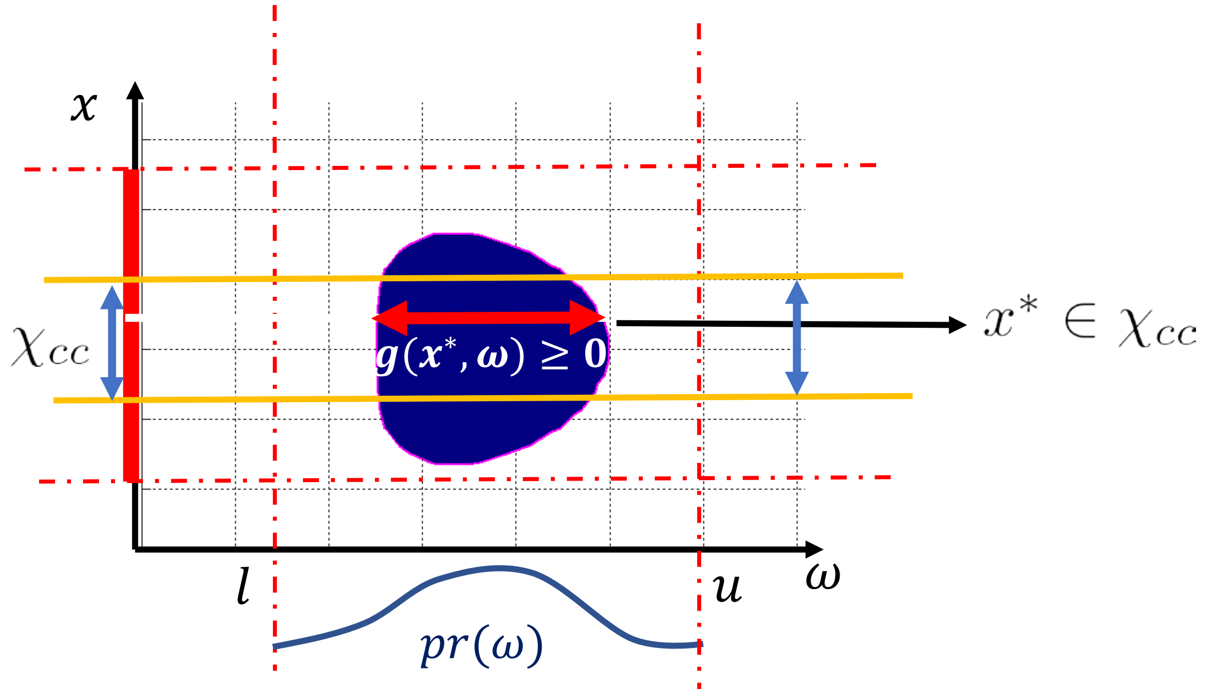
Example: Probabilistic Safety Constraint

Design parameter x should satisfy the probabilistic safety constraint:

- ***Success Set*** $= \{g(x,\omega) \geq 0\}$

$$g(x,\omega) = 0.5\ \omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2 (x-0.5)^2 + (x-0.5)^4\right)$$

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{\text{pr}(\omega)}(\ g(x,\omega) \geq 0\ ) \geq 1-\Delta \end{aligned}$$



$$Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega \geq 1-\Delta$$

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g(x^*,\omega) \geq 0\}$

Risk Aware Optimization

Chance Constrained Optimization

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & \text{Probability}_{\text{pr}(\omega)}\big(\, g_i(x,\omega) \geq 0,\ i = 1,\ldots,n_g \,\big) \geq 1-\Delta \end{array}$$

Deterministic optimization:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & x \in \chi_{cc} \end{array}$$

- Find set of design parameters χ_{cc} such that

For any $x^* \in \chi_{cc}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega \geq 1-\Delta$

Chance Constrained Set

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g_i(x^*,\omega) \geq 0, i = 1, \ldots, n_g\}$

Chance Constrained Set:

- $\{x \in \mathbb{R}^n : Prob(Success) \geq 1-\Delta\} \xrightarrow{\text{semialgebraic set approximation}} \chi_{cc} = \{x \in \mathbb{R}^n : \mathcal{P}(x) \geq 1-\Delta\}$

Topics

- Formulation of Chance Optimization and Chance Constrained Optimization
- Geometrical Interpretation
- Challenges
- Moment Based SDP for Chance Optimization
- Dual of Moment-SDP (Sum-of-Squares Program)
- SOS Based SDP for Chance Constrained Optimization
- Outer and Inner approximations of Chance Constrained Sets

Chance / Chance Constrained Optimization

- **Challenges**

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ ) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i = 1, \ldots, n_g \end{aligned}$$

Objective function and constraints are polynomial functions.

Existing Methods For Chance/Chance Constrained Optimization:

- **Sampling based Approaches**

  A. Nemirovski and A. Shapiro, Scenario approximations of chance constraints, in Probabilistic and Randomized Methods for Design under Uncertainty, Springer, New York, pp. 3–48, 2004.

  R. Tempo, G. Calafiore, and F. Dabbene, Randomized Algorithms for Analysis and Control of Uncertain Systems, Communications and Control Engineering Series, Springer-Verlag, London, 2013.

- **Particular Classes of Constraints and Uncertainties**

  - Linear Chance Constraints with Gaussian Uncertainties

    H. Xu, C. Caramanis, and S. Mannor, Optimization under probabilistic envelope constraints, Oper. Res., 60 pp. 682–699, 2012.

    C. M. Lagoa, X. Li, and M. Sznaier, Probabilistically constrained linear programs and risk-adjusted controller design, SIAM J. Optim., 15 (2005), pp. 938–951.

  - Convex Constraints:

    A. Nemirovski, A. Shapiro , “Convex Approximations of Chance Constrained Programs”, SIAM J. OPTIM., Vol. 17, No. 4, pp. 969–996, 2006.

  - …

In This Lecture

- ➢ We develop a comprehensive approach to address a general class of Chance and Chance Constrained Optimizations.

- ➢ Provided approach deals with:
  - Nonlinear/Linear Chance Constrained/Chance Optimizations.
  - Bonded and Unbounded Probabilistic Uncertainties.

- ➢ Provided approach relies on
  - Measure-Moment SDP
  - Duality of Measure-Moment SDP
  - Sum-Of-Squares Programming

- Ashkan Jasour, Necdet S. Aybat, and Constantino Lagoa, ”Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets”, SIAM Journal on Optimization, 25(3), 1411–1440, 2015.
- Ashkan Jasour, "Convex Approximation of Chance Constrained Problems: Application in Systems and Control", School of Electrical Engineering and Computer Science, The Pennsylvania State University, 2016.

Chance Optimization

$$\underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega)\geq 0,\ i=1,\ldots,n_p\ )$$
$$\text{subject to} \quad g_i(x)\geq 0,\ i=1,\ldots,n_g$$

Chance Constrained Optimization

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}(\ g_i(x,\omega)\geq 0,\ i=1,\ldots,n_g\ )\geq 1-\Delta$$

Objective function and constraints are polynomial functions.

Goal: Find Convex Relaxations

Tools:

i) **Measure (e.g. probability distribution) and Moments**
ii) **Sum-of-Squares Programming**
iii) **Semidefinite Programs (SDP)**

Topics

- Formulation of Chance Optimization and Chance Constrained Optimization
- Geometrical Interpretation
- Challenges
- Moment Based SDP for Chance Optimization
- Dual of Moment-SDP (Sum-of-Squares Program)
- SOS Based SDP for Chance Constrained Optimization
- Outer and Inner approximations of Chance Constrained Sets

Chance Optimization

- ➢ **Moment SDP Formulation**

Chance Optimization

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{maximize}} && \text{Probability}_{\text{pr}(\omega)}(\; p_i(x,\omega) \geq 0, \;\; i = 1, ..., n_p \;) \\ &\text{subject to} && g_i(x) \geq 0, \;\; i = 1, ..., n_g \end{aligned}$$

Tools: i) Measure and Moments ii) Semidefinite Programs

- We will follow the same steps described in Lecture 4 (moment SDP for deterministic Optimization).
- We will consider uncertain parameters at each steps.

Chance Optimization

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0,\;\; i = 1, \ldots, n_p \,\right) \\ \text{subject to} & g_i(x) \geq 0,\;\; i = 1, \ldots, n_g \end{array}$$

Tools: i) Measure and Moments ii) Semidefinite Programs

Step 1: Infinite-dimensional LP

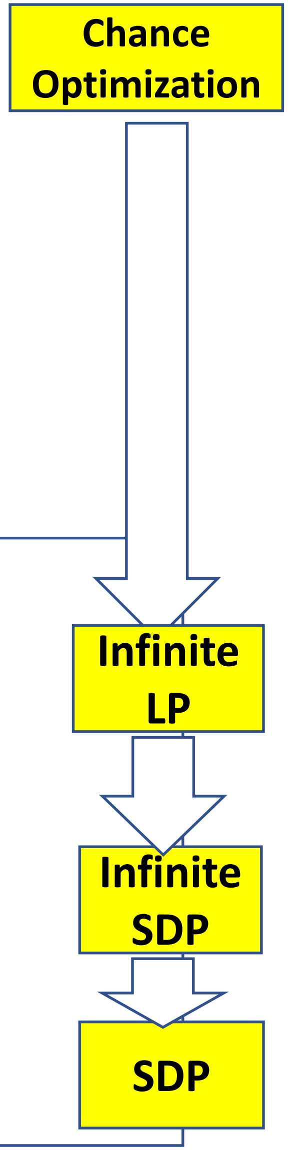
Reformulate **Chance Optimization** problem in terms of **Probability Distributions** (**measures**)

Step 2: Infinite-dimensional SDP

Reformulate **Problem of Step 1** in terms of **Moments (higher order statistics)**

Step 3: Finite SDP

Truncate matrices of **SDP of Step 2 (truncated moments)**



Review of Measure and Moment Approach for Nonlinear Optimization

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization
  $\Omega = \mathbb{R}^n$
- Constrained Optimization
  $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i = 1, ..., m\}$

Review of Measure and Moment Approach for Nonlinear Optimization

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Optimization in terms of Probability distributions (measures):

- treat x as a random variable.
- **Decision variable:** μ Probability measure associated with x

Review of Measure and Moment Approach for Nonlinear Optimization

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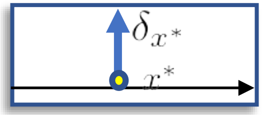
Optimization in terms of Probability distributions (measures):

- treat x as a random variable.
- **Decision variable:** μ Probability measure associated with x
- Infinite dimensional Linear Program in measures
  - Optimal solution:

$x^* \in \mathbf{\Omega}$, $p(x^*) = \mathrm{P}^*$

Unique global optimal solution of the original problem.

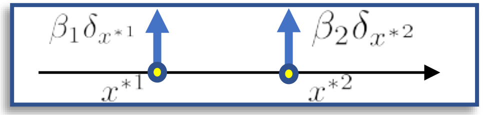
$$\mu^* = \delta_{x^*}$$



$x^{*i} \in \mathbf{\Omega}, \quad i = 1, ..., r$, $p(x^{*i}) = \mathrm{P}^*$

r global optimal solution of the original problem.

$$\mu^* = \sum_{i=1}^{r} \beta_i \delta_{x^{*i}}, \ \ \beta_i > 0, \sum_{i=1}^{r} \beta_i = 1$$



Review of Measure and Moment Approach for Nonlinear Optimization

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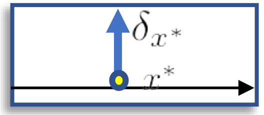

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r global optimal solution of the original problem.

$\mu^* = \sum_{i=1}^{r} \beta_i \delta_{x^{*i}}, \ \beta_i > 0, \sum_{i=1}^{r} \beta_i = 1$

Optimization in Truncated Moment Space

- ➢ Approximate measure with a **finite** moment sequence. $y_\alpha = E_\mu[x^\alpha]$
- ➢ Moment representation of a measure supported on $\mathbf{K}$ $\longrightarrow$

Moment Matrix
$\mathbf{M}_d(y) \succeq 0$

Localizing Matrix
$\mathbf{M}_d(g_i y) \succeq 0$

Review of Measure and Moment Approach for Nonlinear Optimization

$$\mathrm{P}^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

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- Constrained Optimization
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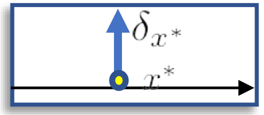

$x^{*i} \in \mathbf{\Omega}, \quad i = 1, ..., r$, $p(x^{*i}) = \mathrm{P}^*$
r global optimal solution of the original problem.

$\mu^* = \sum_{i=1}^{r} \beta_i \delta_{x^{*i}}, \ \ \beta_i > 0, \sum_{i=1}^{r} \beta_i = 1$

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- ➢ Approximate measure with a **finite** moment sequence. $y_\alpha = \mathrm{E}_\mu[x^\alpha]$
- ➢ Moment representation of a measure supported on $\mathbf{K}$ $\longrightarrow$
- ➢ Optimal solution is the moment sequence of Dirac measures.

Moment Matrix
$\mathbf{M}_d(y) \succcurlyeq 0$

Localizing Matrix
$\mathbf{M}_d(g_i y) \succcurlyeq 0$

Review of Measure and Moment Approach for Nonlinear Optimization

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

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- ➢ Infinite dimensional Linear Program in measures
  - Optimal solution: $x^* \in \mathbf{\Omega}$, $p(x^*) = \mathrm{P}^*$

    Unique global optimal solution of the original problem.

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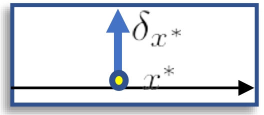

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Optimization in Truncated Moment Space

- ➢ Approximate measure with a **finite** moment sequence. $y_\alpha = \mathrm{E}_\mu[x^\alpha]$
- ➢ Moment representation of a measure supported on $\mathbf{K}$ $\longrightarrow$ Moment Matrix $\mathbf{M}_d(y) \succcurlyeq 0$ Localizing Matrix $\mathbf{M}_d(g_i y) \succcurlyeq 0$
- ➢ Optimal solution is the moment sequence of Dirac measures.
- ➢ **Rank condition** to identify the moments of Dirac measures. (**Finite Convergence**)
- ➢ We can **extract** x^* from the **moments** of **Dirac** measure.

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0,\ \ i = 1,\ldots,n_p \,\right) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i = 1,\ldots,n_g \end{aligned}$$

Tools: i) Measure and Moments ii) Semidefinite Programs

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **Measures**

Step 2: Infinite-dimensional SDP

Reformulate **Problem of Step 1** in terms of **Moments (higher order statistics)**

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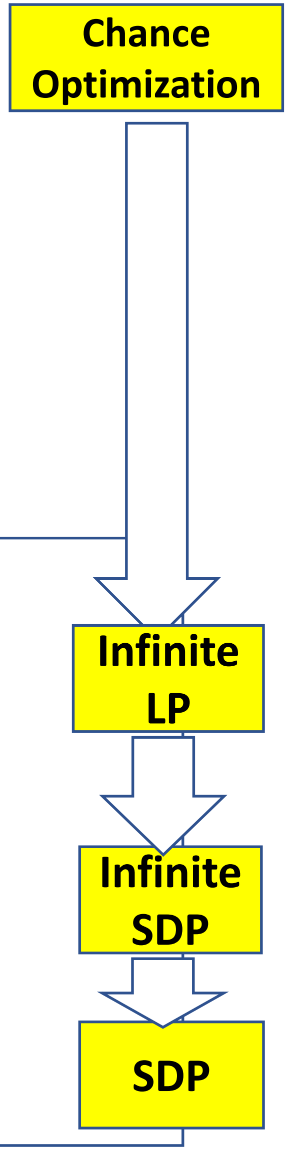
Truncate matrices of **SDP of Step 2 (truncated moments)**



Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

$$
\begin{aligned}
\mathbf{P}^* = &\underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0, \;\; i = 1, \ldots, n_p \, \right) \\
&\text{subject to} \quad g_i(x) \geq 0, \;\; i = 1, \ldots, n_g
\end{aligned}
$$

- We treat deign variables x as random variables.
- Instead of looking for “x”, we look for its probability measure.
- Two Sets of probability measures:
  - Unknown Probability measure of Design parameters: $x \sim \mu_x$
  - Given Probability measure of uncertain parameters: $\omega \sim \mu_\omega$

➢ Equivalent optimization in terms of μ_x and μ_ω

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

$$\begin{aligned} \mathbf{P}^* = & \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega)\geq 0,\ \ i=1,\ldots,n_p\ ) \\ & \text{subject to} \quad g_i(x)\geq 0,\ \ i=1,\ldots,n_g \end{aligned}$$

Given Success Set in (x,ω) Space:

- $\mathcal{K}=\{(x,\omega)\in\mathbb{R}^n\times\mathbb{R}^m:\ p_i(x,\omega)\geq 0,\ \ i=1,\ldots,n_p\ \}$

Given Feasible Set:

$$\chi=\{x\in\mathbb{R}^n: g_i(x)\geq 0,\ \ i=1,\ldots,n_g\}$$

Assumption:

- Sets $\mathcal{K}$ and χ are **Archimedean** (Compact).
- and with out loss of generality (after rescaling of polynomials) $\mathcal{K}\subset[-1,1]^{n+m}$ $\chi\subset[-1,1]^n$

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

$$\begin{aligned} \mathbf{P}^* = & \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i=1,\dots,n_p\ ) \\ & \text{subject to} \quad g_i(x) \geq 0,\ \ i=1,\dots,n_g \end{aligned}$$

Given Success Set in (x, ω) Space:

- $\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ p_i(x,\omega) \geq 0,\ \ i=1,\dots,n_p\ \}$

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- and with out loss of generality (after rescaling of polynomials) $\underbrace{\mathcal{K} \subset [-1,1]^{n+m} \quad \chi \subset [-1,1]^n}_{\text{To avoid numerical issues in solving SDPs}}$

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

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- and with out loss of generality (after rescaling of polynomials) $\mathcal{K} \subset [-1,1]^{n+m}$ $\chi \subset [-1,1]^n$

We will revisit this assumption (Noncompact Sets: Page 99)

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

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Infinite dimensional Linear Program in Measures

μ: Slack Measure
μ_x: Probability Measure Assigned to x
μ_ω: Known Probability Measure of ω
$\mu_x \times \mu_\omega$: joint measure of x and ω

$$\begin{aligned} \mathbf{P}^*_\mu := \; & \text{maximize}_{\mu_x,\mu} \int d\mu, \\ \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \text{(Upper bound measure)} \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

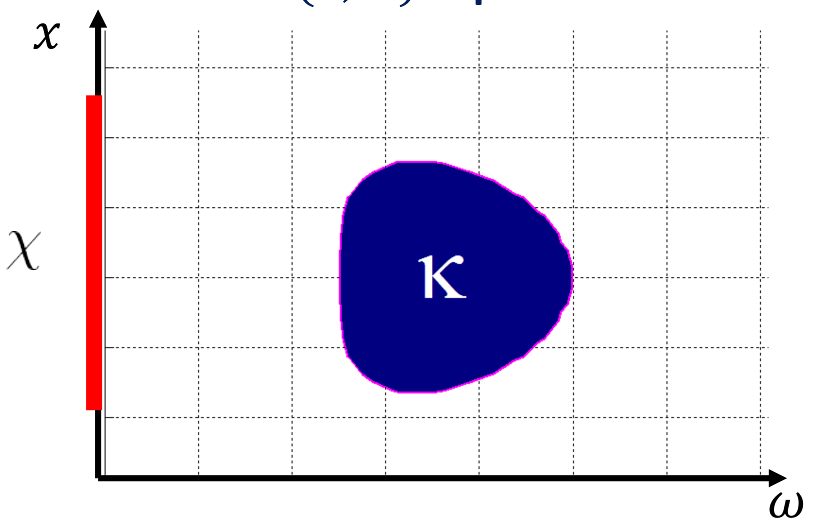
Interpretation of Measure LP:

Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ )$$
$$\text{subject to} \quad g_i(x) \geq 0,\ \ i = 1, \ldots, n_g$$

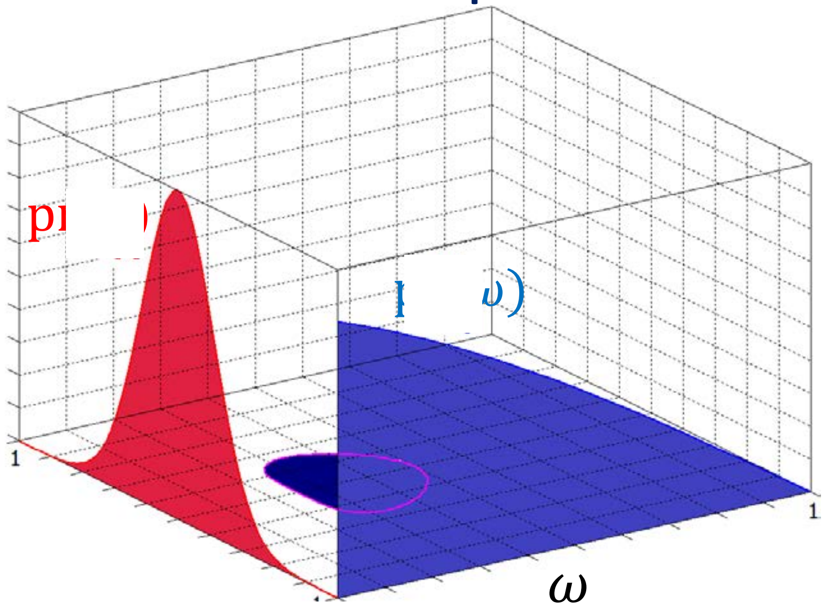
$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ \ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ \}$ $\quad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ \ i = 1, \ldots, n_g\}$

(x, ω) sapce



ω

Measure space



- Measure μ_x with pdf $\text{pr}(x)$
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- **Suppose** that measure μ_x assigned to the x is **given**:

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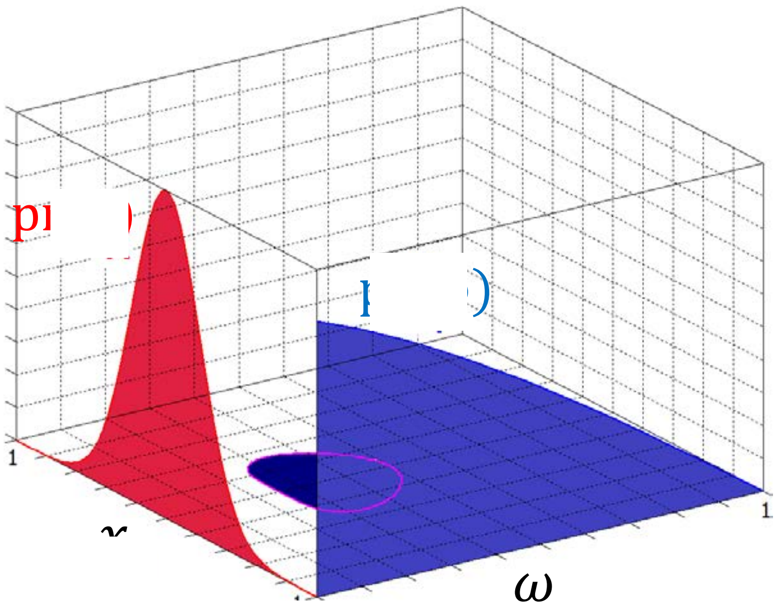
$$\text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ )$$



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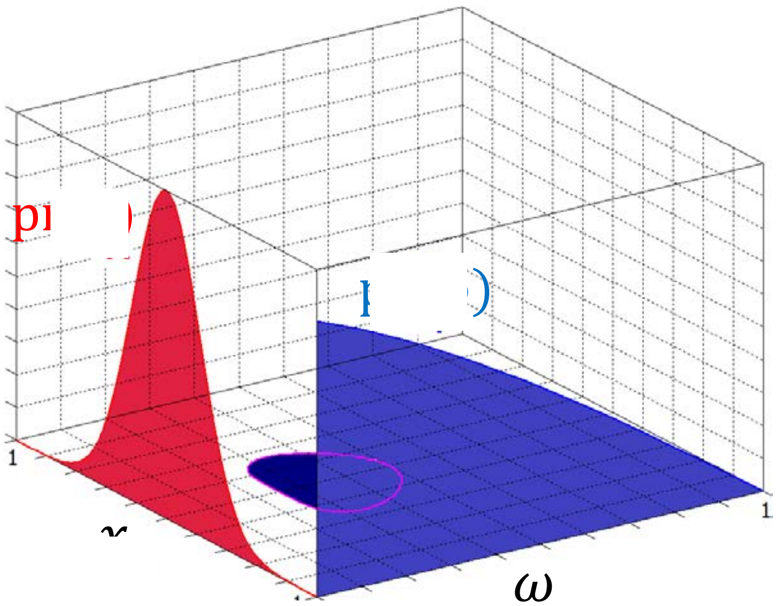
Joint probability measure of (x, ω)



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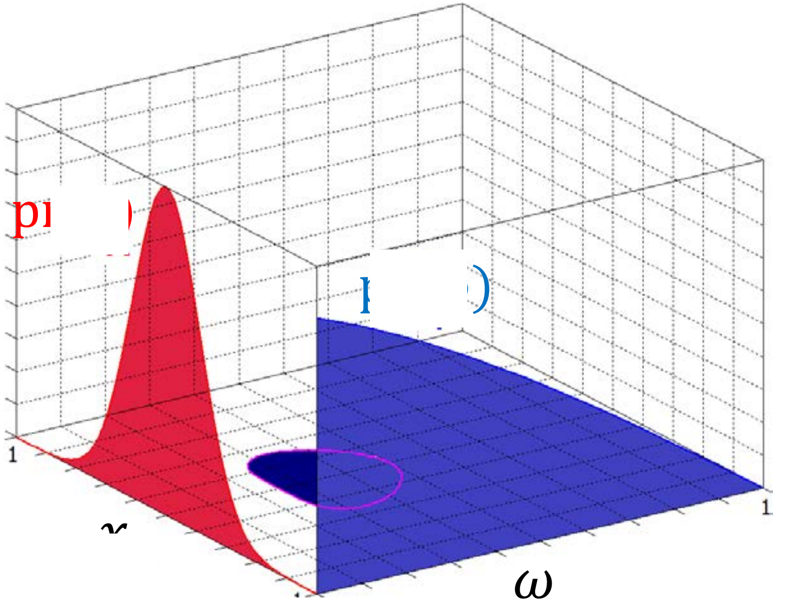

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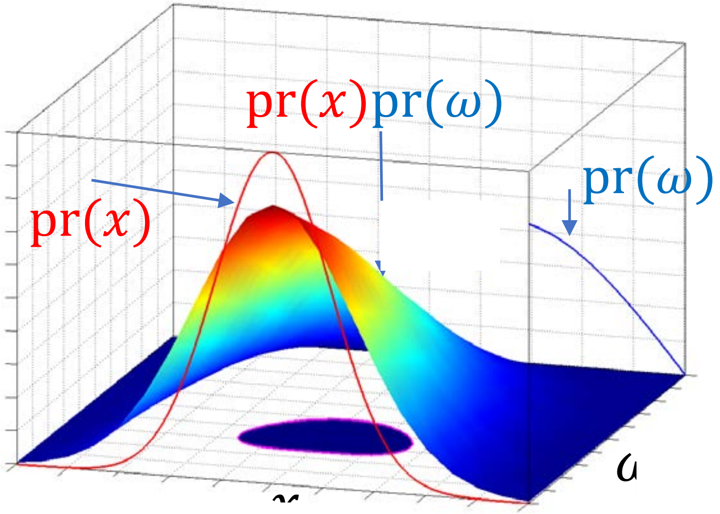

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- Measure $\mu_x \times \mu_\omega$ with pdf $\text{pr}(x)\text{pr}(\omega)$
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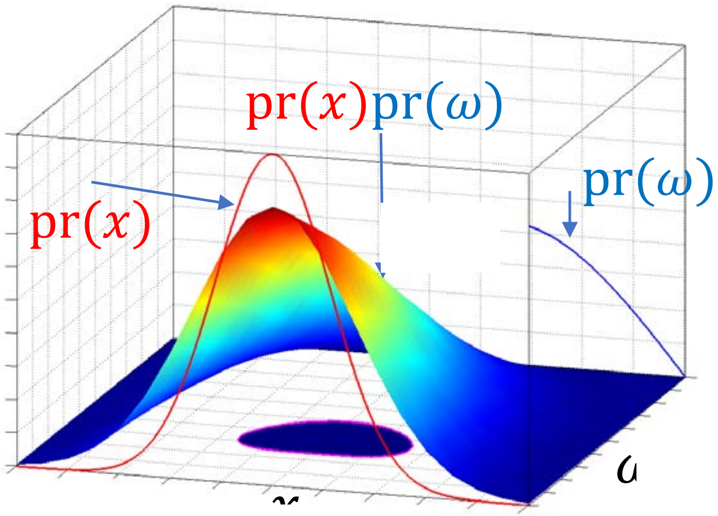

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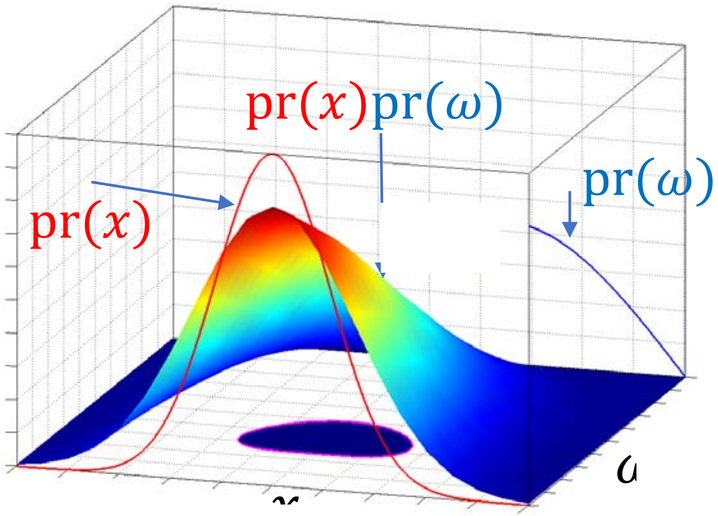

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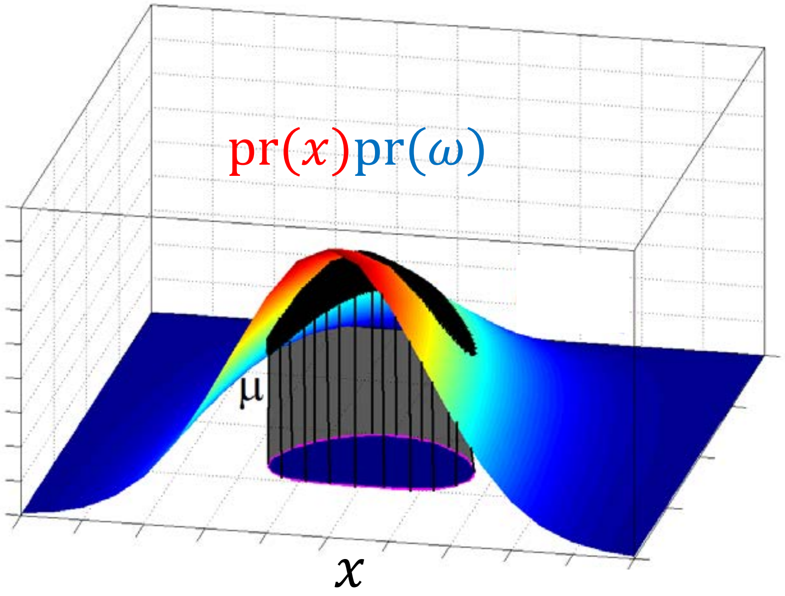

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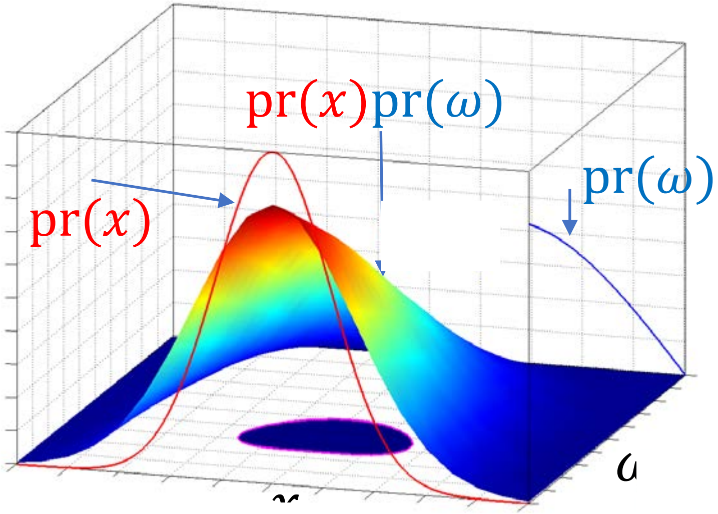

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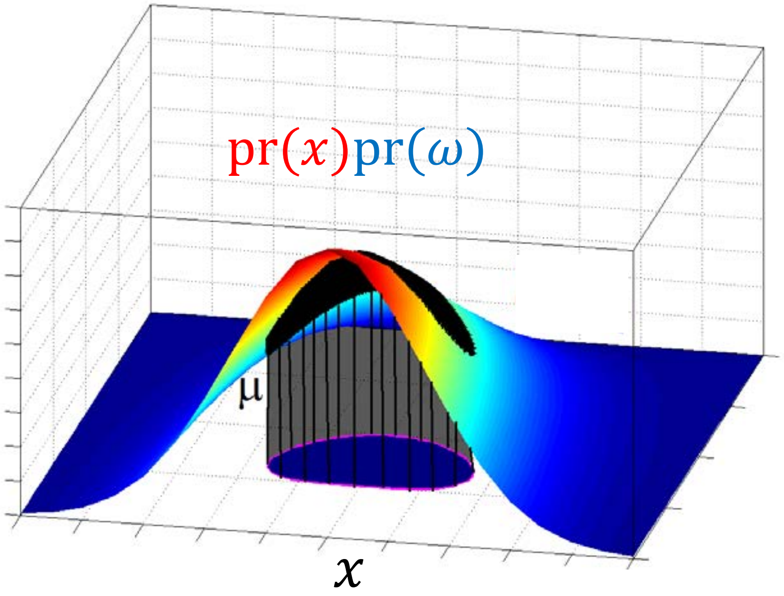

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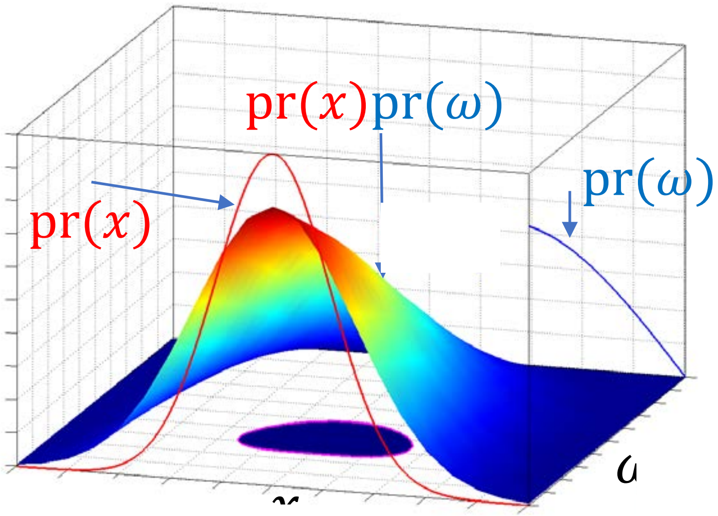

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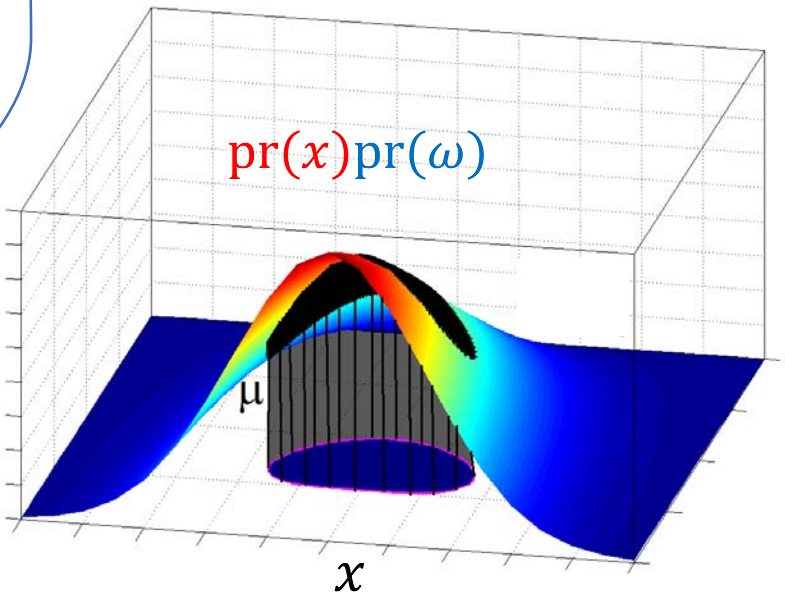
$$\boxed{= \int_{\mathcal{K}} \underbrace{d\mu}_{\mu_x \times \mu_\omega}} + \underbrace{\int_{\text{complement of set } \mathcal{K}} d\mu}_{\substack{=0 \\ supp(\mu) \subset \mathcal{K}}}$$



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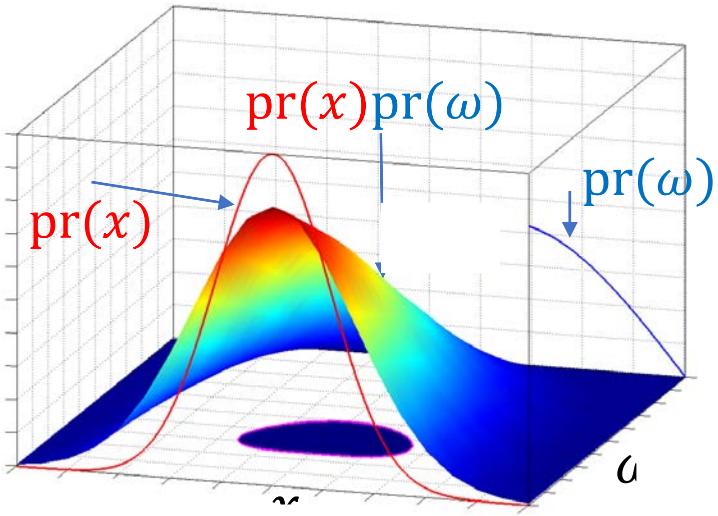

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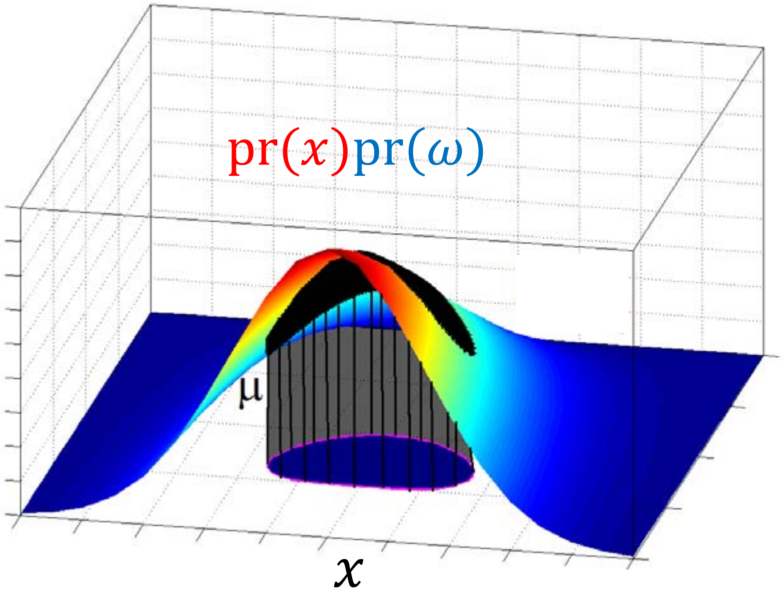

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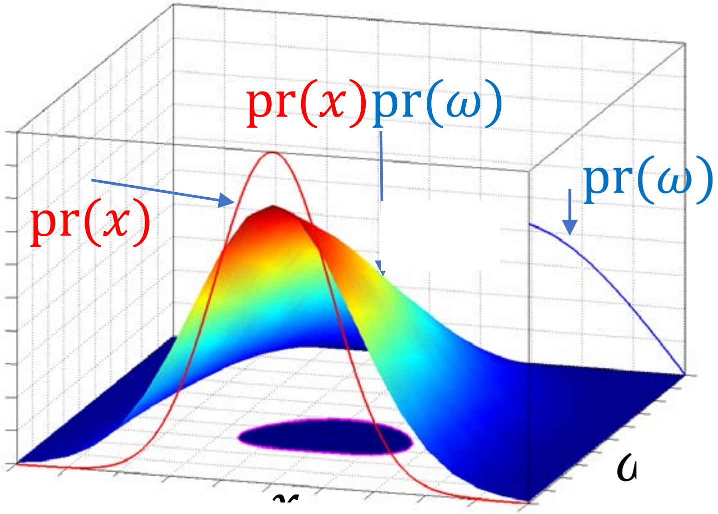

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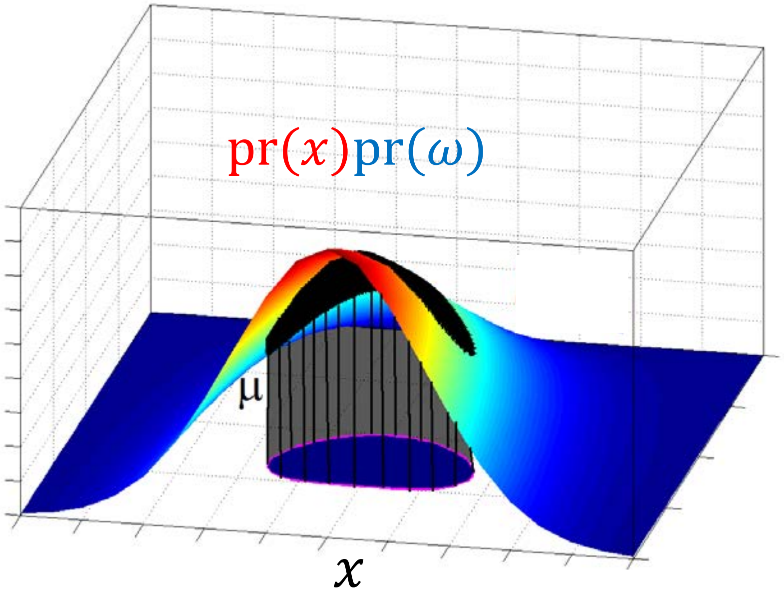
y_0 : zero-order moment of μ



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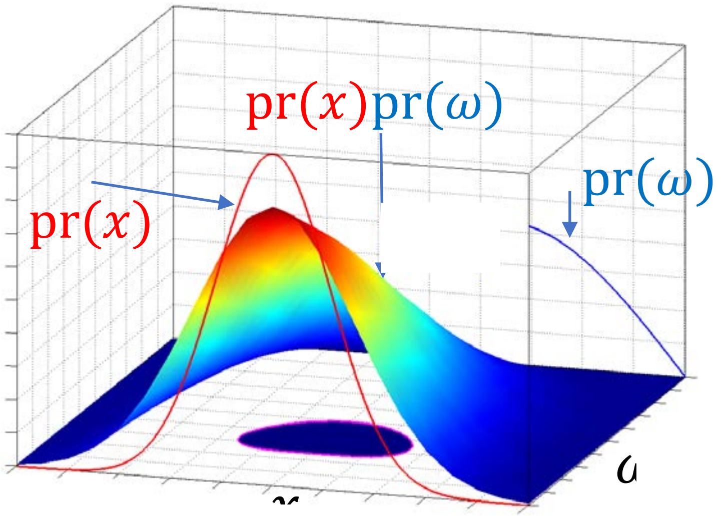

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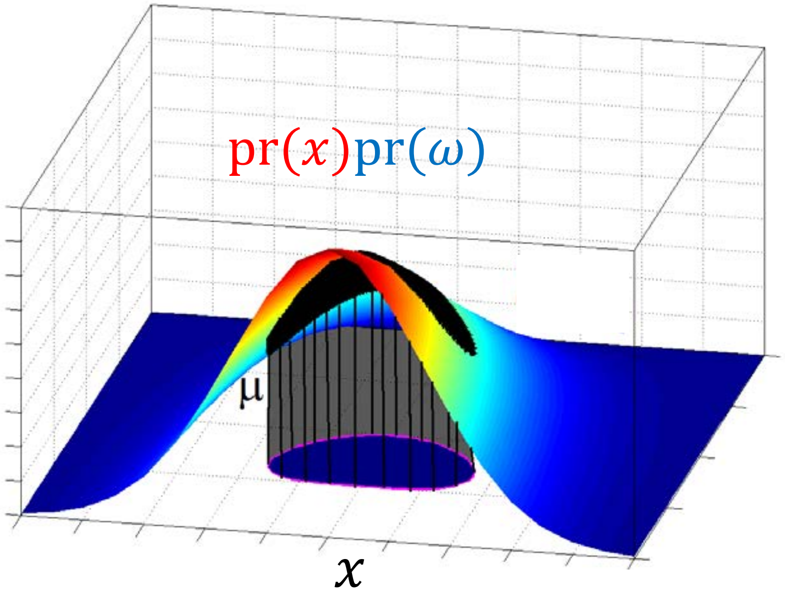

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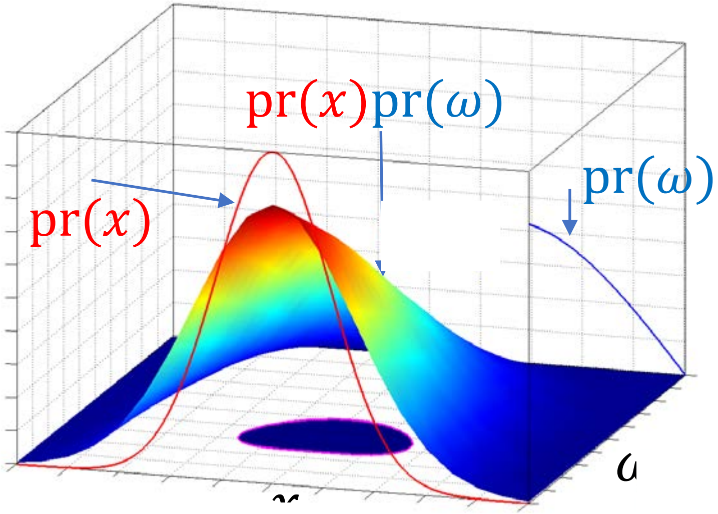

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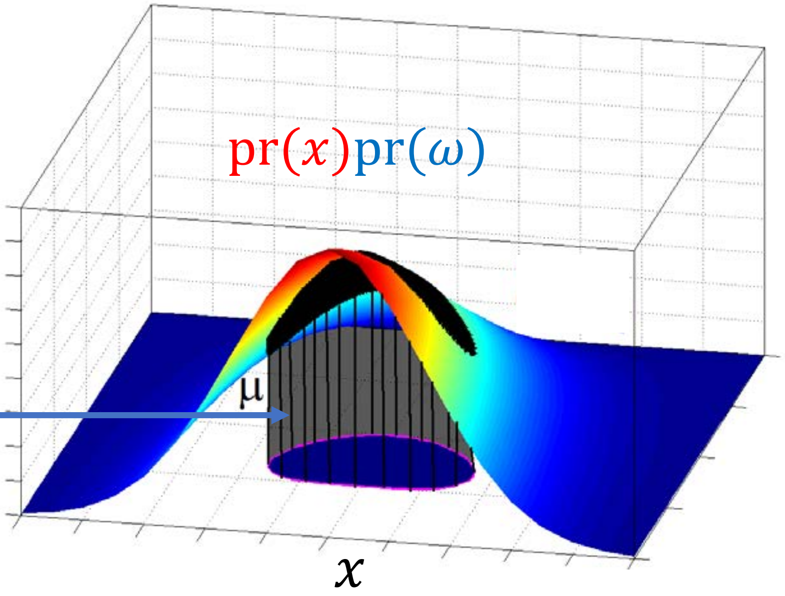
- To construct such **measure μ**, we solve the following optimization

$$\max_{\mu} \int d\mu$$

$$\mu \preccurlyeq \mu_x \times \mu_\omega \quad supp(\mu) \in \mathcal{K}$$

Optimal Solution



➢ **To maximize the probability of success:**

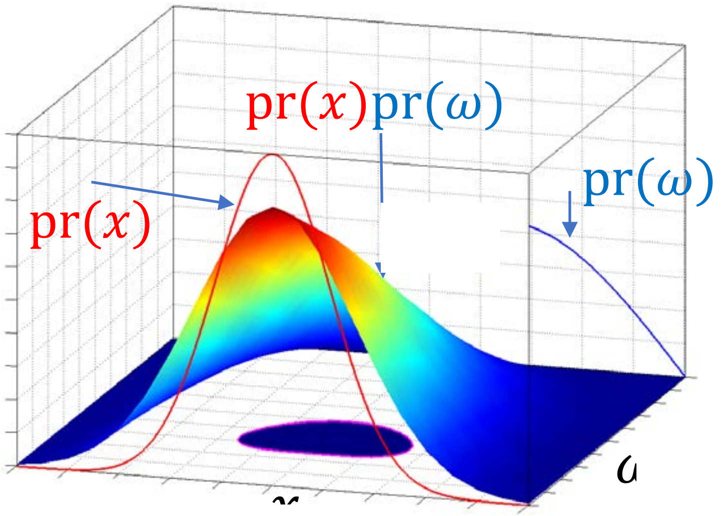
,at the same time, we look for measure $\boldsymbol{\mu}$ and measure $\boldsymbol{\mu_x}$

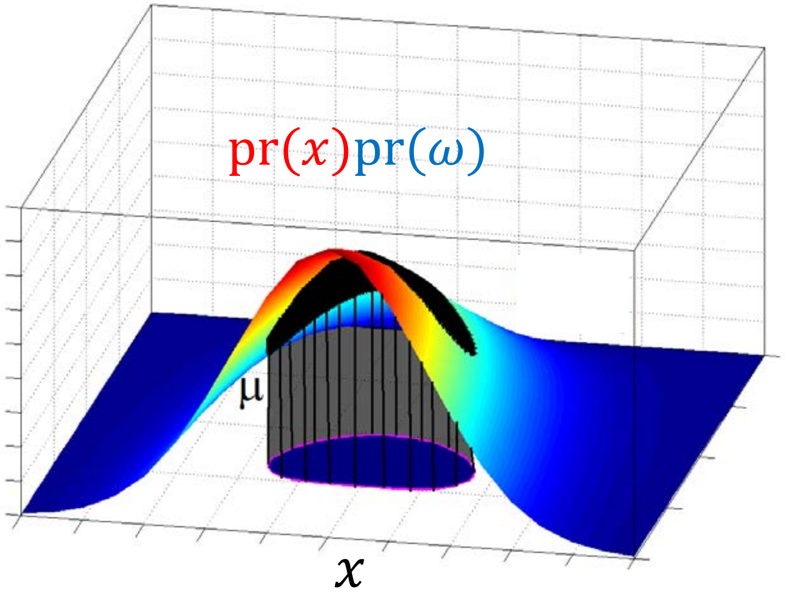
$$\text{maximize Probability}_{\mathrm{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ ) =$$

$$\max_{\mu, \boxed{\mu_x}} \int d\mu$$

$$\mu \preccurlyeq \mu_x \times \mu_\omega \quad supp(\mu) \in \mathcal{K}$$

$$\boxed{\begin{array}{l} \mu_x \text{ is a probability measure} \\ supp(\mu_x) \subset \chi \end{array}}$$





Step 1: Infinite-dimensional LP

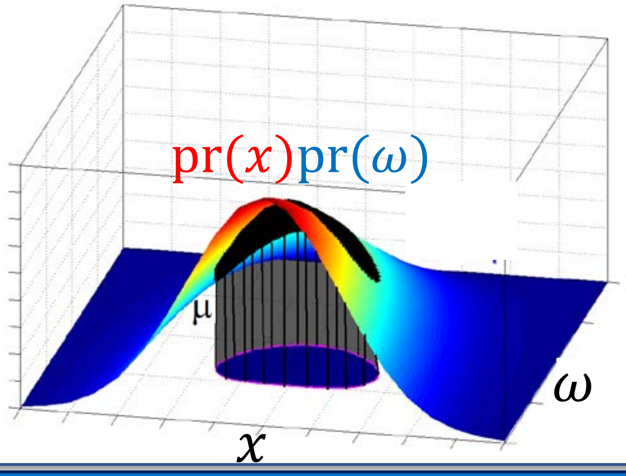
Reformulate **Chance Optimization** problem in terms of **measures**

$$\begin{aligned} \mathbf{P}^* = &\underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(\; p_i(x,\omega) \geq 0, \;\; i=1,\ldots,n_p \; \right) \\ &\text{subject to} \quad g_i(x) \geq 0, \;\; i=1,\ldots,n_g \end{aligned}$$

- $\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \;\; p_i(x,\omega) \geq 0, \;\; i=1,\ldots,n_p \; \}$ $\qquad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i=1,\ldots,n_g\}$

Infinite dimensional Linear Program in Measures

$$\begin{aligned} \mathbf{P}^*_\mu := \; & \text{maximize}_{\mu_x,\mu} \int d\mu, \\ \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$



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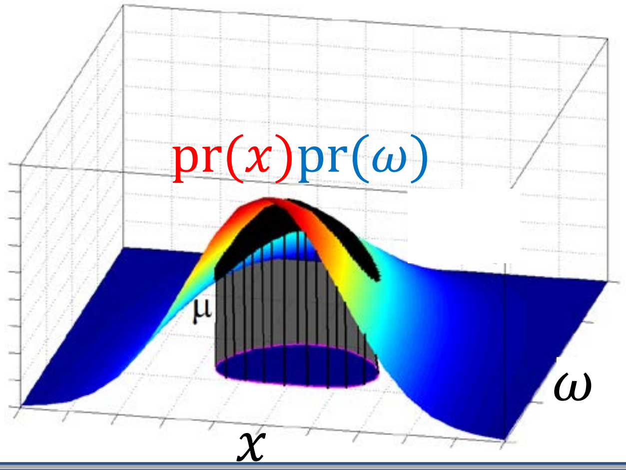

μ: Responsible for Probability Estimation Problem

μ_x: Responsible for Design Problem

1 Chance Optimization

$$\mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, ..., n_p\ )$$

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Equivalent

2 Equivalent Problem in the measure space:

Infinite dimensional LP

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Theorem: The optimization problems in (1) and (2) are equivalent in the following sense (**Appendix I**):

- Theorem 3.1 : A. Jasour, N. S. Aybat, and C. Lagoa ”*Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets*”, SIAM Journal on Optimization, 25(3), 1411–1440, 2015.
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  Then $\mu^* = \delta_{x^*}$ is unique optimal solution of optimization in measures.
- If $x^{*i} \in \chi, \quad i=1,\ldots,r$ are "r" global optimal solution of the original problem,
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3 Equivalent Problem in the moment space:

2 Equivalent Problem in the measure space:

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$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$

μ_x is a probability measure

$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

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$$\chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, ..., n_g\}$$

3 Moment Representation of Measures:

Measure:

$$\mu_x \qquad supp(\mu_x) \in \chi \qquad \int d\mu_x = 1$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y_x) \succcurlyeq 0 \qquad \mathbf{M}_\infty(g_i y_x) \succcurlyeq 0, \; |_{i=1}^{n_g} \qquad y_{x_0} = 1$$

Measure:

$$\mu \qquad supp(\mu) \in \mathcal{K}$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y) \succcurlyeq 0 \qquad \mathbf{M}_\infty(p_i y) \succcurlyeq 0, |_{i=1}^{n_p}$$

Measure:

$$\underbrace{\mu_x \times \mu_\omega - \mu}_{\text{(nonnegative) measure}} \succcurlyeq 0$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y_x \times y_\omega - y) \succcurlyeq 0$$

moments of joint measure $\mu_x \times \mu_\omega$

moments of μ

1 Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ i = 1, ..., n_p\ )$$
$$\text{subject to} \quad g_i(x) \geq 0,\ i = 1, ..., n_g$$

Equivalent

2 Equivalent Problem in the measure space:

Infinite dimensional LP

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_x, \mu} \int d\mu,$$
$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$
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Equivalent

3 Equivalent Problem in the moment space:

$$\mathbf{P}^*_{\text{mom}} := \text{maximize}_{y, y_x} \quad y_0$$
$$\text{s.t.} \quad M_\infty(\mathbf{y}) \succcurlyeq 0,\ M_\infty(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p$$
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Lemma 3.2 : A. Jasour, N. S. Aybat, and C. Lagoa "*Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets*", SIAM Journal on Optimization, 25(3), 1411–1440, 2015.

3 **Equivalent Problem in the moment space:**

Infinite dimensional SDP

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\end{aligned}$$

Relaxation

4 **Finite SDP in moments:**

Truncated moment SDP
in terms of moment up to order $2d$

$$\begin{aligned}
\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := &\text{maximize}_{y,y_x} \quad y_0 \\
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Theorem: $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \geq \mathbf{P}^*$

Upper bound of optimal objective function of chance optimization

$$\underbrace{\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \geq \mathbf{P}^{*\mathbf{d+1}}_{\mathbf{mom}} \quad lim_{d\to\infty} \mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} = \mathbf{P}^*}_{\text{monotonically converges}}$$

- Theorem 3.3 : A. Jasour, N. S. Aybat, and C. Lagoa "*Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets*", SIAM Journal on Optimization, 25(3), 1411–1440, 2015.
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Relaxation

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Truncated moment SDP
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Infinite dimensional SDP

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Relaxation

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- If $x^* \in \chi$ is unique global optimal solution of the original problem, Then moments of $\mu^* = \delta_{x^*}$ is unique optimal solution of optimization in moments. (Moment Rank Condition for **finite convergence** of x^*)
- If $x^{*i} \in \chi, \quad i = 1, \ldots, r$ are "r" global optimal solution of the original problem, Then moments of $\mu^* = \sum_{i=1}^r \beta_i \delta_{x^{*i}},\ \beta_i > 0$ is unique optimal solution of optimization in moments. (Moment rank Condition for **finite convergence** of x^*)

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Results for Probability Estimation Problem

Results for Design Problem

- If $x^* \in \chi$ is unique global optimal solution of the original problem, Then moments of $\mu^* = \delta_{x^*}$ is unique optimal solution of optimization in moments. (Moment Rank Condition for **finite convergence** of x^*)
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1 **Chance Optimization**

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(p_i(x,\omega) \geq 0, \; i = 1, ..., n_p \right)$$
$$\text{subject to} \quad g_i(x) \geq 0, \; i = 1, ..., n_g$$

2 **Equivalent Problem in the measure space:**

Infinite dimensional LP

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x, \mu} \int d\mu,$$
$$\begin{aligned} \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

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Infinite dimensional SDP

$$\mathbf{P}^*_{\mathbf{mom}} := \text{maximize}_{y, y_x} \quad y_0$$
$$\begin{aligned} \text{s.t.} \quad & M_\infty(\mathbf{y}) \succcurlyeq 0, \; M_\infty(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p \\ & M_\infty(y_x) \succcurlyeq 0, \; M_\infty(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, n_g, \; y_{x_0} = 1 \\ & M_\infty(y_\omega \times y_x - y) \succcurlyeq 0 \end{aligned}$$

Relaxation

4 **Finite SDP in moments:**

Truncated moment SDP

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y, y_x} \quad y_0$$
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Chance Optimization

$$\mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, ..., n_p\ )$$
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Moment Relaxation (SDP)

$$\begin{aligned}
\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} :=&\text{maximize}_{y,y_x} \quad y_0 \\
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\end{aligned}$$

$d_{g_i} = \lceil \frac{deg(g_i(x))}{2} \rceil \ \ d_{p_i} = \lceil \frac{deg(p_i(x))}{2} \rceil$

$2d \geq \max\,(deg(p_i(x)), deg(g_i(x)))$

- As $d \to \infty$ $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \downarrow \mathrm{P}^*$
- Finite SDP of order d: $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}}$ is an upper bound of P^*
  - ➢ If obtained solution y_x^* satisfies rank condition $\text{Rank } M_d(y_x^*) = \text{Rank } M_{d-v}(y_x^*) = r \quad v = \max\{d_{g_i}\}$

    $x_i^*, i = 1, ..., r,$ global solutions can be extracted by linear algebra from y_x^*
  - ➢ Otherwise, increase d and solve new SDP.

- A. Jasour ,"Finite Convergence of Moment-SDP Hierarchy for Nonlinear Chance Constrained Optimization"(to appear)

Example: Probabilistic Safety Constraint

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \qquad -1 \leq x \leq 1$$

$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$

$\omega \sim Uniform[-1,1]$

Measure LP:

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
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Moment SDP:

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y,y_x} \quad y_0$$
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Moment vector of measure μ $\quad y = [y_{00}|y_{10}, y_{01}|y_{20}, y_{11}, y_{02}|y_{30}, y_{21}, y_{12}, y_{03}|y_{40}, y_{31}, y_{22}, y_{13}, y_{04}]$

Moment vector of measure μ_x $\quad y_x = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}]$

Moment vector of measure μ_ω $\quad y_\omega = [1, y_{\omega_1}, y_{\omega_2}, y_{\omega_3}, y_{\omega_4}] = [1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0]$

Moment vector of measure $\mu_x \times \mu_\omega$

$$\begin{aligned} y_x y_\omega &= \mathrm{E}[1, x, \omega, x^2, x\omega, \omega^2, x^3, x^2\omega, x\omega^2, \omega^3, x^4, x^3\omega, x^2\omega^2, x\omega^3, \omega^4] \\ &= [1, y_{x_1}, y_{\omega_1}, y_{x_2}, y_{x_1}y_{\omega_1}, y_{\omega_2}, y_{x_3}, y_{x_2}y_{\omega_1}, y_{x_1}y_{\omega_2}, y_{\omega_3}, y_{x_4}, y_{x_3}y_{\omega_1}, y_{x_2}y_{\omega_2}, y_{x_1}y_{\omega_3}, y_{\omega_4}] \\ &= [1, y_{x_1}, 0, y_{x_2}, 0, \tfrac{1}{3}1, y_{x_3}, 0, \tfrac{1}{3}y_{x_1}, 0, y_{x_4}, 0, \tfrac{1}{3}y_{x_2}, 0, \tfrac{1}{5}1] \end{aligned}$$

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Moment vector of measure μ_x $\quad y_x = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}]$

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Moment vector of measure $\mu_x \times \mu_\omega$

$$\begin{aligned} y_x y_\omega &= \mathrm{E}[1, x, \omega, x^2, x\omega, \omega^2, x^3, x^2\omega, x\omega^2, \omega^3, x^4, x^3\omega, x^2\omega^2, x\omega^3, \omega^4] \\ &= [1, y_{x_1}, y_{\omega_1}, y_{x_2}, y_{x_1}y_{\omega_1}, y_{\omega_2}, y_{x_3}, y_{x_2}y_{\omega_1}, y_{x_1}y_{\omega_2}, y_{\omega_3}, y_{x_4}, y_{x_3}y_{\omega_1}, y_{x_2}y_{\omega_2}, y_{x_1}y_{\omega_3}, y_{\omega_4}] \\ &= [1, y_{x_1}, 0, y_{x_2}, 0, \tfrac{1}{3}1, y_{x_3}, 0, \tfrac{1}{3}y_{x_1}, 0, y_{x_4}, 0, \tfrac{1}{3}y_{x_2}, 0, \tfrac{1}{5}1] \end{aligned}$$

Moment SDP

$$\max_{y,y_x} y_{00}$$

$$M_2(y) \succeq 0 \Rightarrow \left(\begin{array}{c|cc|ccc} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right) \succeq 0$$

$$M_1(py) \succeq 0 \Rightarrow -y_{04} + \frac{1}{2}y_{03} - y_{22} + y_{12} - \frac{1}{4}y_{02} + \frac{1}{2}y_{21} - \frac{1}{2}y_{11} + \frac{1}{8}y_{01} - y_{40} + 2y_{30} - \frac{3}{2}y_{20} + \frac{1}{2}y_{10} - \frac{1}{16} \succeq 0$$

$$M_2(y_x y_q) - M_2(y) = \succeq 0 \Rightarrow \begin{pmatrix} 1 & y_{x1} & 0 & y_{x2} & 0 & 1/3 \\ y_{x1} & y_{x2} & 0 & y_{x3} & 0 & 1/3y_{x1} \\ 0 & 0 & 1/3 & 0 & 1/3y_{x1} & 0 \\ y_{x2} & y_{x3} & 0 & y_{x4} & 0 & 1/3y_{x2} \\ 0 & 0 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 \\ 1/3 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 & 2/5 \end{pmatrix} - M_2(y) = \succeq 0$$

$$M_2(y_x) \succeq 0 \Rightarrow \begin{pmatrix} y_{x_0} & y_{x_1} & y_{x_2} \\ y_{x_1} & y_{x_2} & y_{x_3} \\ y_{x_2} & y_{x_3} & y_{x_4} \end{pmatrix} \succeq 0 \qquad y_{x_0} = 1 \qquad |y_x| \leq 1$$

Implementation using GloptiPoly

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_Moment_ChanceOpt

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

$$\omega \sim Uniform[-1,1]$$

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
$$\text{s.t.} \quad \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega}$$
$$\mu_x \text{ is a probability measure}$$
$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
$$\text{s.t.} \quad \boxed{\mu + \mu_s = \mu_x \times \mu_\omega}$$
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μ_s (nonnegative) measure

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$\omega \sim Uniform[-1,1]$

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\boxed{\mu \preccurlyeq \mu_x \times \mu_\omega}$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\boxed{\mu + \mu_s = \mu_x \times \mu_\omega}$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

μ_s (nonnegative) measure

```
nx=1, nq=1
```
x: design parameter, q: uncertain parameter

Measure μ:
```
mpol('x',nx); mpol('q',nq); mu = meas([x;q]); y=mom(mmon([x;q],2*d));
```
Space x, q — μ — Moments of μ up to order 2d

```
K=[0.5*q(1)*(q(1)^2+(x(1)-0.5)^2)-(q(1)^4+q(1)^2*(x(1)-0.5)^2+(x(1)-0.5)^4)];
```
Set K: Support of measure μ

Measure μ_x:
```
mpol('x2',nx);mux= meas([xm]); yx=mom(mmon([x2],2*d));
```
Space x — μ_x — Moments of μ_x up to order 2d

Measure μ_s:
```
mpol('x_s',nx); mpol('q_s',nq); mu_s = meas([x_s;q_s]); y_s=mom(mmon([x_s;q_s],2*d));
```
Space x, q — μ_s — Moments of μ_s up to order 2d

Moments of uncertain parameter ω (uniform distribution):
```
yq=[1 0  0.3333   0  0.2]
```

Moments of $\mu_x \times \mu_\omega$:
```
yxq=  mom_xq(nx,nq,d,yq,yx);
```

Construct moment SDP from measures:
```
Opt=msdp(max(mass(mu)),mass(mux)==1,K>=0,y_s==(yxq - y),-1<=yx,yx<=1,d);      msol(Opt);
```
max $\int d\mu$ — $\int d\mu_x = 1$ — Support of μ — $\mu_s = \mu_x \times \mu_\omega - \mu$ — $supp(\mu_x) \subset [-1,1]$ — Solve moment SDP

Obtained result:

$$y = [0.6610,\ 0.3305,\ 0.1484,\ 0.1687,\ 0.0742,\ 0.1022,\ 0.0878,\ 0.0387,\ 0.0511,\ 0.0406,\ 0.0465,\ 0.0210,\ 0.0263,\ 0.0203,\ 0.0203]$$

Upper bound of probability of success

$$y_x = [1,\ 0.5,\ 0.2558,\ 0.1330,\ 0.8521]$$

Rank Test:

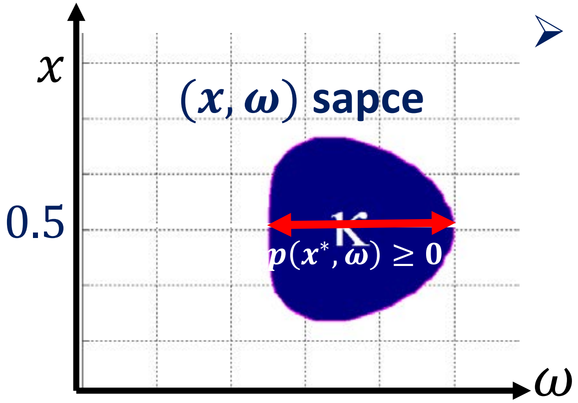
$$M_2(y_x) = \begin{pmatrix} 1.0000 & 0.5 & 0.2558 \\ 0.5000 & 0.2558 & 0.1330 \\ 0.2558 & 0.1330 & 0.8521 \end{pmatrix} \longrightarrow \text{Rank} \approx 1 \longleftarrow M_1(y_x) = \begin{pmatrix} 1.0000 & 0.5 \\ 0.5000 & 0.2558 \end{pmatrix} \Rightarrow$$

Moment of Dirac probability measure $\mu_x = \delta_{x^*}$

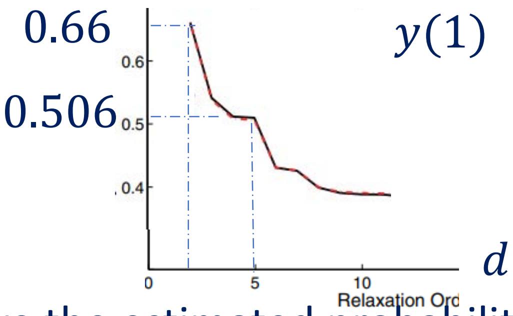
eigenvalues = 0.0046, 0.7011, 1.4022

eigenvalues =0.0046, 1.2512

$$\Rightarrow\ x^* = y_{x_1} = E[x] = 0.5$$



- As relaxation order d increase $y(1)$ converges to the true probability



(we will improve the estimated probability in Lecture 10)

Example: Probabilistic Safety Constraint 2

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \qquad -1 \leq x \leq 1$$

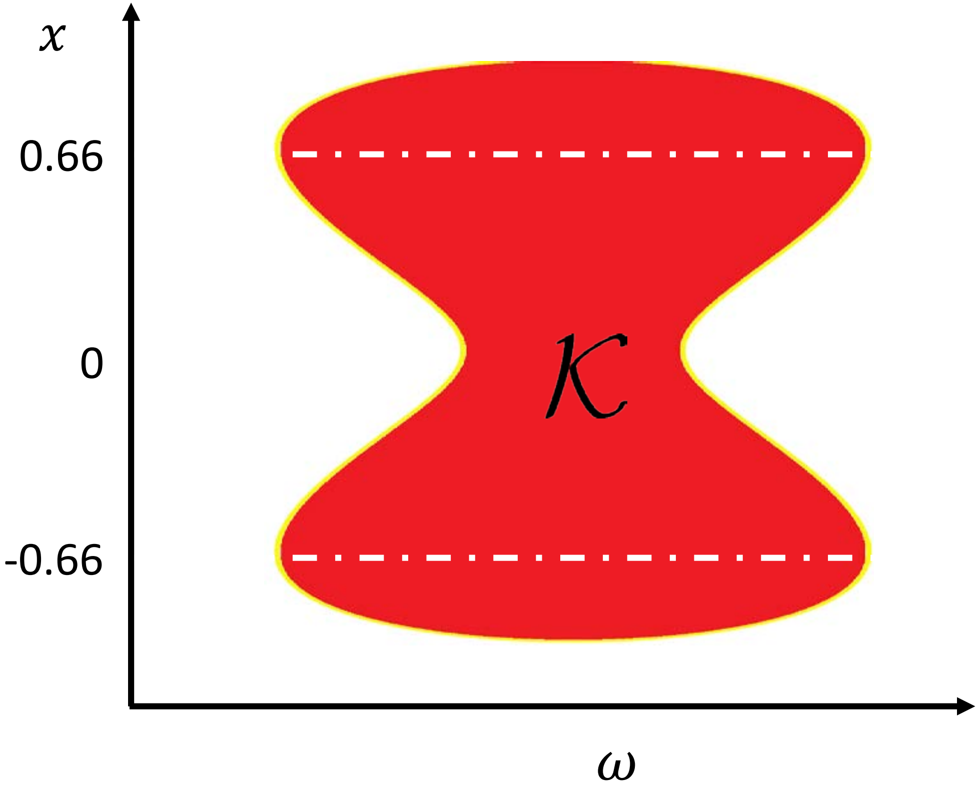
$$p(x,\omega) = \{x \in \mathbb{R}^2 : -\tfrac{1}{16}x_1^4 + \tfrac{1}{4}x_1^3 - \tfrac{1}{4}x_1^2 - \tfrac{9}{100}x_2^2 + \tfrac{29}{400} \geq 0\}$$
$$\omega \sim Uniform[-1,1]$$

d=5 $\quad y_x = [1, 0, 0.44, 0, 0.19, 00, 0.08, 0, 0.06, 0, 0.91]$

Rank Test:

$\text{Rank}\ M_d(y_x) = \text{Rank}\ M_{d-1}(y_x) \approx 2$

$\Rightarrow$ $x_2^* = -0.66, x_1^* = 0.66$



https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_2_Moment_ChanceOpt

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $\big(x_1(0), x_2(0), x_3(0)\big) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
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Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

- Suppose at time k: $\left(x_1(k), x_2(k), x_3(k)\right) \sim U([-0.1\,, 0.1]^3)$ $\qquad \omega(k) \sim Beta(2,5)$
- We want to find the control input at time k, i.e., $u(k)$, such that states $(x_1(k+1), x_2(k+1), x_3(k+1))$ reach the neighborhood of the given way-point $(0,0,0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1 - 0}{0.03}\right)^2 - \left(\frac{x_2 - 0}{0.02}\right)^2 - \left(\frac{x_3 - 0.9}{0.4}\right)^3 \geq 0$, with a high probability.

Example: Control of Uncertain Nonlinear System

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$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{x_1(k+1)}{0.03}\right)^2 - \left(\frac{x_2(k+1)}{0.02}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0\right)$$
$$\text{subject to} \qquad -1 \leq u(k) \leq 1$$

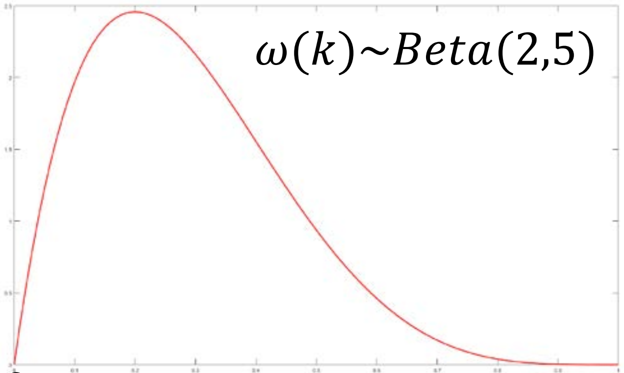
Example: Control of Uncertain Nonlinear System

- $(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\,, 0.1]^3)$

  $i-th$ moment of $Uniform([a,b])$: $y_i = \frac{1}{b-a}\frac{b^{i+1}-a^{i+1}}{i+1}$

- $\omega_k \sim Beta(5,2)$

  $i-th$ moment of $Beta(\alpha,\beta)$: $y_i = \frac{\alpha+i-1}{\alpha+\beta+i-1} y_{i-1}, y_0 = 1$



$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{\omega(k)x_2(k)}{0.03}\right)^2 - \left(\frac{x_1(k)x_3(k)}{0.02}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k)}{0.4}\right)^2 \geq 0 \right)$$

$$\text{subject to} \qquad -1 \leq u(k) \leq 1$$

$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\,, 0.1]^3)$$

$$\omega(k) \sim Beta(5,2)$$

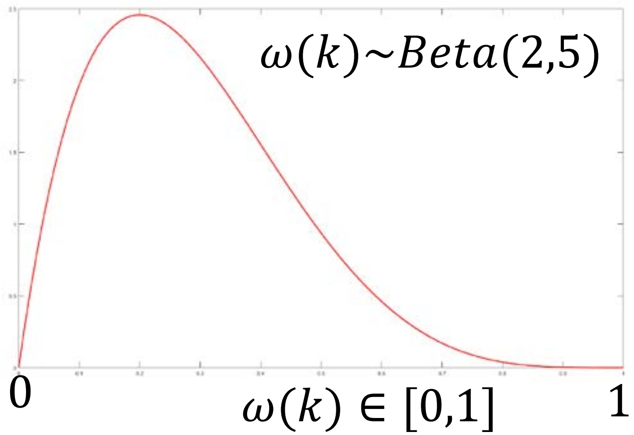
Example: Control of Uncertain Nonlinear System

- $\big(x_1(k), x_2(k), x_3(k)\big) \sim U([-0.1\,, 0.1]^3)$

  $i-th$ moment of $\;Uniform([a,b])\;:\;\; y_i = \frac{1}{b-a}\frac{b^{i+1}-a^{i+1}}{i+1}$

- $\omega_k \sim Beta(5{,}2)$

  $i-th$ moment of $Beta(\alpha,\beta)\;:\;\; y_i = \frac{\alpha+i-1}{\alpha+\beta+i-1}y_{i-1}, y_0 = 1$



$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{\omega(k)x_2(k)}{0.03}\right)^2 - \left(\frac{x_1(k)x_3(k)}{0.02}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k)}{0.4}\right)^2 \geq 0\right)$$

$$\text{subject to} \qquad -1 \leq u(k) \leq 1$$

$$\big(x_1(k), x_2(k), x_3(k)\big) \sim U([-0.1\,, 0.1]^3)$$

$$\omega(k) \sim Beta(5{,}2)$$

d=2 $\quad y_u = [1, 0.476, 0.2601, 0.1260, 0.4934]$

Rank Test:

$$\text{Rank } M_d(y_u) = \text{Rank } M_{d-1}(y_u) \approx 1$$

eigenvalues = 0.0273, 0.3939, 1.3324 eigenvalues = 0.0273, 1.2328

$u(k) = y_{u_1} = 0.476$

$Prob\ of\ Success = y(1) = 1$

True Prob for u=0.476 obtained by Monte-Carlo = 1

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_3_Moment_ChanceOpt

Noncompact Sets:

Noncompact Sets:

Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0, \; i = 1, \ldots, n_p \, \right)$$
$$\text{subject to} \quad g_i(x) \geq 0, \; i = 1, \ldots, n_g$$

SDP Relaxation

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := & \text{maximize}_{y, y_x} \quad y_0 \\ \text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0, \; M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p \\ & M_d(y_x) \succcurlyeq 0, \; M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, n_g, \; y_{x_0} = 1 \\ & M_d(y_\omega \times y_x - y) \succcurlyeq 0 \end{aligned}$$

- $\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \; p_i(x,\omega) \geq 0, \; i = 1, \ldots, n_p \,\}$ $\quad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, \ldots, n_g\}$

Assumption: Sets $\mathcal{K}$ and χ are **Archimedean** (implies Compactness).

Noncompact Sets:

Chance Optimization

$$\mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ i=1,\dots,n_p\ )$$
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$$M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i=1,\dots,n_g,\ y_{x_0}=1$$
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- For **unbounded uncertainty** ω, set $\mathcal{K}$ is not compact (closed and bounded).
  Hence, moment machinery can not be applied directly.

Noncompact Sets:

Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ i = 1, ..., n_p\ )$$
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- $\mathcal{K} = \{(x, \omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ p_i(x,\omega) \geq 0,\ i = 1, ..., n_p\ \}$ $\quad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., n_g\}$

Assumption: Sets $\mathcal{K}$ and χ are **Archimedean** (implies Compactness).

- ➢ For **unbounded uncertainty** ω, set $\mathcal{K}$ is not compact (closed and bounded). Hence, moment machinery can not be applied directly.
- ➢ However, we can apply the moment machinery to “moment determinant” unbounded uncertainties.
- ➢ Probability measure is “moment determinant” if it can completely be determined by its (finite) moments.

- J.B. Lasserre, Lebesgue decomposition in action via semidefinite relaxations, Adv. Comput. Math. 42, pp. 1129–1148, 2016.
- J.B. Lasserre, Computing gaussian and exponential measures of semialgebraic sets, arXiv:1508.06132. Submitted

Noncompact Sets:

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$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(p_i(x,\omega) \geq 0,\ i = 1, ..., n_p \right)$$
$$\text{subject to} \quad g_i(x) \geq 0,\ i = 1, ..., n_g$$

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$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y, y_x} \quad y_0$$
$$\text{s.t.} \quad M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p$$
$$M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, n_g,\ y_{x_0} = 1$$
$$M_d(y_\omega \times y_x - y) \succcurlyeq 0$$

- $\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ p_i(x,\omega) \geq 0,\ i = 1, ..., n_p\}$ $\quad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., n_g\}$

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  Hence, moment machinery can not be applied directly.

- However, we can apply the moment machinery to "moment determinant" unbounded uncertainties.
- Probability measure is "moment determinant" if it can completely be determined by its (finite) moments.

- This includes the important case where probability measure is the normal distribution.

We can represent normal distribution using its first and second moments $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-m)^2}{2\sigma^2}}$

- J.B. Lasserre, Lebesgue decomposition in action via semidefinite relaxations, Adv. Comput. Math. 42, pp. 1129–1148, 2016.
- J.B. Lasserre, Computing gaussian and exponential measures of semialgebraic sets, arXiv:1508.06132. Submitted

Topics

- Formulation of Chance Optimization and Chance Constrained Optimization
- Geometrical Interpretation
- Challenges
- Moment Based SDP for Chance Optimization
- Dual of Moment-SDP (Sum-of-Squares Program)
- SOS Based SDP for Chance Constrained Optimization
- Outer and Inner approximations of Chance Constrained Sets

Chance Constrained Optimization

- **SOS SDP Formulation**

Risk Aware Optimization

Chance Constrained Optimization

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}\big(\ g_i(x,\omega) \geq 0,\ \ i = 1,\ldots,n_g\ \big) \geq 1-\Delta$$

Deterministic optimization:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad x \in \chi_{cc}$$

- Find set of design parameters χ_{cc} such that

For any $x^* \in \chi_{cc}$ $\quad Prob\{Success\} = \int_{S(x^*)} pr(\omega)d\omega \geq 1-\Delta$

Chance Constrained Set

where $\boldsymbol{S}(\boldsymbol{x}^*) = \{g_i(x^*,\omega) \geq 0, i = 1,\ldots,n_g\}$

Chance Constrained Set:

- $\{x \in \mathbb{R}^n : Prob(Success) \geq 1-\Delta\}$ $\xrightarrow{\text{semialgebraic set approximation}}$ $\chi_{cc} = \{x \in \mathbb{R}^n : \mathcal{P}(x) \geq 1-\Delta\}$

Chance Constrained Optimization

- Semialgebraic set approximation of chance constrained set $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

  relies on **i)** **dual optimization** of moment SDP (for chance optimization),
  ii) Polynomial approximation of **Indicator function.**

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} & \int d\mu, \\ \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\chi)$

space of (nonnegative) measures supported on $\mathcal{K}$ $\times$ space of (nonnegative) measures supported on χ

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\chi)$

space of nonnegative continuous functions on $\mathcal{K}$ $\times$ space of nonnegative continuous functions on χ

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

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$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
$$\begin{aligned} \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} \mathbf{P}^*_{\mathbf{dual}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\chi)$

space of (nonnegative) measures supported on $\mathcal{K}$ $\times$ space of (nonnegative) measures supported on χ

(Appendix II)

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\chi)$

space of nonnegative continuous functions on $\mathcal{K}$ $\times$ space of nonnegative continuous functions on χ

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x, \mu} & \int d\mu, \\ \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} \mathbf{P}^*_{\text{dual}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathcal{C}}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\chi)$

space of (nonnegative) measures supported on $\mathcal{K}$ $\times$ space of (nonnegative) measures supported on χ

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\chi)$

space of nonnegative continuous functions on $\mathcal{K}$ $\times$ space of nonnegative continuous functions on χ

- Dual optimization
  - Looks for continuous function $\mathcal{W}(x,\omega)$ and scalar β
  - Minimizes scalar $\beta \geq 0$
  - β is upper bound of $\int \mathcal{W}(x,\omega) d\mu_\omega$

Dual of moment SDP

Primal Conic Program

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \langle c, x\rangle_{V_1} \\ \text{subject to} & A^*(x) = b \\ & x \in K^*. \end{array}$$

Dual Conic Program

$$\begin{array}{ll} \underset{y,s}{\text{maximize}} & \langle y, b\rangle_{V_2} \\ \text{subject to} & c - A(y) = s \\ & s \in K. \end{array}$$

LP in Measure

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
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Dual

$$\mathbf{P}_{\text{dual}}^* = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad \beta$$
$$\begin{array}{ll} \text{subject to} & \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{array}$$

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- Dual optimization
  - Looks for continuous function $\mathcal{W}(x,\omega)$ and scaler β
  - Minimizes scaler $\beta \geq 0$
  - β is upper bound of $\int \mathcal{W}(x,\omega) d\mu_\omega$

- From strong duality $\mathbf{P}_{\text{dual}}^* = \mathbf{P}_\mu^*$

$\Rightarrow$ $\beta = \text{Probability(success)}$

Interpretation of the dual problem

- For a given design variable x^*

$$\text{Probability}(\text{Success}) = \text{Probability}_{\text{pr}(\omega)}(\ p_i(x^*,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ ) = \int_{\{(x^*,\omega):\ p_i(x^*,\omega)\geq 0,\ i=1,\ldots,n_p\}} d\mu_\omega$$

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Indicator function:

$$\mathbf{I}_\omega = \begin{cases} 1 & \forall\omega \in \{(x^*,\omega):\ p_i(x^*,\omega) \geq 0,\ i = 1,\ldots,n_p\} \\ 0 & \text{otherwise} \end{cases}$$

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$$\text{Probability(Success)} = \text{Probability}_{\text{pr}(\omega)}\left(p_i(x^*, \omega) \geq 0,\ i = 1, \ldots, n_p \right) = \int_{\{(x^*,\omega):\ p_i(x^*,\omega)\geq 0,\ i=1,\ldots,n_p\}} d\mu_\omega$$

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  μ_ω :probability measure

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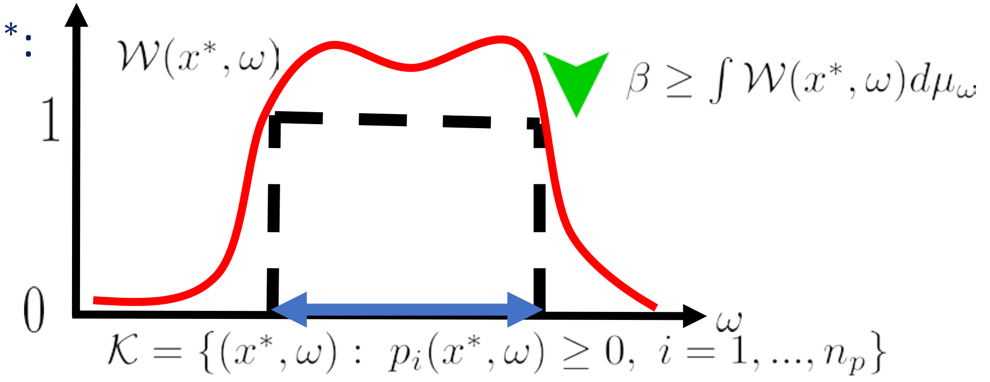

Interpretation of the dual problem

- For a given design variable x^*

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Interpretation of the dual problem

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  μ_ω :probability measure

Optimal solution Dual Opt $\Rightarrow$

- For a given design variable x^*:

$$\begin{cases} \mathcal{W}^*(x^*,\omega) = 1 \quad \forall (x^*,\omega) \in \mathcal{K} = \{(x^*,\omega):\ p_i(x^*,\omega) \geq 0,\ i=1,\dots,n_p\} \\ \mathcal{W}^*(x^*,\omega) = 0 \quad \text{otherwise} \\ \beta^* = \int \mathcal{W}^*(x,\omega) d\mu_\omega \end{cases}$$

Interpretation of the dual problem

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$$\text{Probability}(\text{Success}) = \text{Probability}_{\text{pr}(\omega)}(\ p_i(x^*,\omega) \geq 0,\ i = 1, \ldots, n_p\ ) = \int_{\{(x^*,\omega):\ p_i(x^*,\omega)\geq 0,\ i=1,\ldots,n_p\}} d\mu_\omega$$

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Optimal solution Dual Opt

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$$\beta^* = \int \mathcal{W}^*(x,\omega) d\mu_\omega \quad = \text{Probability}(\text{Success})$$

Interpretation of the dual problem

- For a given design variable x^*

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$$\beta^* = \int \mathcal{W}^*(x,\omega) d\mu_\omega = \text{Probability(Success)}$$

$$\mathcal{W}^*(x,\omega) = \mathbf{I}_\omega(x)$$

- $\mathcal{W}(x,\omega) \geq 0 \qquad \mathcal{W}(x,\omega) \geq 1\ \ \forall (x,\omega) \in \mathcal{K}$
- Minimizes scaler $\beta \geq 0$
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Optimal solution Dual Opt

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Dual of moment SDP

LP in Measure

$$\mathbf{P}_{\mu}^{*} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\begin{aligned} \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

cone of (nonnegative) measures

Dual

$$\begin{aligned} \mathbf{P}_{\mathbf{dual}}^{*} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathcal{C}}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

cone of nonnegative continuous function

$$\mathbf{P}_{\mu}^{*} = \mathbf{P}_{\mathbf{dual}}^{*} = \text{Probability(success)}$$

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cone of nonnegative continuous function

$$\mathbf{P}_\mu^* = \mathbf{P}_{\text{dual}}^* = \text{Probability(success)}$$

Relaxation

Moment SDP

$$\mathbf{P}_{\text{mom}}^{*\mathbf{d}} := \text{maximize}_{y,y_x} \quad y_0$$

$$\begin{aligned} \text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0, \; M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1,\ldots,n_p \\ & M_d(y_x) \succcurlyeq 0, \; M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1,\ldots,n_g, \; y_{x_0} = 1 \\ & M_d(y_\omega \times y_x - y) \succcurlyeq 0 \end{aligned}$$

cone of truncated moments up to order 2d of (nonnegative) measures

Dual of moment SDP

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$

$$\mu_x \text{ is a probability measure}$$

$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

cone of (nonnegative) measures

Dual

$$\mathbf{P}^*_{\mathbf{dual}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad \beta$$

$$\text{subject to} \quad \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega)\in\mathcal{K}$$

$$\beta - \int \mathcal{W}(x,\omega)d\mu_\omega \geq 0 \quad \forall x \in \chi$$

$$\beta \geq 0, \mathcal{W}(x,\omega) \geq 0$$

cone of nonnegative continuous function

$$\mathbf{P}^*_\mu = \mathbf{P}^*_{\mathbf{dual}} = \text{Probability(success)}$$

Relaxation

Relaxation

Moment SDP

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y,y_x} \quad y_0$$

$$\text{s.t.} \quad M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1,\dots,n_p$$

$$M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1,\dots,n_g,\ y_{x_0} = 1$$

$$M_d(y_\omega \times y_x - y) \succcurlyeq 0$$

cone of truncated moments up to order 2d of (nonnegative) measures

Dual

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \beta$$

$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega)\in\mathcal{K} \quad \textbf{:SOS}$$

$$\beta - \int \mathcal{W}(x,\omega)d\mu_\omega \geq 0 \quad \forall x \in \chi \quad \textbf{:SOS}$$

$$\beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \quad \textbf{:SOS}$$

cone of SOS poynomials up to order 2d

Dual of moment SDP

LP in Measure

$$\mathbf{P}_{\mu}^{*} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
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cone of (nonnegative) measures

Dual

$$\mathbf{P}_{\mathbf{dual}}^{*} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad \beta$$
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$$\beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi$$
$$\beta \geq 0, \mathcal{W}(x,\omega) \geq 0$$

cone of nonnegative continuous function

$$\mathcal{W}^*(x,\omega) = \mathbf{I}_\omega(x)$$
$$\beta^* = \int \mathcal{W}^*(x,\omega) d\mu_\omega = \text{Probability(Success)}$$

$$\mathbf{P}_{\mu}^{*} = \mathbf{P}_{\mathbf{dual}}^{*} = \text{Probability(success)}$$

Relaxation

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$$\mathbf{P}_{\mathbf{mom}}^{*\mathbf{d}} := \text{maximize}_{y,y_x} \quad y_0$$
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cone of truncated moments up to order 2d of (nonnegative) measures

Dual

$$\mathbf{P}_{\mathbf{sos}}^{*\mathbf{d}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \beta$$
$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \quad \text{:SOS}$$
$$\beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \quad \text{:SOS}$$
$$\beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \quad \text{:SOS}$$

cone of SOS poynomials up to order 2d

$$\mathcal{W}^*(x,\omega) \geq 0 \quad \mathcal{W}^*(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K}$$

polynomial $\mathcal{W}^*(x,\omega)$ = Upper bound of $\mathbf{I}_\omega(x)$

$$\mathbf{P}_{\mathbf{mom}}^{*\mathbf{d}} = \mathbf{P}_{\mathbf{sos}}^{*\mathbf{d}} \geq \text{Probability(success)}$$

Example: Moment SDP

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

$$\omega \sim Uniform[-1,1]$$

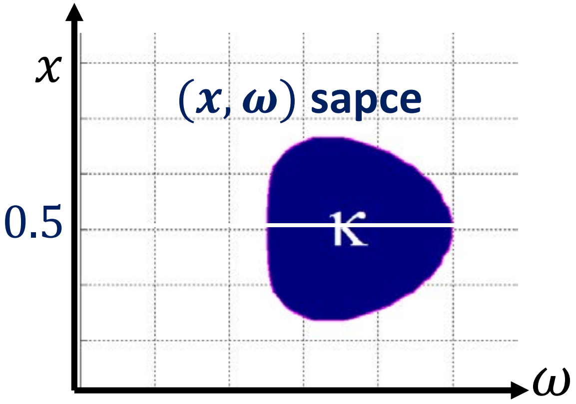
Measure LP:

$$\mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$
$$\mu_x \text{ is a probability measure}$$
$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

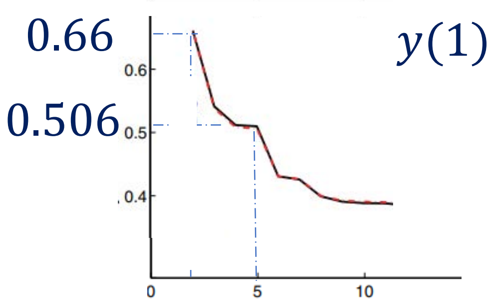
Moment SDP:

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y,y_x} \quad y_0$$
$$\text{s.t.} \quad M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0|_{i=1}^{n_p}$$
$$M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0|_{j=1}^{n_g},\ y_{x_0} = 1$$
$$M_d(y_\omega \times y_x - y) \succcurlyeq 0$$

$$x^* = y_{x_1} = 0.5$$



- As relaxation order d increase $y(1)$ converges to the true probability



Example: Dual Optimization

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

$$\omega \sim Uniform[-1,1]$$

Dual SOS Program:

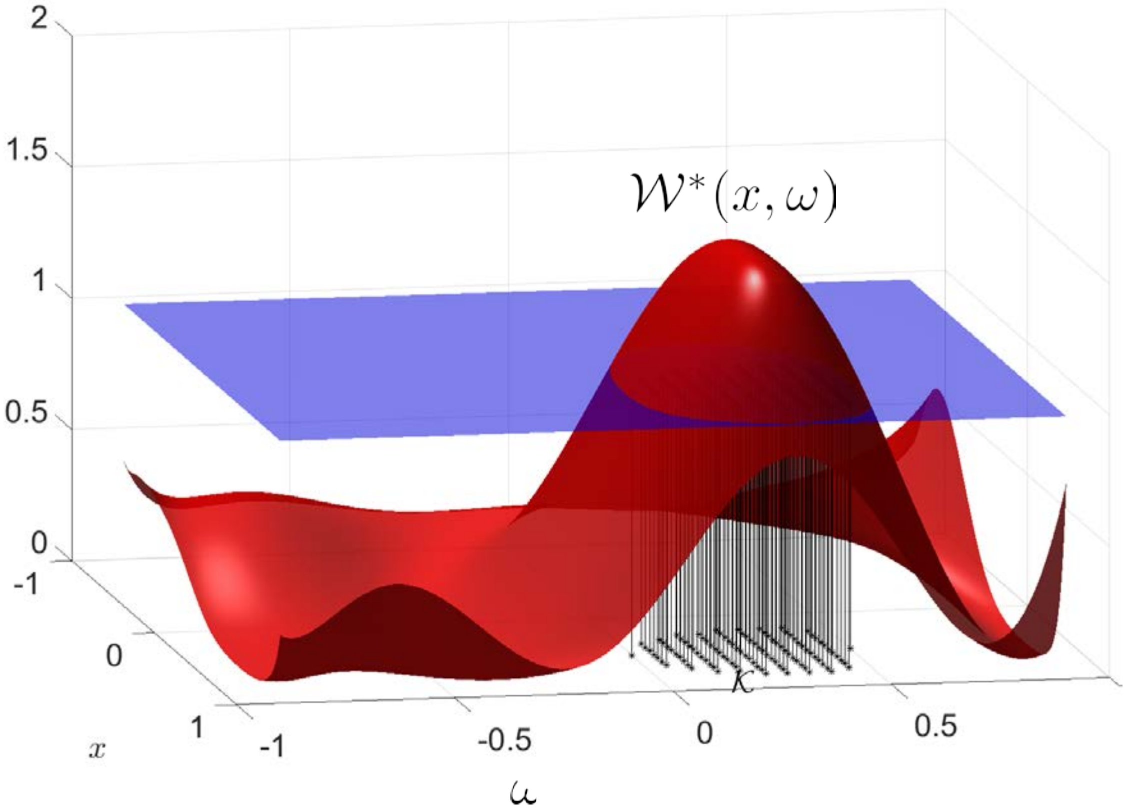
$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \beta$$

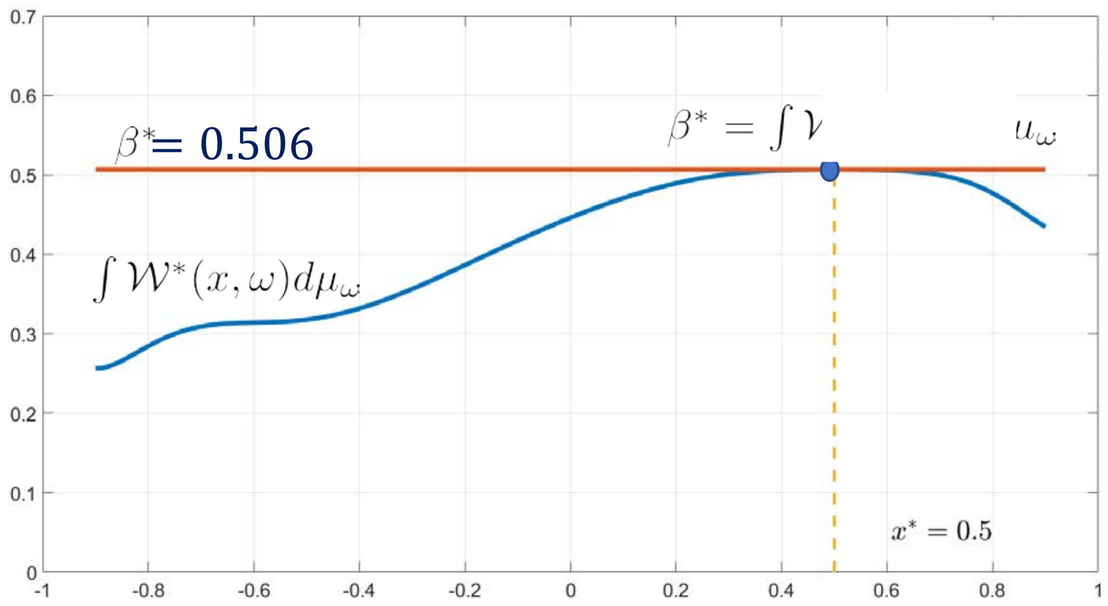
$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}$$

$$\beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi$$

$$\beta \geq 0, \mathcal{W}(x,\omega) \geq 0$$

Obtained Results using Yalmip $d = 5$:





https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_Dual_ChanceOpt

Example: Dual Optimization

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

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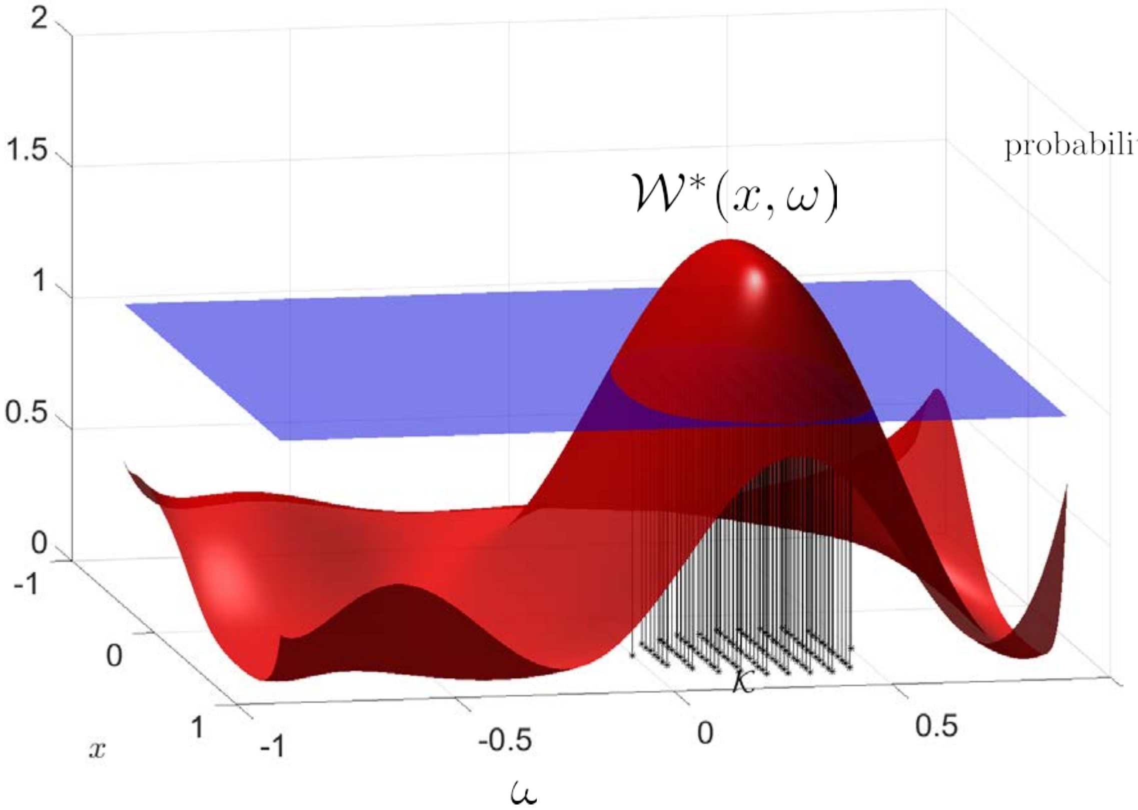

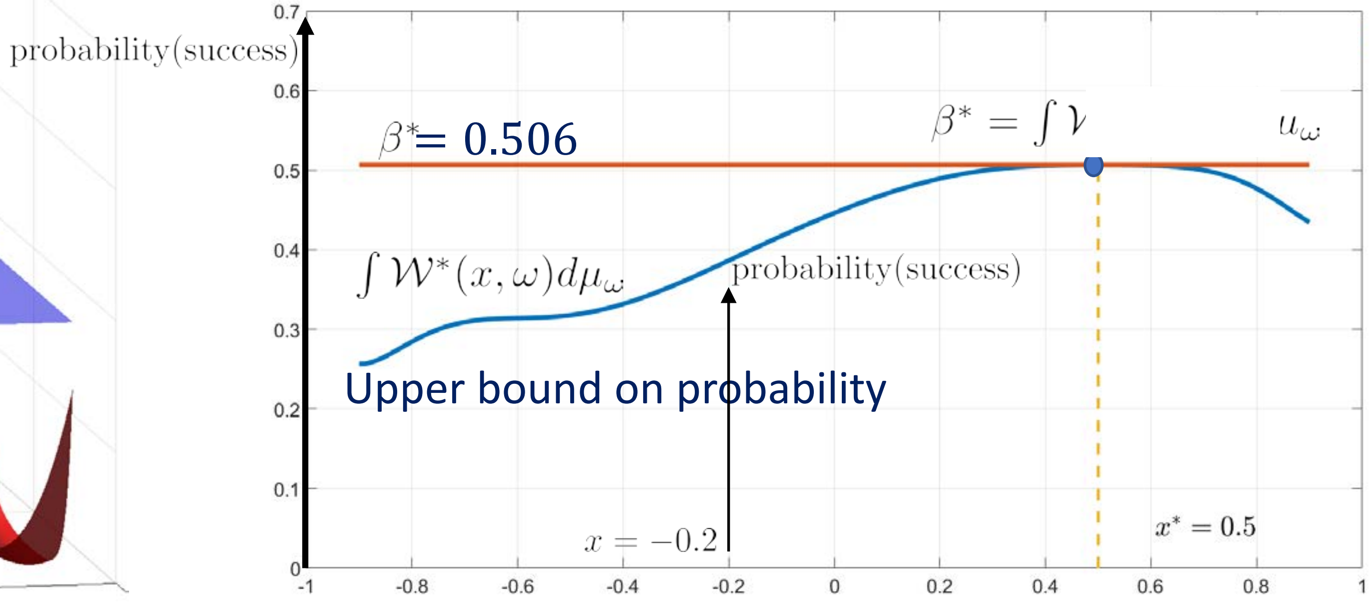

Example: Dual Optimization

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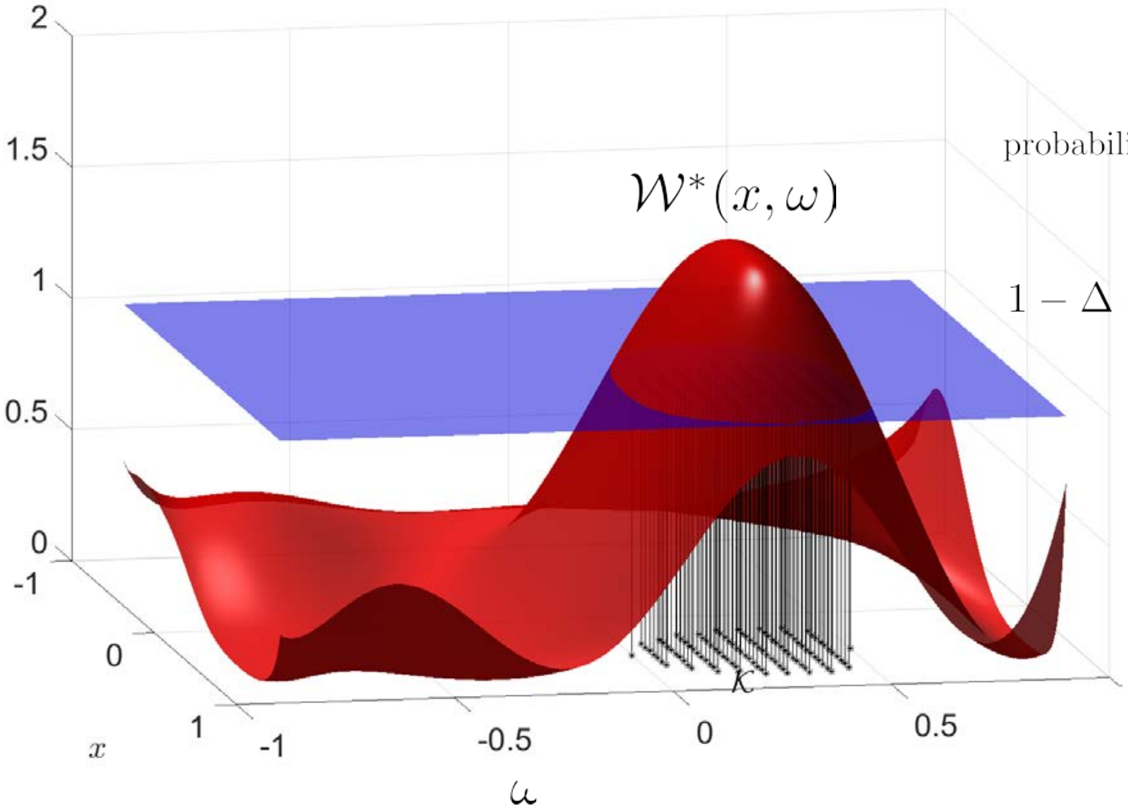

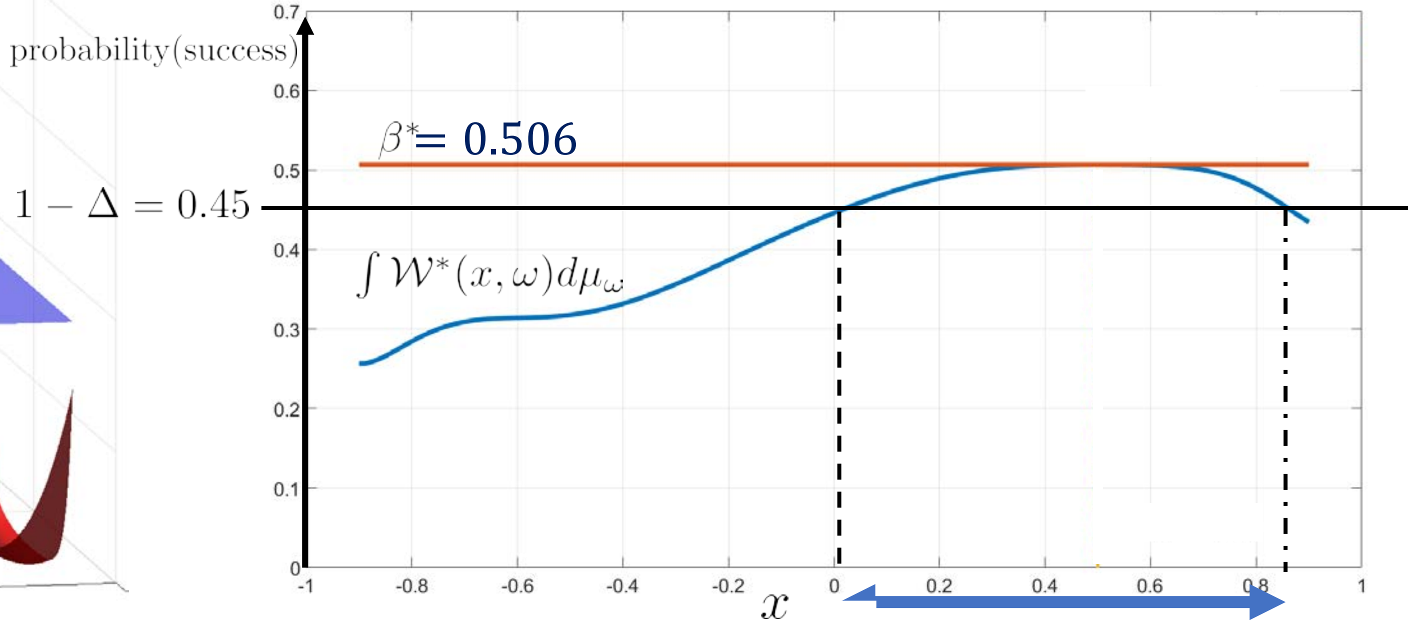

Outer approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1-\Delta\}$

$$\chi_{cc} = \left\{x \in \mathbb{R}^n : \int \mathcal{W}^*(x,\omega)\ d\mu_\omega \geq 1-\Delta\right\}$$

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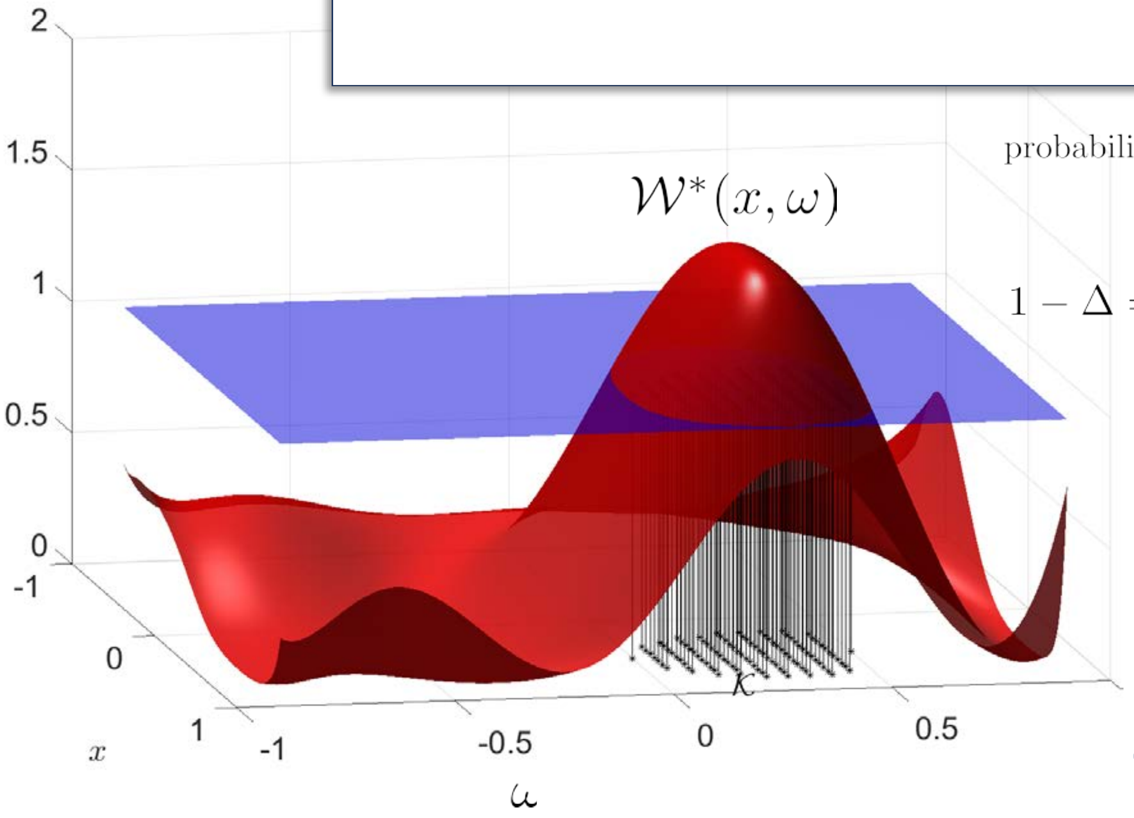
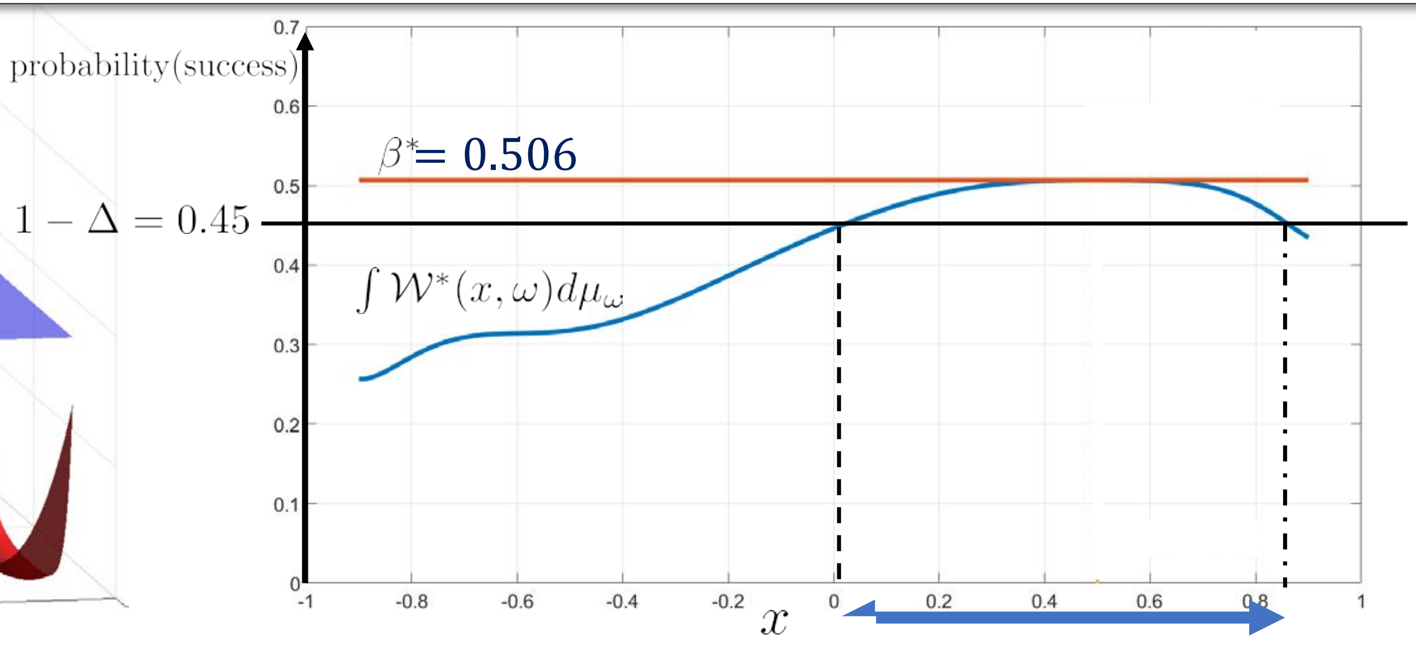
Obtained Results us





Outer approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

$$\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}^*(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$$

Chance Constrained Set:

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\} = \{x \in \mathbb{R}^n : \int \mathbf{I}(x)_\omega \, d\mu_\omega \geq 1 - \Delta\}$$

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

polynomial $\mathcal{W}(x,\omega)$ = Upper bound of $\mathbf{I}_\omega(x)$

$\int \mathcal{W}(x,\omega) d\mu_\omega$ = Upper bound of probability for design variable x

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega) \, d\mu_\omega \geq 1 - \Delta\}$

Chance Constrained Set:

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1-\Delta\} = \{x \in \mathbb{R}^n : \int \mathbf{I}(x)_\omega \, d\mu_\omega \geq 1-\Delta\}$$

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Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega) \, d\mu_\omega \geq 1-\Delta\}$

- To obtain chance constrained set, we need to find polynomial approximation of **indicator function of set** $\mathcal{K}$.

  Hence, in the SOS optimization, we can directly minimize the values of $\mathcal{W}(x,\omega)$ in (x,ω) space, $i.e., \int \mathcal{W}(x,\omega) d\mu_\omega dx$

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$\int \mathcal{W}(x,\omega) d\mu_\omega =$ Upper bound of probability for design variable x

Less SOS Constraints

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega) \, d\mu_\omega \geq 1 - \Delta\}$

Example: Dual Optimization

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$$\text{subject to} \qquad -1 \leq x \leq 1$$

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Obtained Results using Yalmip $d = 5$:

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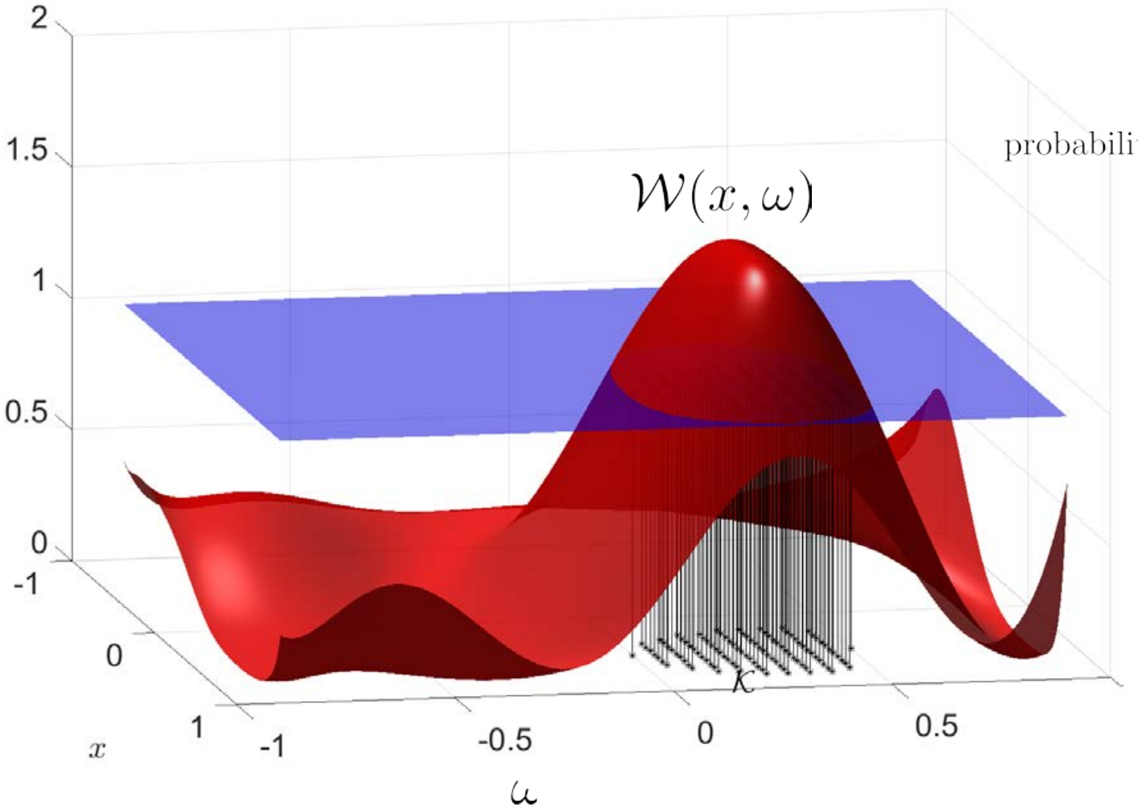

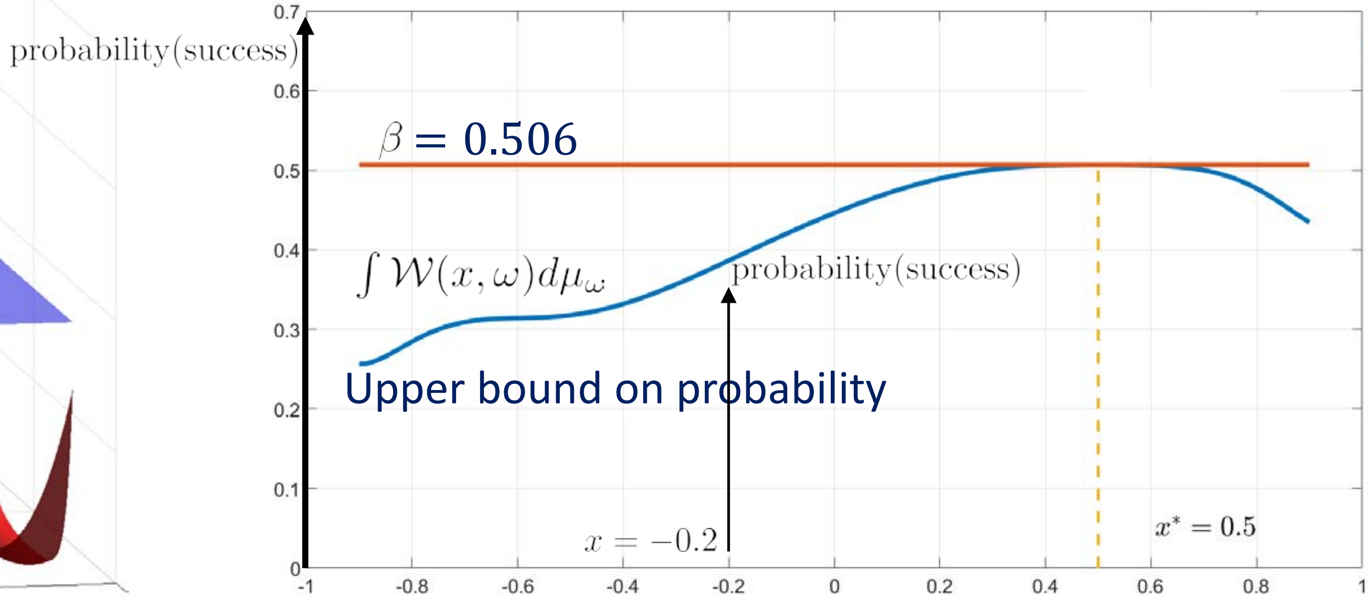

Example: Modified SOS Program for Chance Constrained Set

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
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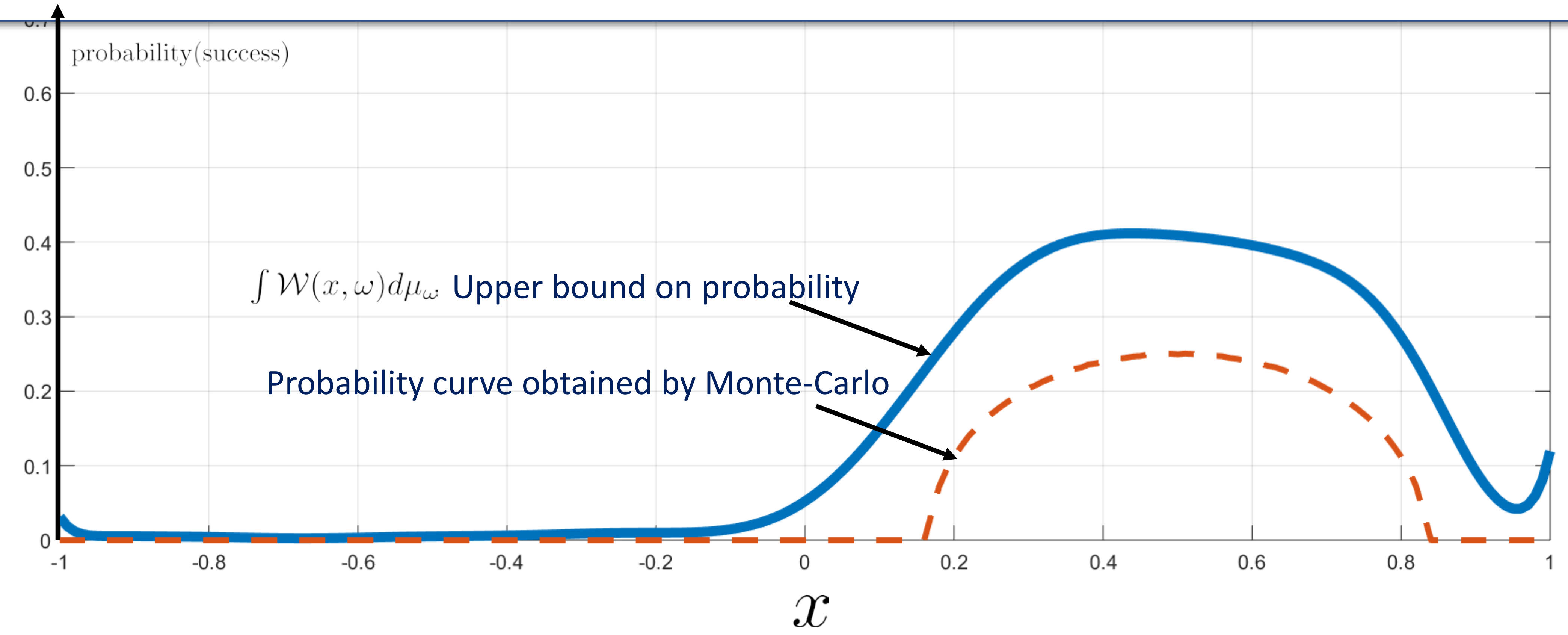
$$\mathcal{W}(x,\omega) \geq 0$$

d=10



https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_SOS_ChanceConstrained

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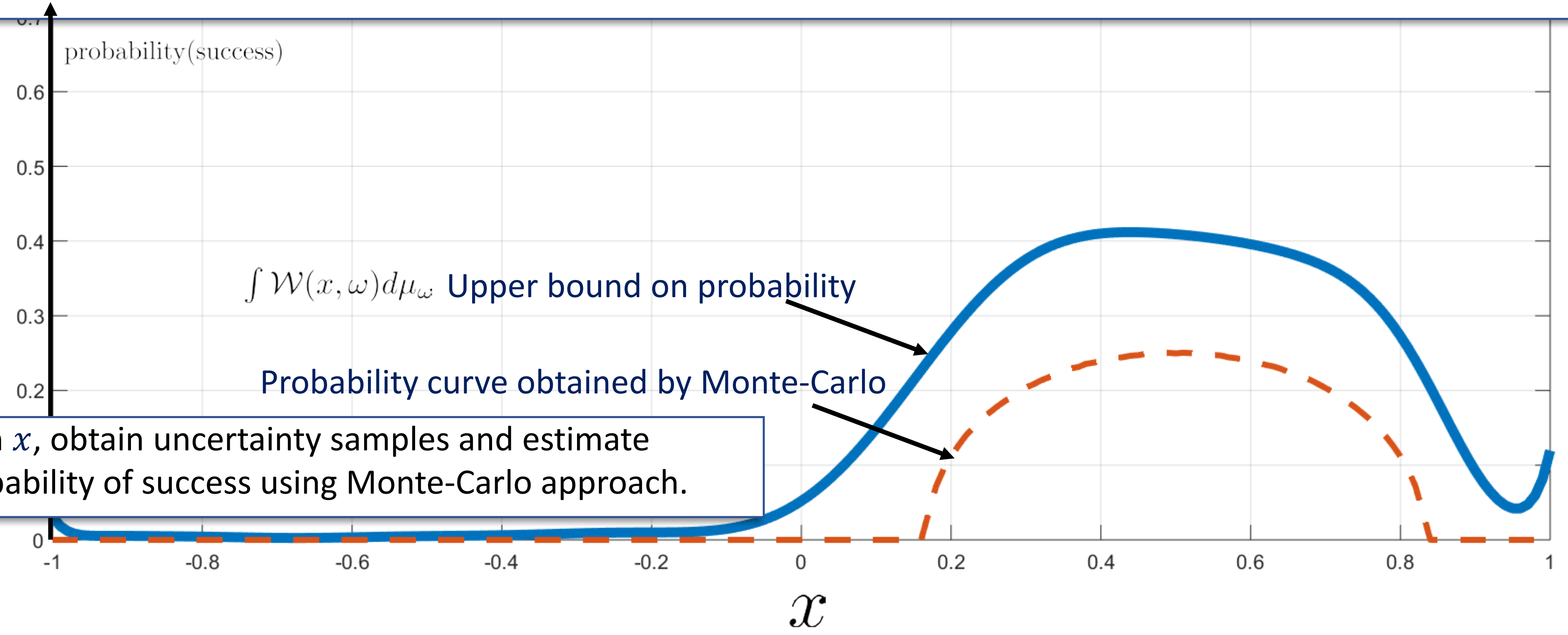

For each x, obtain uncertainty samples and estimate the probability of success using Monte-Carlo approach.

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_SOS_ChanceConstrained

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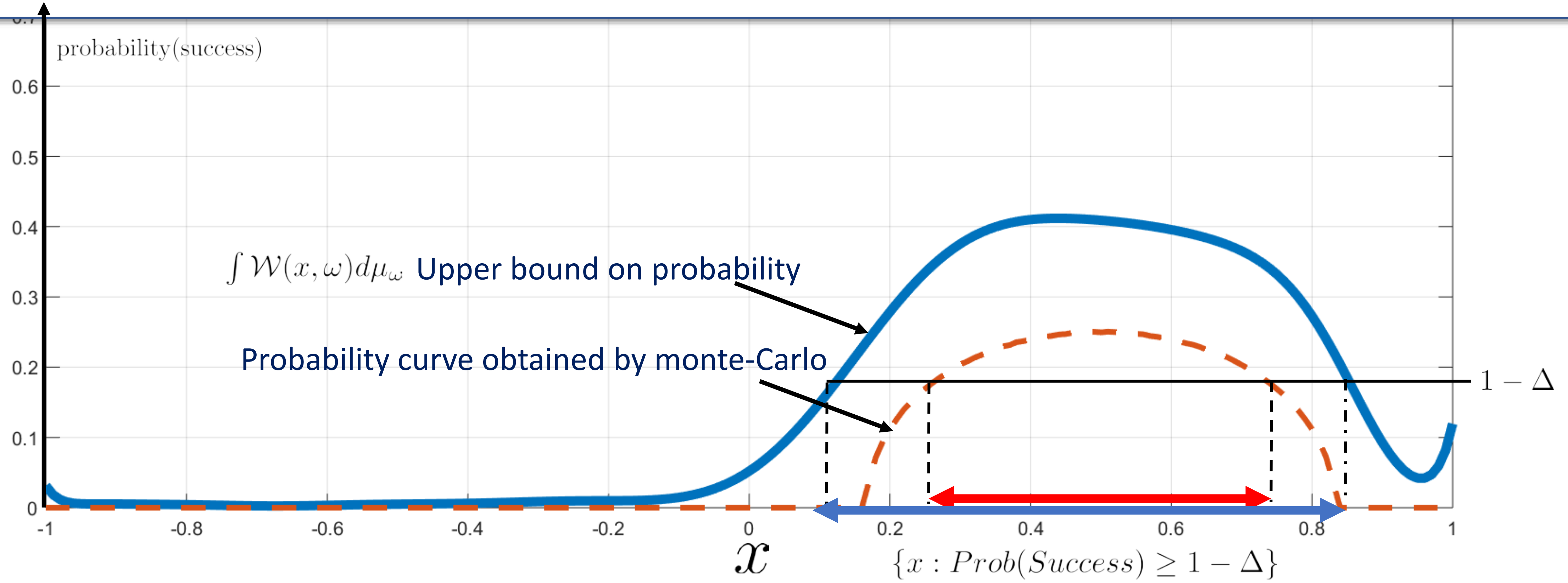

Outer approximation : $\chi_{cc} = \left\{x : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1-\Delta\right\}$

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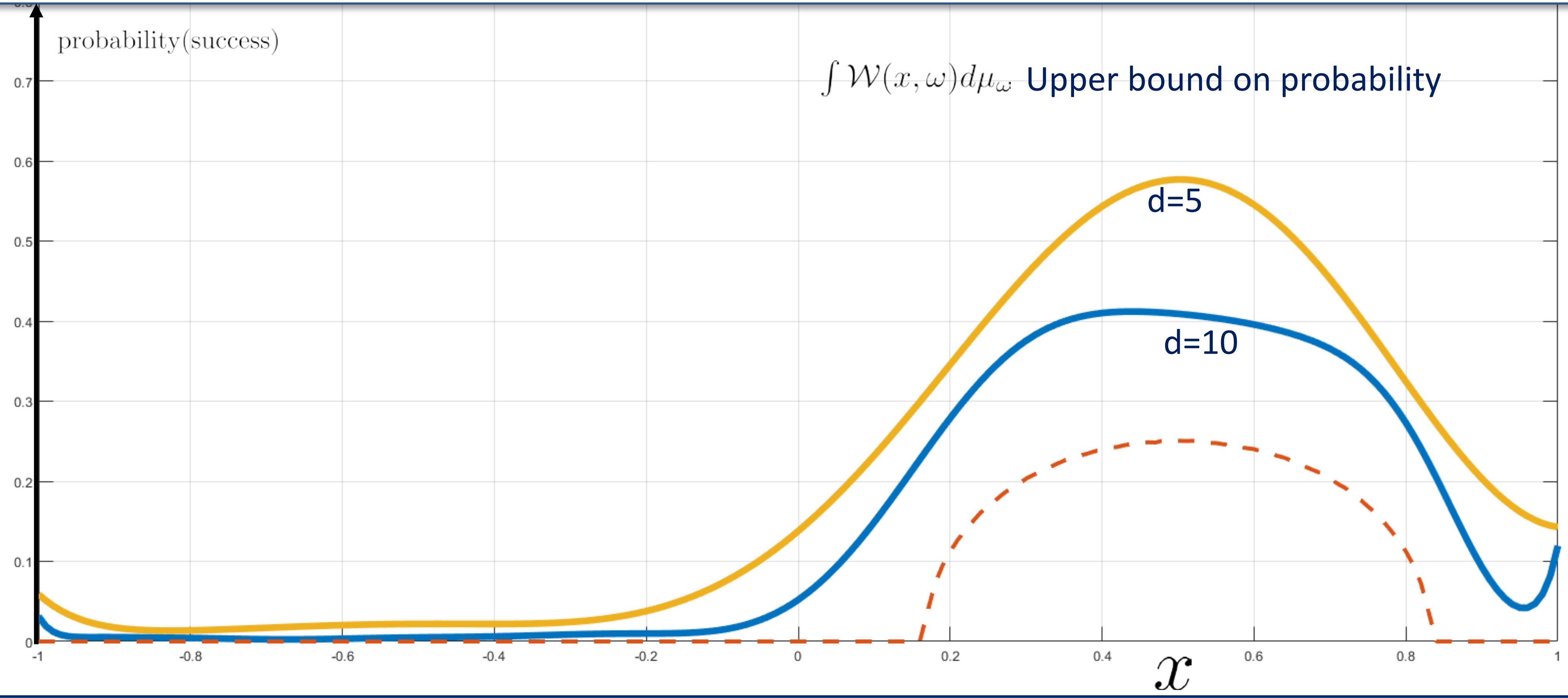

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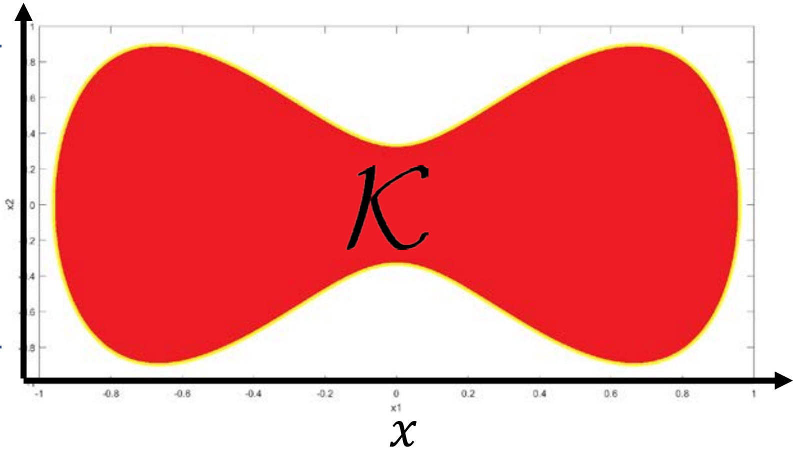
$$p(x,\omega) = \{x \in \mathbb{R}^2 : -\tfrac{1}{16}x_1^4 + \tfrac{1}{4}x_1^3 - \tfrac{1}{4}x_1^2 - \tfrac{9}{100}x_2^2 + \tfrac{29}{400} \geq 0\}$$

$$\omega \sim Uniform[-1,1]$$

d=5

$x_2^* = -0.66, x_1^* = 0.66$



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https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_2_SOS_ChanceConstrained

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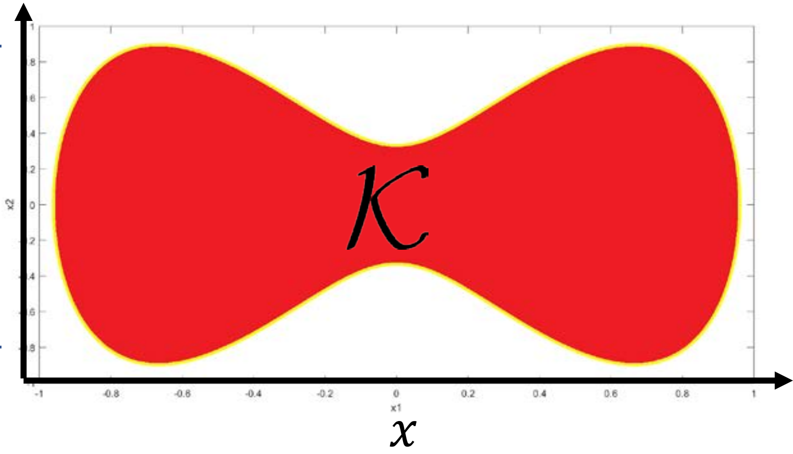

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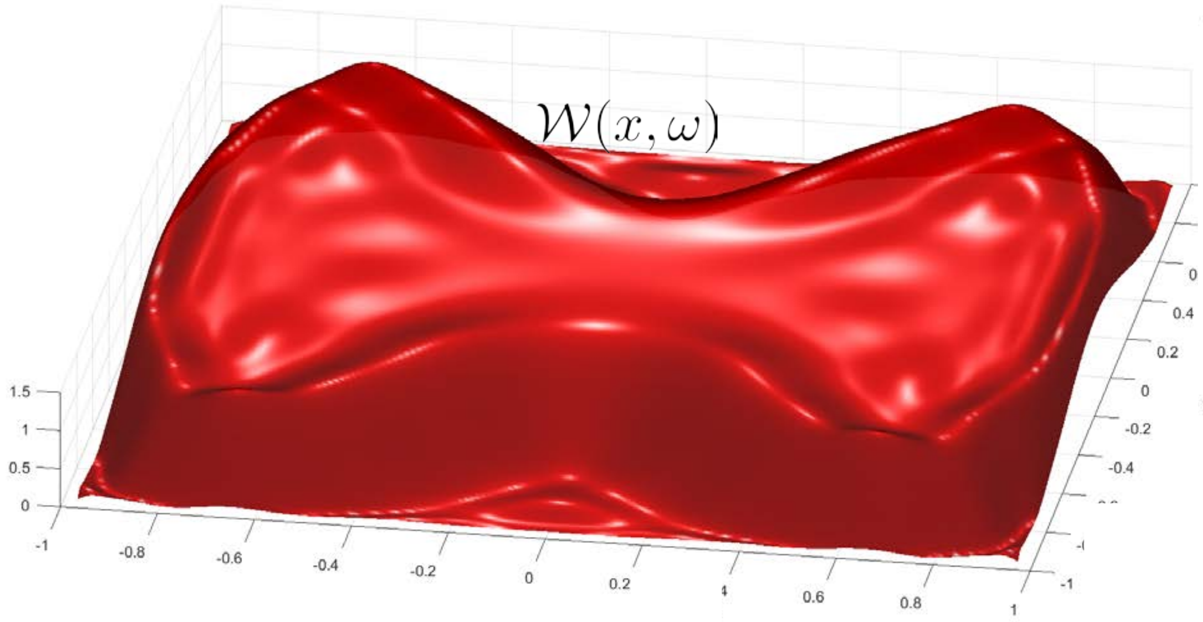

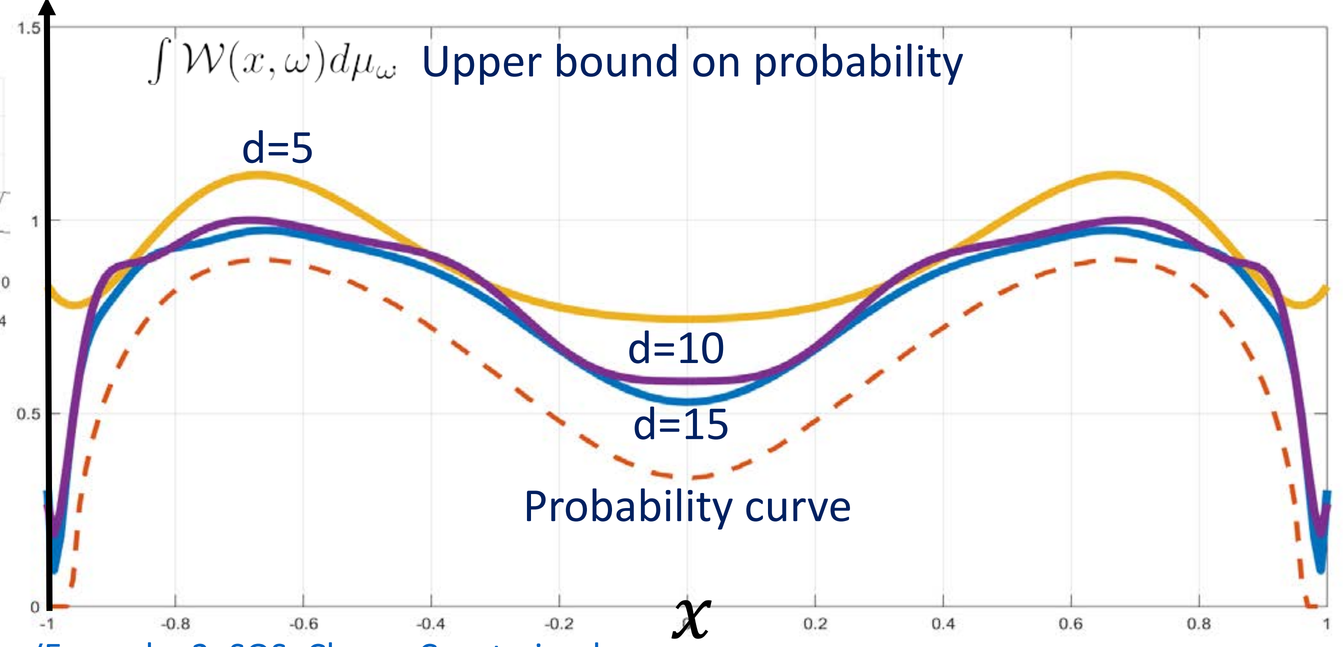

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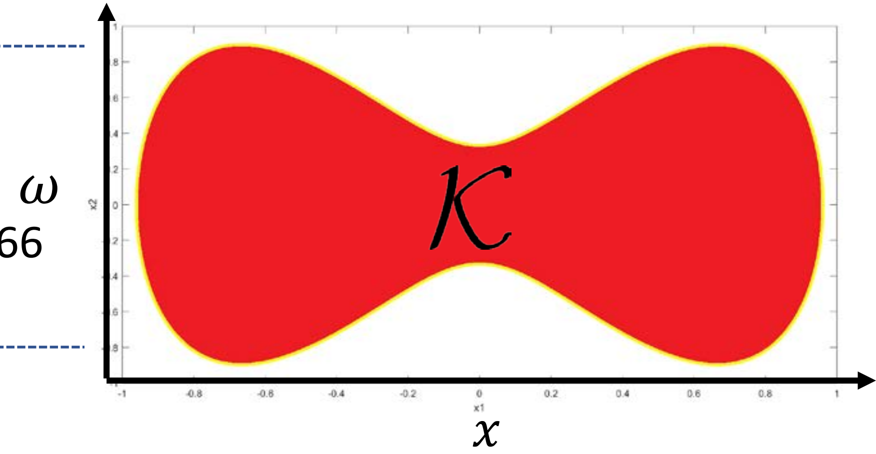

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Outer approximation of $\{x : Prob(Success) \geq 1 - \Delta\}$ **:**

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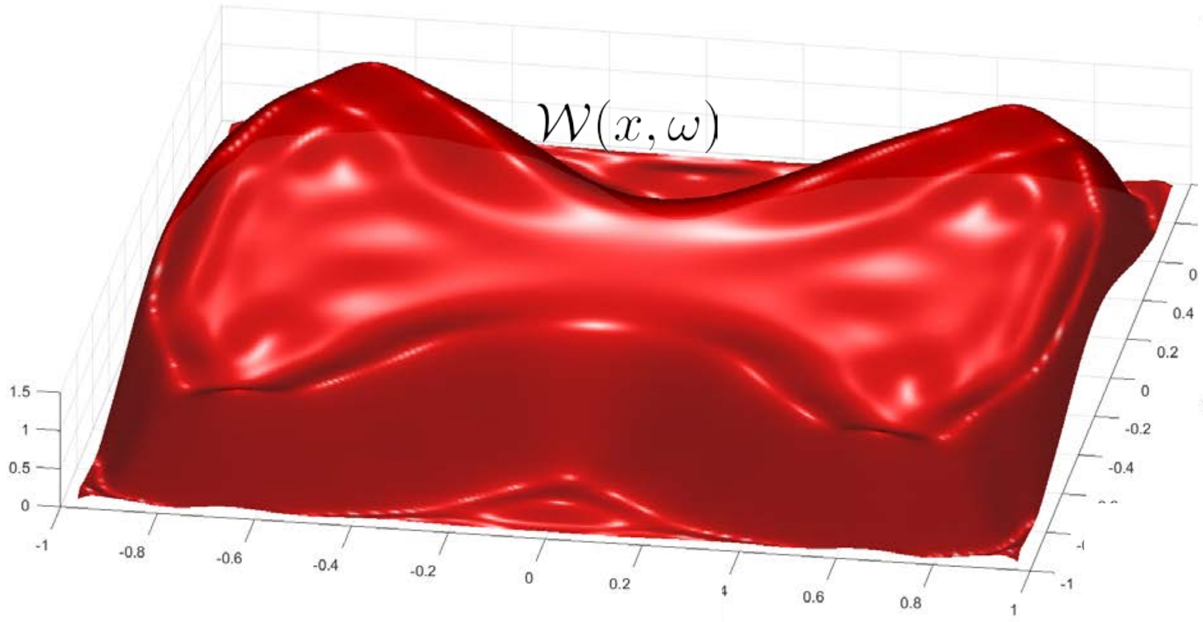
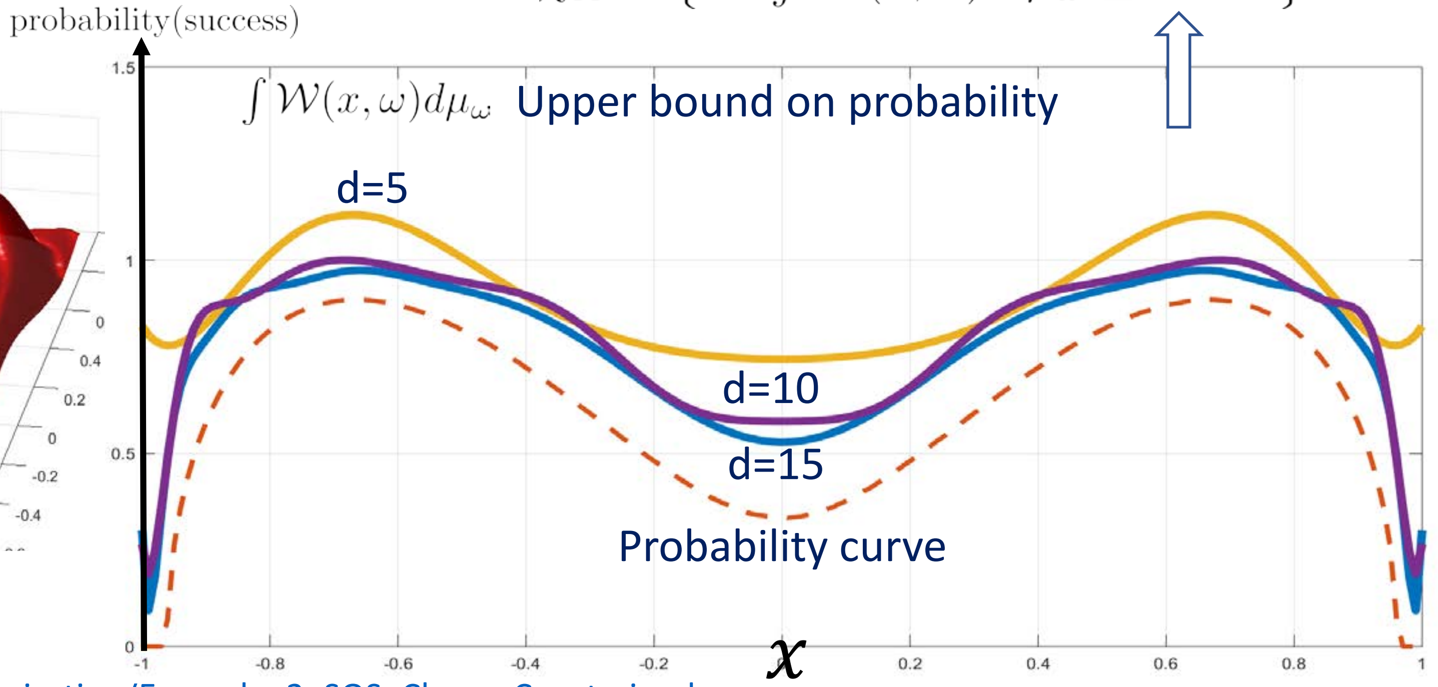




https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_2_SOS_ChanceConstrained

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $\big(x_1(0), x_2(0), x_3(0)\big) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

- Suppose at time k: $\big(x_1(k), x_2(k), x_3(k)\big) \sim U([-0.1\,, 0.1]^3)$ $\quad\quad \omega_k \sim Beta(2,5)$

- We want to find a set of control inputs at time k that steer states $(x_1(k+1), x_2(k+1), x_3(k+1))$ to the neighborhood of the given way-point $(0,0,0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1 - 0}{0.03}\right)^2 - \left(\frac{x_2 - 0}{0.02}\right)^2 - \left(\frac{x_3 - 0.9}{0.4}\right)^3 \geq 0$, with a probability greater or equal to $1 - \Delta$.

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $(x_1(0), x_2(0), x_3(0)) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

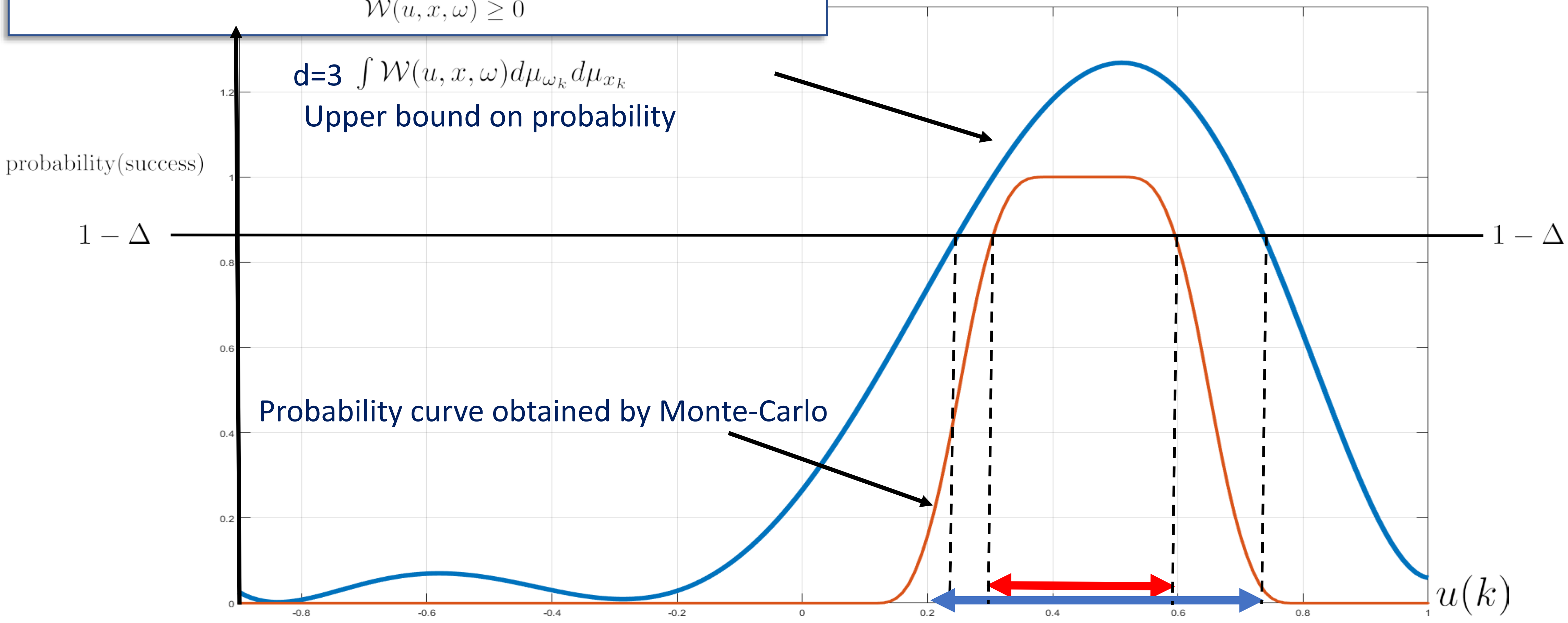
- Suppose at time k: $(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\,, 0.1]^3)$ $\omega_k \sim Beta(2{,}5)$
- We want to find a set of control inputs at time k that steer states $(x_1(k+1), x_2(k+1), x_3(k+1))$ to the neighborhood of the given way-point $(0{,}0{,}0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1-0}{0.03}\right)^2 - \left(\frac{x_2-0}{0.02}\right)^2 - \left(\frac{x_3-0.9}{0.4}\right)^3 \geq 0$, with a probability greater or equal to $1-\Delta$.

$$U_{cc} = \{u(k) : \text{Probability}(Success) \geq 1-\Delta\}$$
$$= \left\{ u(k) :\ \text{Probability}\left(1 - \left(\frac{x_1(k+1)}{0.03}\right)^2 - \left(\frac{x_2(k+1)}{0.02}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0\right) \geq 1-\Delta \right\}$$

Example: Control of Uncertain Nonlinear System

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(u,x,\omega) \in \mathbb{R}_d[u,\omega]}{\text{minimize}} \quad \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} du$$
$$\text{subject to} \quad \mathcal{W}(u,x,\omega) - 1 \geq 0 \quad \forall (u,x,\omega) \in \mathcal{K}$$
$$\mathcal{W}(u,x,\omega) \geq 0$$

$$x = [x_1(k), x_2(k)]$$



Outer approximation of $U_{cc} = \{u(k) : Prob(Success) \geq 1-\Delta\}$ **:** $\{u(k) : \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} \geq 1-\Delta\}$

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_3_SOS_ChanceConstrained

Example: Safe Control of Uncertain Nonlinear System

Consider the uncertain nonlinear system of the previous example. Unsafe set is defined as

$$X_{obs} = \{\ (x_1, x_2, x_3): 1^2 - \left(\frac{x_1 - 0.1}{0.2}\right)^2 - \left(\frac{x_2 - 0.1}{0.2}\right)^2 - \left(\frac{x_3 - 0.4}{0.3}\right)^3 \geq 0\}$$

Example: Safe Control of Uncertain Nonlinear System

Consider the uncertain nonlinear system of the previous example. Unsafe set is defined as

$$X_{obs} = \{\ (x_1, x_2, x_3) \colon 1^2 - \left(\frac{x_1 - 0.1}{0.2}\right)^2 - \left(\frac{x_2 - 0.1}{0.2}\right)^2 - \left(\frac{x_3 - 0.4}{0.3}\right)^3 \geq 0\}$$

- Chance constrained set for control input at time k is defined as follows:

$$U_{cc} = \{u(k) : \text{Probability}(x(k+1) \in X_{obs}) \geq \Delta\}$$

$$U_{cc} = \left\{u(k) : \text{Probability}\left(\underbrace{1 - \left(\frac{\omega(k)x_2(k) - 0.1}{0.2}\right)^2 - \left(\frac{x_1(k)x_3(k) - 0.1}{0.2}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k) - 0.4}{0.3}\right)^2 \geq 0}_{\mathcal{K}} \right) \geq \Delta\right\}$$

$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\ , 0.1]^3)$$

$$\omega(k) \sim Beta(5,2)$$

Example: Safe Control of Uncertain Nonlinear System

Consider the uncertain nonlinear system of the previous example. Unsafe set is defined as

$$X_{obs} = \{ \ (x_1, x_2, x_3): 1^2 - \left(\frac{x_1 - 0.1}{0.2}\right)^2 - \left(\frac{x_2 - 0.1}{0.2}\right)^2 - \left(\frac{x_3 - 0.4}{0.3}\right)^3 \geq 0\}$$

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$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\ , 0.1]^3)$$

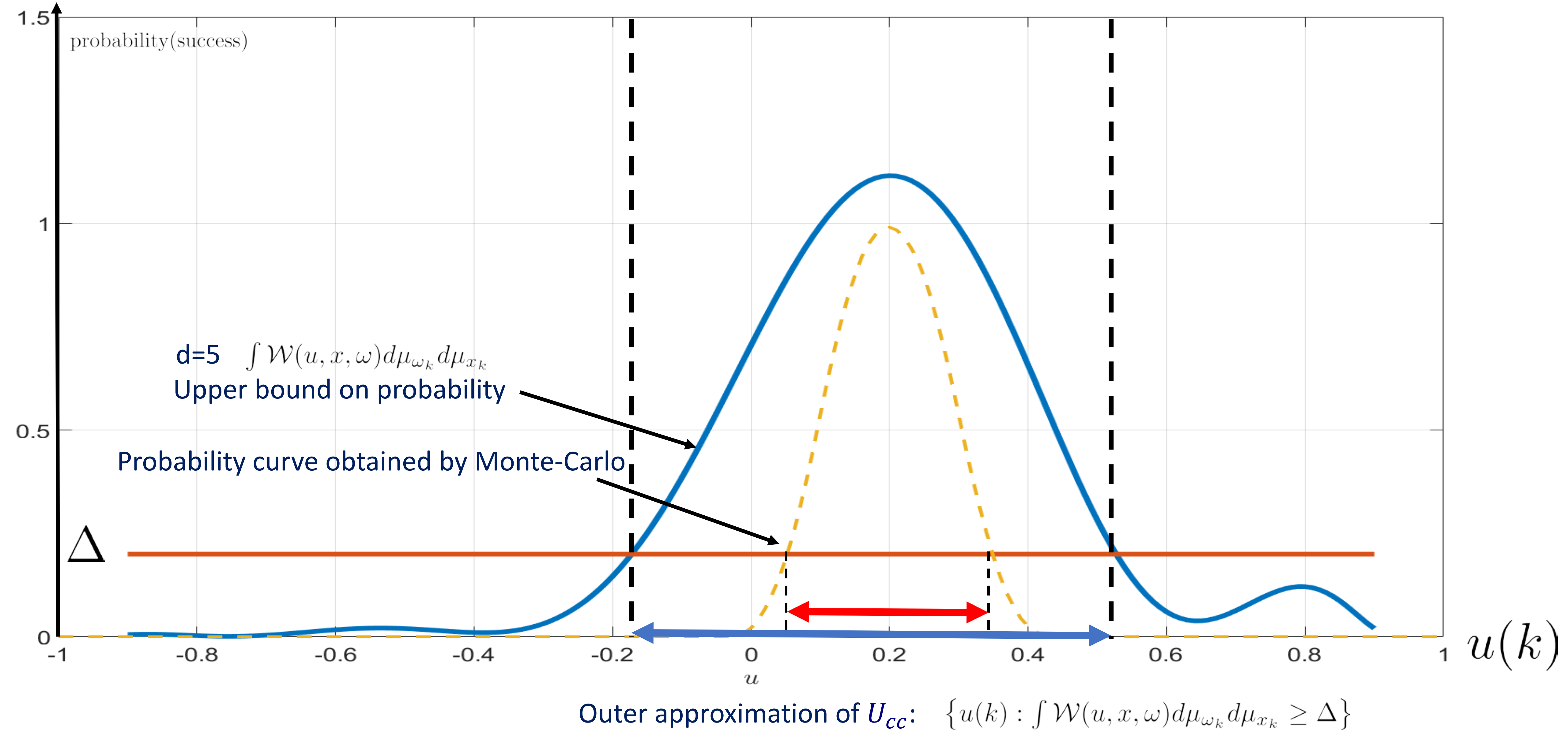
$$\omega(k) \sim Beta(5,2)$$

Outer approximation:

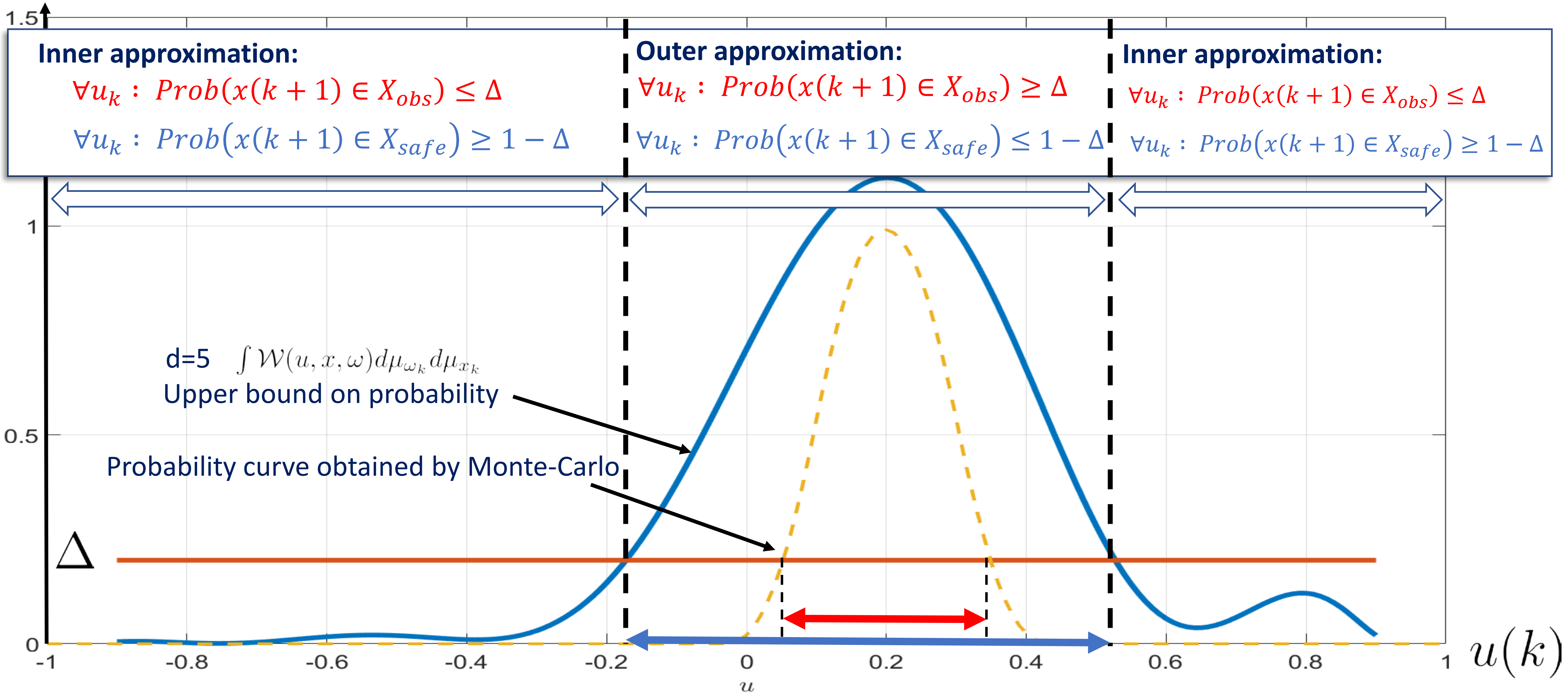
$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\text{sos}} = \underset{\beta\in\mathbb{R}, \mathcal{W}(u,x,\omega)\in\mathbb{R}_d[u,\omega]}{\text{minimize}} \quad & \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} du \\ \text{subject to} \quad & \mathcal{W}(u,x,\omega) - 1 \geq 0 \quad \forall (u,x,\omega) \in \mathcal{K} \\ & \mathcal{W}(u,x,\omega) \geq 0 \end{aligned}$$

Outer approximation of U_{cc}:

$$\{u \in \mathbb{R}^n : \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} \geq \Delta\}$$



probability(success)
d=5 $\int \mathcal{W}(u,x,\omega)d\mu_{\omega_k}d\mu_{x_k}$
Upper bound on probability
Probability curve obtained by Monte-Carlo
Δ
$u(k)$
u
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Outer approximation of U_{cc}: $\{u(k) : \int \mathcal{W}(u,x,\omega)d\mu_{\omega_k}d\mu_{x_k} \geq \Delta\}$



Outer approximation of U_{cc}: $\{u(k) : \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} \geq \Delta\}$

Inner approximation:

$\forall u_k : Prob(x(k+1) \in X_{obs}) \leq \Delta$

$\forall u_k : Prob\big(x(k+1) \in X_{safe}\big) \geq 1-\Delta$

Outer approximation:

$\forall u_k : Prob(x(k+1) \in X_{obs}) \geq \Delta$

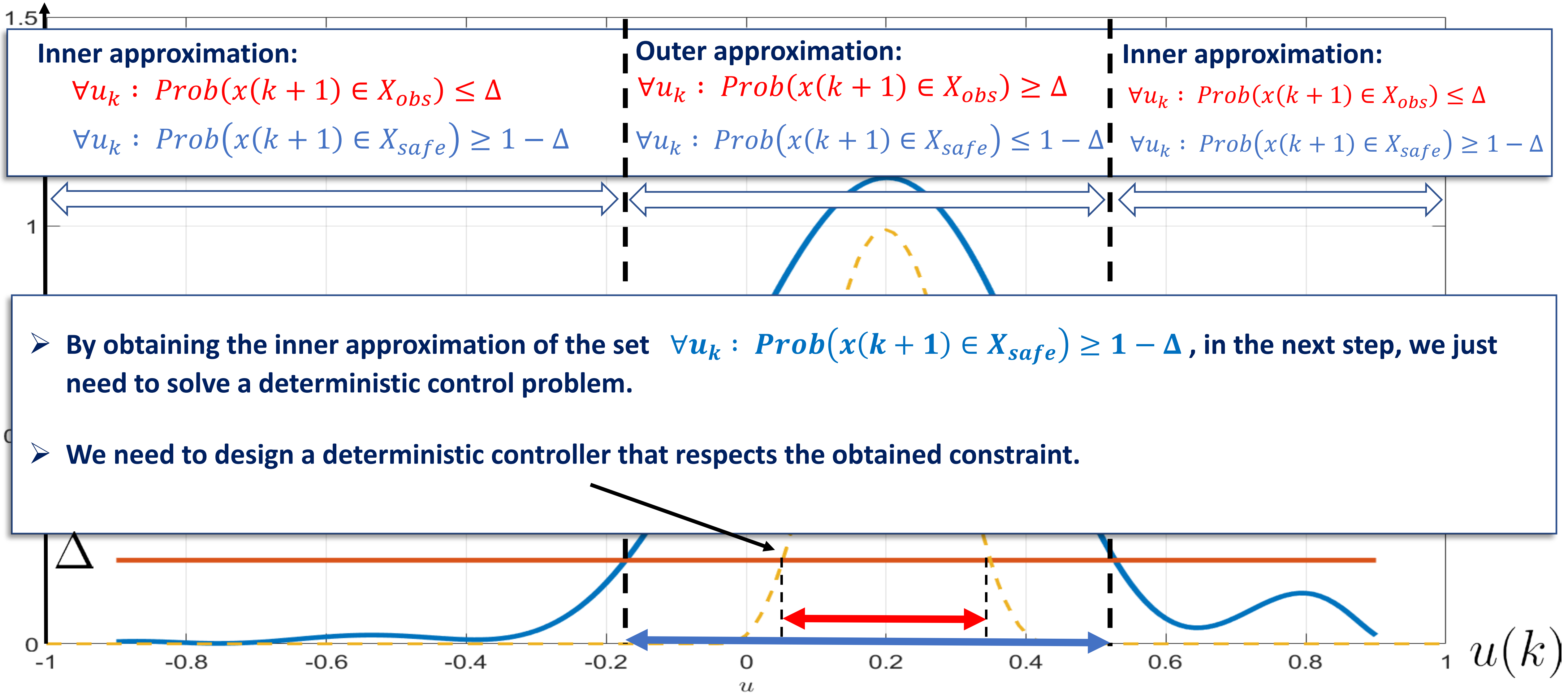
$\forall u_k : Prob\big(x(k+1) \in X_{safe}\big) \leq 1-\Delta$

Inner approximation:

$\forall u_k : Prob(x(k+1) \in X_{obs}) \leq \Delta$

$\forall u_k : Prob\big(x(k+1) \in X_{safe}\big) \geq 1-\Delta$

- **By obtaining the inner approximation of the set $\forall u_k : Prob\big(x(k+1) \in X_{safe}\big) \geq 1-\Delta$, in the next step, we just need to solve a deterministic control problem.**
- **We need to design a deterministic controller that respects the obtained constraint.**



Outer approximation of U_{cc}: $\{u(k) : \int \mathcal{W}(u,x,\omega) d\mu_{\omega_k} d\mu_{x_k} \geq \Delta\}$

Topics

- Formulation of Chance Optimization and Chance Constrained Optimization
- Geometrical Interpretation
- Challenges
- Moment Based SDP for Chance Optimization
- Dual of Moment-SDP (Sum-of-Squares Program)
- SOS Based SDP for Chance Constrained Optimization
- Outer and Inner approximations of Chance Constrained Sets

Inner Approximation of Chance Constrained Sets

Chance Constrained Set: Inner Approximation

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$$

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \mathcal{W}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

polynomial $\mathcal{W}(x,\omega) =$ Upper bound of $\mathbf{I}_\omega(x)$

$\int \mathcal{W}(x,\omega) d\mu_\omega =$ Upper bound of probability for design variable x

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$

Chance Constrained Set: Inner Approximation

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \int \mathcal{W}(x,\omega) d\mu_\omega dx$$

$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}$$

$$\mathcal{W}(x,\omega) \geq 0$$

polynomial $\mathcal{W}(x,\omega) =$ Upper bound of $\mathbf{I}_\omega(x)$

$\int \mathcal{W}(x,\omega) d\mu_\omega =$ Upper bound of probability for design variable x

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$

- We can apply same methodology to the **failure set**

$$\{x : \mathbb{R}^n : Prob(failure) \leq \Delta\} \xrightarrow{\text{Outer approximation:}} \chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \leq \Delta\}$$

Chance Constrained Set: Inner Approximation

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \mathcal{W}(x,\omega) d\mu_\omega dx$$

$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}$$

$$\mathcal{W}(x,\omega) \geq 0$$

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$\int \mathcal{W}(x,\omega) d\mu_\omega =$ Upper bound of probability for design variable x

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$

➢ We can apply same methodology to the **failure set**

$$\{x : \mathbb{R}^n : Prob(failure) \leq \Delta\} \xrightarrow{\text{Outer approximation:}} \chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \leq \Delta\}$$

Inner approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

Chance Constrained Set: Inner Approximation

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \mathcal{W}(x,\omega) d\mu_\omega dx$$

$$\text{subject to} \quad \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}$$

$$\mathcal{W}(x,\omega) \geq 0$$

polynomial $\mathcal{W}(x,\omega)$ = Upper bound of $\mathbf{I}_\omega(x)$

$\int \mathcal{W}(x,\omega) d\mu_\omega$ = Upper bound of probability for design variable x

Outer approximation : $\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$

➢ We can apply same methodology to the **failure set**

$$\{x : \mathbb{R}^n : Prob(failure) \leq \Delta\} \xrightarrow{\text{Outer approximation:}} \chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \leq \Delta\}$$

Inner approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

Chance Constrained Set

- We need to find the **“outer”** approximation of the set of design parameters that results in failure

- We need to find the **“inner”** approximation of the set of design parameters that results in success.

Example: Inner approximation

$$\mathcal{K} = \{(x,\omega) : p(x,\omega) = (0.8x-0.1)^4 - 0.27\omega((0.8x-0.1)^2 + 0.3\omega^2) + 0.3\omega^2(0.8x-0.1)^2 + +0.09\omega^4 \geq 0\}$$

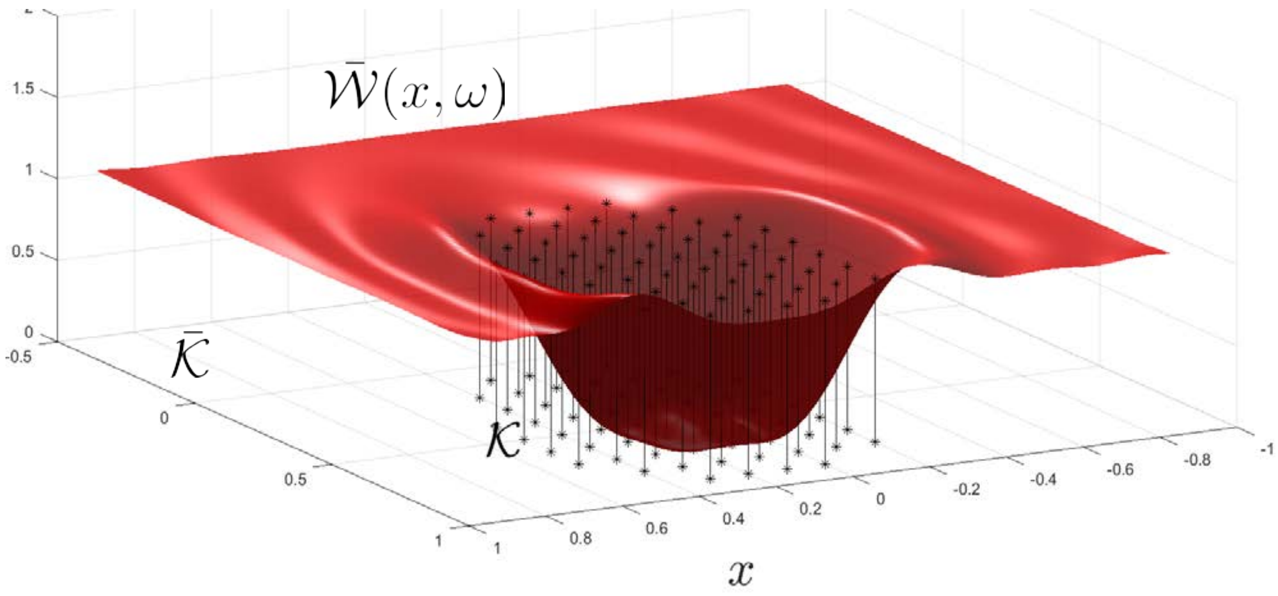
$\omega \sim Uniform[-1,1]$ $\{x : Prob(Success) \geq 1-\Delta\}$

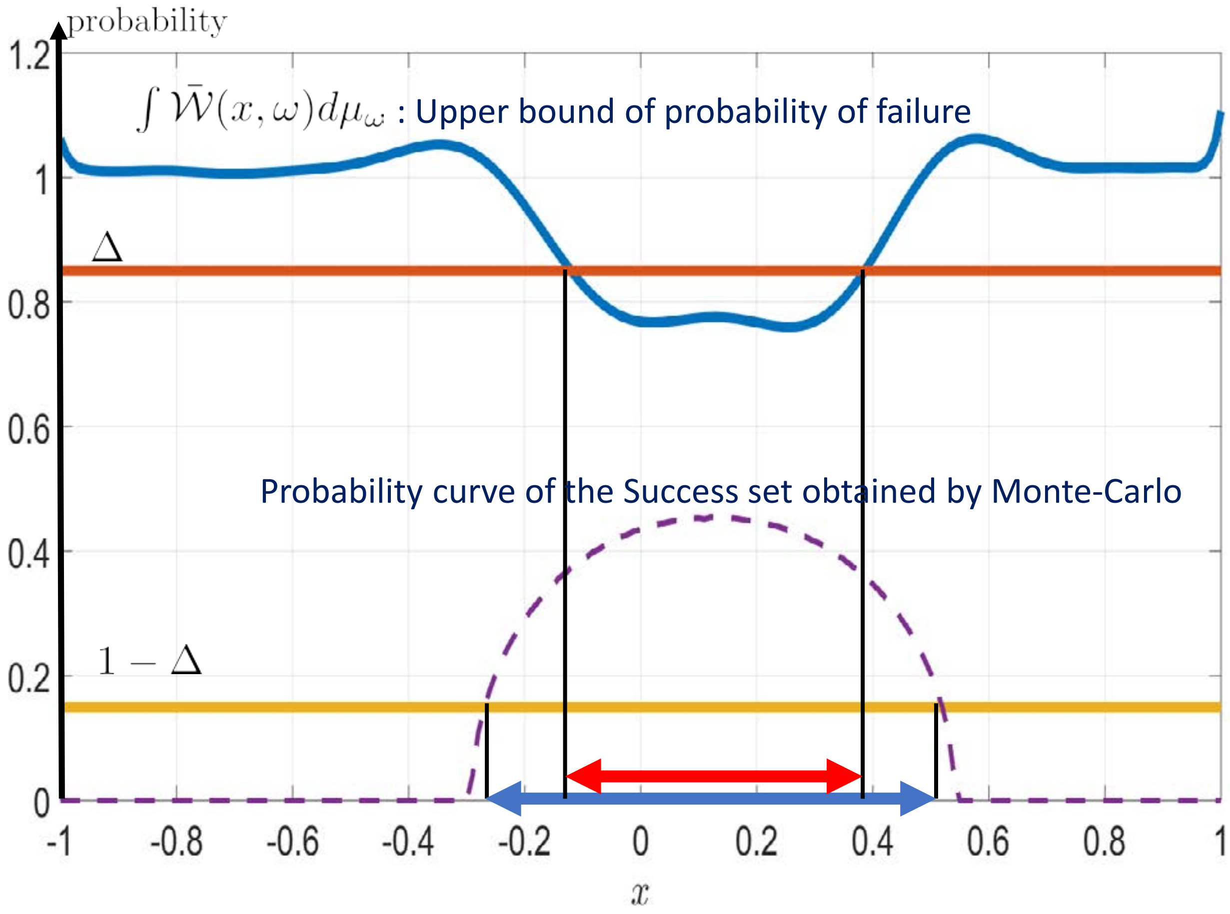
$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \int \bar{\mathcal{W}}(x,\omega)d\mu_\omega dx$$

subject to $\bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$

$\bar{\mathcal{W}}(x,\omega) \geq 0$

Complement Set

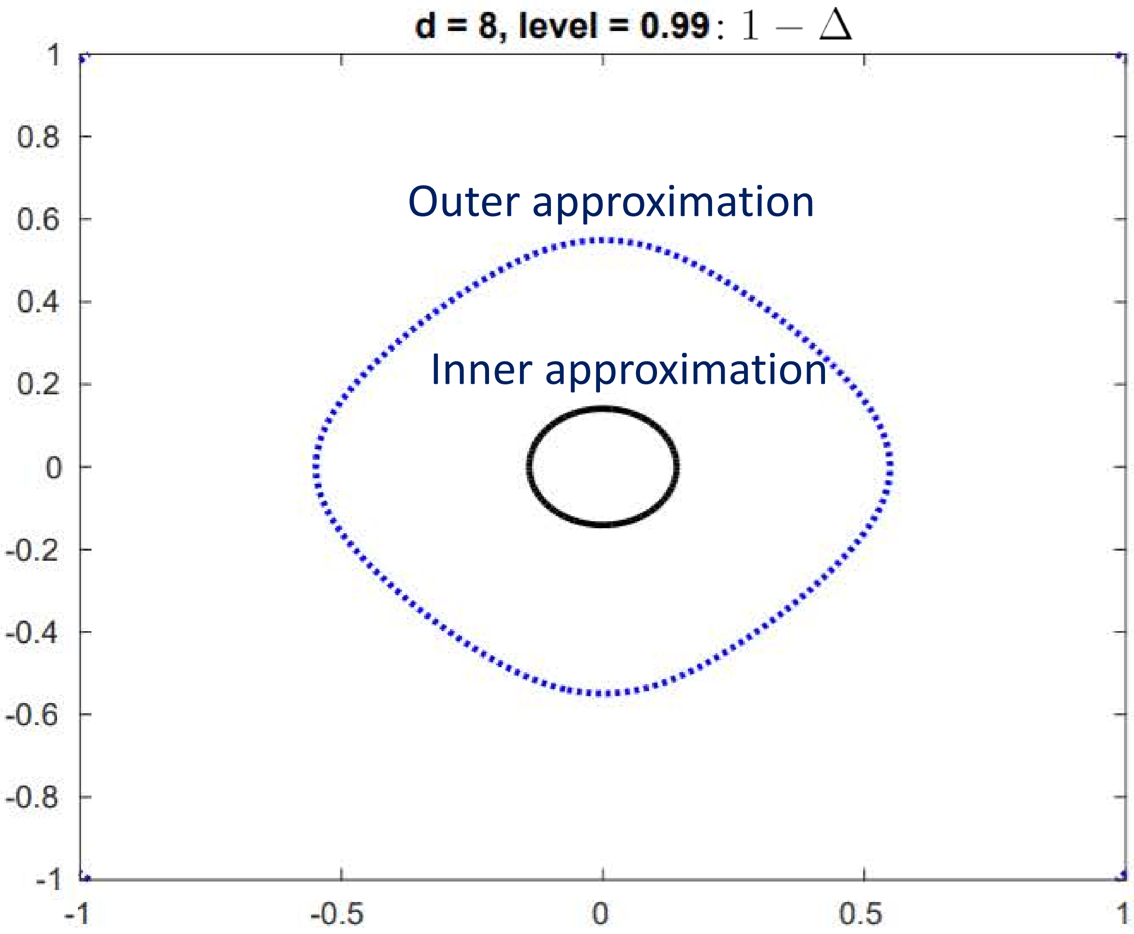


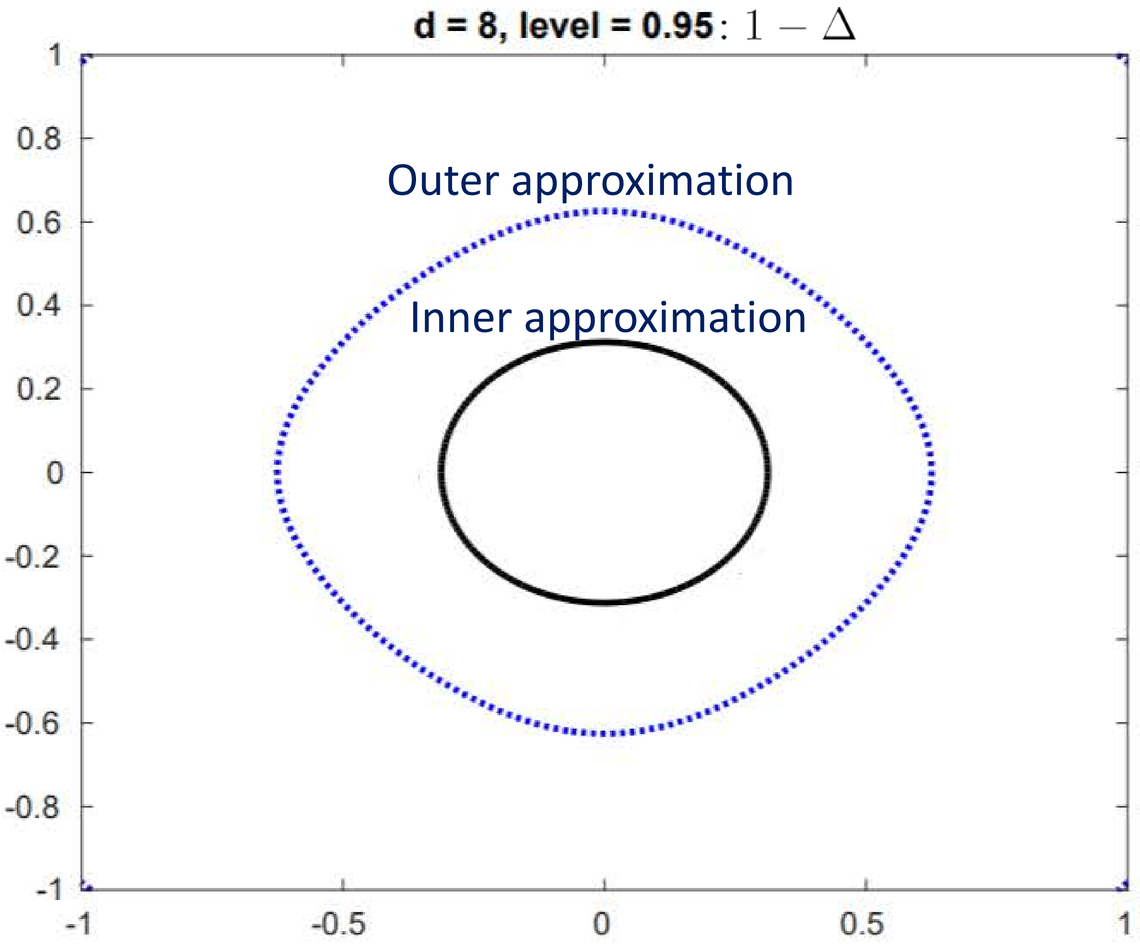


Inner approximation of $\{x : Prob(Success) \geq 1-\Delta\}$: $\chi_{cc} = \{x \in \mathbb{R}^n : \int \bar{\mathcal{W}}(x,\omega)\, d\mu_\omega \geq 1-\Delta\}$

Example: Probabilistic Safety Constraints

$$\chi_{cc} = \{(x_1, x_2) : Prob(\{1 - x_1^2 - x_2^2 - \omega^2 \geq 0\}) \geq 1 - \Delta\}$$





- Jean B. Lasserre, "Representation of Chance-Constraints With Strong Asymptotic Guarantees", IEEE Control Systems Letters , Volume: 1 , Issue: 1, 2017.

Summary

Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, ..., n_p\ )$$

$$\text{subject to} \qquad g_i(x) \geq 0,\ \ i = 1, ..., n_g$$

Moment Relaxation (SDP)

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} :=& \text{maximize}_{y, y_x} \quad y_0 \\ \text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p \\ & M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, n_g,\ y_{x_0} = 1 \\ & M_d(y_\omega \times y_x - y) \succcurlyeq 0 \end{aligned}$$

$d_{g_i} = \lceil \frac{deg(g_i(x))}{2} \rceil$ $d_{p_i} = \lceil \frac{deg(p_i(x))}{2} \rceil$

$2d \geq \max\left(deg(p_i(x)), deg(g_i(x))\right)$

- As $d \to \infty$ $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \downarrow \mathrm{P}^*$
- Finite SDP of order d: $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}}$ is upper bound of P^*
  - ➢ If obtained solution y_x^* satisfies rank condition $\text{Rank } M_d(y_x^*) = \text{Rank } M_{d-v}(y_x^*) = r \quad v = \max\{d_{g_i}\}$

    $x_i^*, i = 1, ..., r,$ global solutions can be extracted by linear algebra from y_x^*
  - ➢ Otherwise, increase d and solve new SDP.

Chance Constrained Optimization

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}\left(g_i(x,\omega) \geq 0,\ i=1,...,n_g \right) \geq 1-\Delta$$

Deterministic Opt $\Rightarrow$

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad x \in \chi_{cc}$$

SOS Programing for Chance Constrained Set

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \mathcal{W}(x,\omega)d\mu_\omega dx$$
$$\text{subject to} \quad \mathcal{W}(x,\omega)-1\geq 0 \quad \forall(x,\omega)\in\mathcal{K}$$
$$\mathcal{W}(x,\omega)\geq 0$$

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega)d\mu_\omega dx$$
$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega)-1\geq 0 \quad \forall(x,\omega)\in\bar{\mathcal{K}}$$
$$\bar{\mathcal{W}}(x,\omega)\geq 0$$

complement set

Chance Constrained Set $\{x\in\mathbb{R}^n : \text{Probability}(Success) \geq 1-\Delta\}$

- **Outer approximation :**

$$\chi_{cc} = \left\{x\in\mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1-\Delta\right\}$$

- As polynomial order $d \to \infty$

χ_{cc} Convergences to the true chance constrained set

- **Inner approximation :**

$$\bar{\chi}_{cc} = \left\{x\in\mathbb{R}^n : \int \bar{\mathcal{W}}(x,\omega)\ d\mu_\omega \leq \Delta\right\}$$

- As polynomial order $d \to \infty$

$\bar{\chi}_{cc}$ Convergences to the true chance constrained set

Chance Optimization and Dual SOS Programming

- A. Jasour, N. S. Aybat, and C. Lagoa ”Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets”, SIAM Journal on Optimization, 25(3), 1411–1440, 2015.
https://epubs.siam.org/doi/pdf/10.1137/140958736

- A. Jasour, "Convex Approximation of Chance Constrained Problems: Application in Systems and Control", School of Electrical Engineering and Computer Science, The Pennsylvania State University, 2016.
https://etda.libraries.psu.edu/catalog/13313aim5346

- A. Jasour, C. Lagoa, ”Semidefinite Relaxations of Chance Constrained Algebraic Problems”, 51st IEEE Conference on Decision and Control, Maui, Hawaii, 2012.
https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=6426305

- A. Jasour ,“Finite Convergence of Moment-SDP Hierarchy for Nonlinear Chance Constrained Optimization”(to appear)

- A. Jasour, C. Lagoa, "Convex Constrained Semialgebraic Volume Optimization: Application in Systems and Control” https://arxiv.org/abs/1701.08910

Chance Constrained Optimization (modified formulation)

- J. B. Lasserre, Representation of Chance-Constraints With Strong Asymptotic Guarantees, IEEE Control Systems Letters , Volume: 1 , Issue: 1, 2017.
https://ieeexplore.ieee.org/abstract/document/7927696

Appendix I:
Chance Optimization and Measure-LP

- We rewrite the chance optimization as a standard nonlinear optimization as follows:

$$\begin{aligned} \mathrm{P}^* = & \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad F(x) \\ & \text{subject to} \quad g_i(x) \geq 0,\ i = 1, ..., n_g \end{aligned}$$

$$\text{where} \quad F(x) = \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ i = 1, ..., n_p\ )$$

Case 1: ➢ $x^* \in \mathbf{K}\,, F(x^*) = \mathrm{P}^*$:Unique global optimal solution of the original problem.

Case 2: ➢ $x^{*i} \in \mathbf{K},\quad i = 1, ..., r\,, F(x^{*i}) = \mathrm{P}^*$: r global optimal solution of the original problem.

- Equivalent LP in measures (similar to measure-LP of nonlinear optimization problems, Lecture 4, page 41)

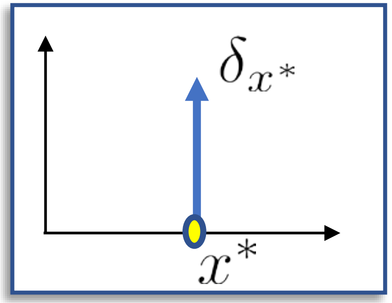
$$\begin{aligned} \mathrm{P}^*_\mu = & \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \mathrm{E}_{\mu_x}[F(x)] \\ & \text{subject to} \quad \int d\mu_x = 1 \\ & \qquad\qquad supp(\mu_x) \subset \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., n_g\} \end{aligned}$$

Dirac Measures

- Optimal solutions in measure space are Dirac measures.

- $x^* \in \mathbf{K}$, $F(x^*) = \mathrm{P}^*$:Unique global optimal solution of the original problem.
- $\mu^* = \delta_{x^*}$:Optimal solution of optimization in measures.

$$\mathrm{P}^*_\mu = \mathrm{E}_\mu[F(x)] = \int F(x)\delta_{x^*}dx = F(x^*) = \mathrm{P}^* \Longrightarrow \mathrm{P}^* = \mathrm{P}^*_\mu$$



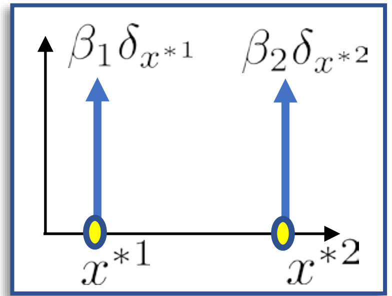
$\mu^* = \delta_{x^*}$

- $x^{*i} \in \mathbf{K}$, $i = 1, \ldots, r$, $F(x^{*i}) = \mathrm{P}^*$: r global optimal solution of the original problem.
- $\mu^* = \sum_{i=1}^r \beta_i \delta_{x^{*i}}$, $\beta_i > 0, \sum_{i=1}^r \beta_i = 1$:Optimal solution of optimization in measures.(r -atomic measure)

$$y_0 = 1 \rightarrow \quad y_0 = \sum_i^r \beta_i (x^{*i})^0 \rightarrow \sum_{i=1}^r \beta_i = 1$$

$$\mathrm{P}^*_\mu = \mathrm{E}_\mu[F(x)] = \int F(x)d\mu^*$$

$$= \int F(x)\left(\sum_{i=1}^r \beta_i \delta_{x^{*i}}\right)dx = \sum_{i=1}^r \beta_i \left(\int F(x)\delta_{x^{*i}}\right)dx = \sum_{i=1}^r \beta_i F(x^{*i}) = \sum_{i=1}^r \beta_i \mathrm{P}^* = \mathrm{P}^*$$



$\mu^* = \beta_1\delta_{x^{*1}} + \beta_2\delta_{x^{*2}}$

Appendix II:
Dual Optimization of Measure LP

Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x,\mu} & \int d\mu, \\ \text{s.t.} \quad & \mu \preccurlyeq \mu_x \times \mu_\omega \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

Primal Conic Program

$$
\begin{aligned}
P^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\
& \text{subject to} && A^*(x) = b \\
& && x \in K^*.
\end{aligned}
$$

Dual Conic Program

$$
\begin{aligned}
D^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\
& \text{subject to} && c - A(y) = s \\
& && s \in K.
\end{aligned}
$$

LP in Measure

$$
\begin{aligned}
\mathbf{P}^*_{\mu} := \ & \text{maximize}_{\mu_x,\mu} \int d\mu, \\
\text{s.t.} \ & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\
& \mu_x \text{ is a probability measure} \\
& supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}
\end{aligned}
$$

Dual

$$
\begin{aligned}
& \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\
& \text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\
& && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\
& && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0
\end{aligned}
$$

$\mu \in$ $\quad \mu_s \in$ $\quad \mu_x \in$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

Success set

Space of uncertainty $\omega \in \Omega$

Space of design parameters $x \in \chi$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A(y) = s \\ &&& s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\text{s.t.} \quad \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} &\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ &\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &&& \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &&& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle \overbrace{-\begin{bmatrix}1\\0\\0\end{bmatrix}}^{c}, \begin{bmatrix}\mu\\ \mu_s\\ \mu_x\end{bmatrix} \rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \longrightarrow \\ &&& x \in K^*. \quad A^*(.) = -\begin{bmatrix}\mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x\end{bmatrix} \quad b = -\begin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ & \text{subject to} && c - A(y) = s \\ &&& s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathbf{P}^*_\mu := & \underset{\mu_x,\mu}{\text{maximize}} \int d\mu, \\ \text{s.t.} \quad & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} & \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ & \text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &&& \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &&& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$\begin{aligned} \mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle \overbrace{- \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{c}, \begin{bmatrix} \mu \\ \mu_s \\ \mu_x \end{bmatrix} \rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \longrightarrow \\ &&& x \in K^*. \quad A^*(.) = - \begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix} \quad b = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x, \mu} \int d\mu,$$

$$\text{s.t.} \quad \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} &\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ &\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle \underbrace{-\begin{bmatrix}1\\0\\0\end{bmatrix}}_{c}, \begin{bmatrix}\mu\\ \mu_s\\ \mu_x\end{bmatrix}\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \longrightarrow \\ & && x \in K^*. \qquad A^*(.) = -\begin{bmatrix}\mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x\end{bmatrix} \quad b = -\begin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad A^*(.) = -\begin{bmatrix}\mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x\end{bmatrix}$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ & \text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\begin{aligned} \text{s.t. } & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} & \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ & \text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$\mu \in \quad \mu_s \in \quad \mu_x \in$

Linear Map $A^*(.) : V_2^* = \mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi) \rightarrow V_1^* = \mathcal{M}(\Omega \times \chi) \times \mathbb{R} \Longrightarrow A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix}$

Dual Linear Map $A(y) : V_1 = \mathcal{C}(\Omega \times \chi) \times \mathbb{R} \rightarrow V_2 = \mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} \quad \langle c, x\rangle_{V_1} \\ &\text{subject to} \quad A^*(x) = b \\ & \qquad\qquad\quad x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} \quad \langle y, b\rangle_{V_2} \\ &\text{subject to} \quad c - A(y) = s \\ & \qquad\qquad\quad s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_x, \mu} \int d\mu,$$

$$\begin{aligned} \text{s.t.} \quad & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} &\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} \quad \beta \\ &\text{subject to} \quad w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &\qquad\qquad\quad \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &\qquad\qquad\quad \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$\text{Linear Map } A^*(.) : V_2^* = \underbrace{\mathcal{M}(\mathcal{K})}_{\mu \in} \times \underbrace{\mathcal{M}(\Omega \times \chi)}_{\mu_s \in} \times \underbrace{\mathcal{M}(\chi)}_{\mu_x \in} \to V_1^* = \mathcal{M}(\Omega \times \chi) \times \mathbb{R} \Longrightarrow A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix}$$

$$\text{Dual Linear Map } A(y) : V_1 = \mathcal{C}(\Omega \times \chi) \times \mathbb{R} \to V_2 = \mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$$

Primal Conic Program

$$
\begin{aligned}
\mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\
&\text{subject to} && A^*(x) = b \\
& && x \in K^*.
\end{aligned}
$$

Dual Conic Program

$$
\begin{aligned}
\mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\
&\text{subject to} && c - A(y) = s \\
& && s \in K.
\end{aligned}
$$

LP in Measure

$$
\begin{aligned}
\mathbf{P}^*_\mu := &\ \text{maximize}_{\mu_x,\mu} \int d\mu, \\
\text{s.t. } & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\
& \mu_x \text{ is a probability measure} \\
& supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}
\end{aligned}
$$

Dual

$$
\begin{aligned}
&\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\
&\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\
& && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\
& && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0
\end{aligned}
$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

Linear Map $A^*(.) : V_2^* = \underset{\mu \in}{\mathcal{M}(\mathcal{K})} \times \underset{\mu_s \in}{\mathcal{M}(\Omega \times \chi)} \times \underset{\mu_x \in}{\mathcal{M}(\chi)} \rightarrow V_1^* = \mathcal{M}(\Omega \times \chi) \times \mathbb{R} \Longrightarrow A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix}$

Dual Linear Map $A(y) : V_1 = \mathcal{C}(\Omega \times \chi) \times \mathbb{R} \rightarrow V_2 = \mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$

$$< A^*(x), y >_{V_1} = < x, A(y) >_{V_2}$$

$$
\begin{array}{c} \mathcal{M}(\Omega \times \chi) \leftarrow \\ \mathbb{R} \leftarrow \end{array}
- \begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix}
\quad
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\begin{array}{l} \rightarrow \mathcal{C}(\Omega \times \chi) \\ \rightarrow \mathbb{R} \end{array}
\quad
\begin{bmatrix} \mu \\ \mu_s \\ \mu_x \end{bmatrix}
\quad
\begin{bmatrix} A_1(.) \\ A_2(.) \\ A_3(.) \end{bmatrix}
\begin{array}{l} \rightarrow \mathcal{C}(\mathcal{K}) \\ \rightarrow \mathcal{C}(\Omega \times \chi) \\ \rightarrow \mathcal{C}(\chi) \end{array}
$$

Primal Conic Program

$$\begin{aligned} P^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} &\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ &\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

Linear Map $A^*(.) : V_2^* = \underset{\mu \in}{\mathcal{M}(\mathcal{K})} \times \underset{\mu_s \in}{\mathcal{M}(\Omega \times \chi)} \times \underset{\mu_x \in}{\mathcal{M}(\chi)} \rightarrow V_1^* = \mathcal{M}(\Omega \times \chi) \times \mathbb{R} \Longrightarrow A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix}$

Dual Linear Map $A(y) : V_1 = \mathcal{C}(\Omega \times \chi) \times \mathbb{R} \rightarrow V_2 = \mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$

$$< A^*(x), y >_{V_1} = < x, A(y) >_{V_2}$$

$$\begin{array}{c} \mathcal{M}(\Omega \times \chi) \\ \mathbb{R} \end{array} \longleftarrow -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightarrow \begin{array}{c} \mathcal{C}(\Omega \times \chi) \\ \mathbb{R} \end{array} \qquad \begin{bmatrix} \mu \\ \mu_s \\ \mu_x \end{bmatrix} \quad \begin{bmatrix} A_1(.) \\ A_2(.) \\ A_3(.) \end{bmatrix} \rightarrow \begin{array}{c} \mathcal{C}(\mathcal{K}) \\ \mathcal{C}(\Omega \times \chi) \\ \mathcal{C}(\chi) \end{array}$$

$$-\int y_1 d(\mu + \mu_s - \mu_x \times \mu_\omega) - y_2 \int d\mu_x = \int A_1(.) d\mu + \int A_2(.) d\mu_s + \int A_3(.) d\mu_x$$

Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$-\int y_1 d(\mu + \mu_s - \mu_x \times \mu_\omega) - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A(y) = s \\ &&& s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\text{s.t.} \quad \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$$

$$\mu_x \text{ is a probability measure}$$

$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

Dual

$$\begin{aligned} &\underset{\beta,\mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ &\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &&& \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &&& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$-\int y_1 d(\mu + \mu_s - \mu_x \times \mu_\omega) - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

$$\overbrace{-\int y_1 d\mu - \int y_1 d\mu_s + \int y_1 d(\mu_x \times \mu_\omega)} - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

Primal Conic Program

$$\begin{aligned} P^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ & \text{subject to} && c - A(y) = s \\ &&& s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} & \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ & \text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &&& \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &&& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$-\int y_1 d(\mu + \mu_s - \mu_x \times \mu_\omega) - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

$$-\int y_1 d\mu - \int y_1 d\mu_s + \int y_1 d(\mu_x \times \mu_\omega) - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

$$-\int y_1 d\mu - \int y_1 d\mu_s + \overbrace{\int \left(\int y_1 d\mu_\omega\right) d\mu_x} - y_2 \int d\mu_x = \int A_1(.)d\mu + \int A_2(.)d\mu_s + \int A_3(.)d\mu_x$$

Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\text{s.t.} \quad \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega}$$

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$$\begin{aligned} \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

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$$A_1(.) = -y_1 \in \mathcal{C}(\mathcal{K}) \subset \mathcal{C}(\Omega \times \chi)$$

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$$A_1(.) = -y_1 \in \mathcal{C}(\mathcal{K}) \subset \mathcal{C}(\Omega \times \chi) \qquad A_2(.) = -y_1 \in \mathcal{C}(\Omega \times \chi) \qquad A_3(.) = \overbrace{\underbrace{\int y_1 d\mu_\omega}_{\in \mathcal{C}(\Omega \times \chi)}}^{\in \mathcal{C}(\chi)} - \underbrace{y_2}_{\in \mathbb{R}} \in \mathcal{C}(\chi)$$

Primal Conic Program

$$\begin{aligned} P^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

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LP in Measure

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Dual

$$\begin{aligned} &\underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ &\text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &&& \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &&& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

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$$A_1(.) = \underbrace{-y_1}_{\Downarrow} \in \mathcal{C}(\mathcal{K}) \subset \mathcal{C}(\Omega \times \chi)$$

Denoted by $-\mathcal{W}(x,\omega)$

$$A_2(.) = \underbrace{-y_1}_{\Downarrow} \in \mathcal{C}(\Omega \times \chi)$$

Denoted by $-\mathcal{W}(x,\omega)$

$$A_3(.) = \overbrace{\int \underbrace{y_1}_{\in \mathcal{C}(\Omega \times \chi)} d\mu_\omega - \underbrace{y_2}_{\in \mathbb{R}}}^{\in \mathcal{C}(\chi)} \in \mathcal{C}(\chi) \Longrightarrow A_3(.) = -\beta + \int \mathcal{W}(x,\omega) d\mu_\omega$$

Denoted by $-\beta$

Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

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$$\mathbf{P}^*_{\mu} := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} \underset{\beta,\mathcal{W}(x,\omega)}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle -\overbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}^{c}, \begin{bmatrix}\mu\\ \mu_s\\ \mu_x\end{bmatrix}\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \longrightarrow \\ & x \in K^*. \qquad A^*(.) = -\begin{bmatrix}\mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x\end{bmatrix} \quad b = -\begin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \longrightarrow \langle y, b\rangle_{V_1} = \langle \begin{bmatrix}\mathcal{W}(x,\omega)\\ \beta\end{bmatrix}, - \begin{bmatrix}0\\1\end{bmatrix}\rangle_{V_1} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

$$-\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{matrix} -(-\mathcal{W}(x,\omega)) \geq 0 & \forall (x,\omega) \in \mathcal{K} \\ -(-\mathcal{W}(x,\omega)) \geq 0 & \forall (x,\omega) \in \Omega \times \chi \\ -(-\beta + \int \mathcal{W}(x,\omega) d\mu_\omega) \geq 0 & \forall (x,\omega) \in \chi \end{matrix}$$

$A_1(.) = -\mathcal{W}(x,\omega) \in \mathcal{C}(\mathcal{K})$

$A_2(.) = -\mathcal{W}(x,\omega) \in \mathcal{C}(\Omega \times \chi)$

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Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} & \int d\mu, \\ \text{s.t.} \quad & \boxed{\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega} \\ & \mu_x \text{ is a probability measure} \\ & supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K} \end{aligned}$$

Dual

$$\begin{aligned} \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

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$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle -\overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{c}, \begin{bmatrix} \mu \\ \mu_s \\ \mu_x \end{bmatrix} \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \longrightarrow \\ & x \in K^*. \quad A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix} \quad b = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \longrightarrow \langle y, b\rangle_{V_1} = \langle \begin{bmatrix} \mathcal{W}(x,\omega) \\ \beta \end{bmatrix}, - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle_{V_1} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

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Primal Conic Program

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c

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$$b = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

$$\begin{aligned} &\underset{\beta,\mathcal{W}(x,\omega)}{\text{minimize}} \quad \beta \\ &\text{subject to} \quad w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ &\qquad\qquad\quad \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ &\qquad\qquad\quad \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative continuous functions

c

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} \quad \langle c, x\rangle_{V_1} \longrightarrow \\ &\text{subject to} \quad A^*(x) = \underline{b} \\ &\qquad\qquad\quad x \in K^*. \end{aligned}$$

$$b = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}, \quad supp(\mu_s) \subset \Omega \times \chi$

Primal Conic Program

$$\begin{aligned} P^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ & \text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in Measure

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

s.t. $\mu \preccurlyeq \mu_x \times \mu_\omega \longrightarrow \mu + \mu_s = \mu_x \times \mu_\omega$

μ_x is a probability measure

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$

Dual

$$\begin{aligned} & \underset{\beta, \mathcal{W}(x,\omega)}{\text{minimize}} && \beta \\ & \text{subject to} && w(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\ & && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & && \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\Omega \times \chi) \times \mathcal{M}(\chi)$ space of (nonnegative) measures

$\mathcal{C}(\mathcal{K}) \times \mathcal{C}(\Omega \times \chi) \times \mathcal{C}(\chi)$ space of nonnegative conti... ...nctions

$$\begin{aligned} P^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \longrightarrow \langle c, x\rangle_{V_1} = \langle -\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{c}, \begin{bmatrix} \mu \\ \mu_s \\ \mu_x \end{bmatrix} \rangle_{V_1} \\ & \text{subject to} && A^*(x) = b \\ & && x \in K^*. \qquad A^*(.) = -\begin{bmatrix} \mu + \mu_s - \mu_x \times \mu_\omega \\ \int d\mu_x \end{bmatrix} \quad b = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}, \quad supp(\mu_s) \subset \Omega \times \chi$

$$\begin{aligned} D^* = & \text{minimize} && \beta \\ & \text{subject to} && \mathcal{W}(x,\omega) \geq 1 && \forall (x,\omega) \in \mathcal{K} \\ & && \mathcal{W}(x,\omega) \geq 0 && \forall (x,\omega) \in \Omega \times \chi \\ & && \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 && \forall (x,\omega) \in \chi \\ & && \beta \geq 0 \end{aligned}$$