Lecture 13

Occupation Measure Based Control of Continuous-Time Nonlinear Dynamical System

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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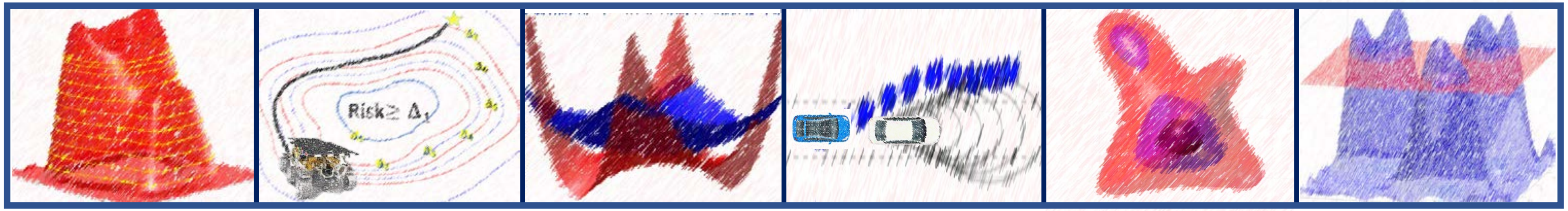


Probabilistic Dynamical Systems and Probabilistic Safety Constraints

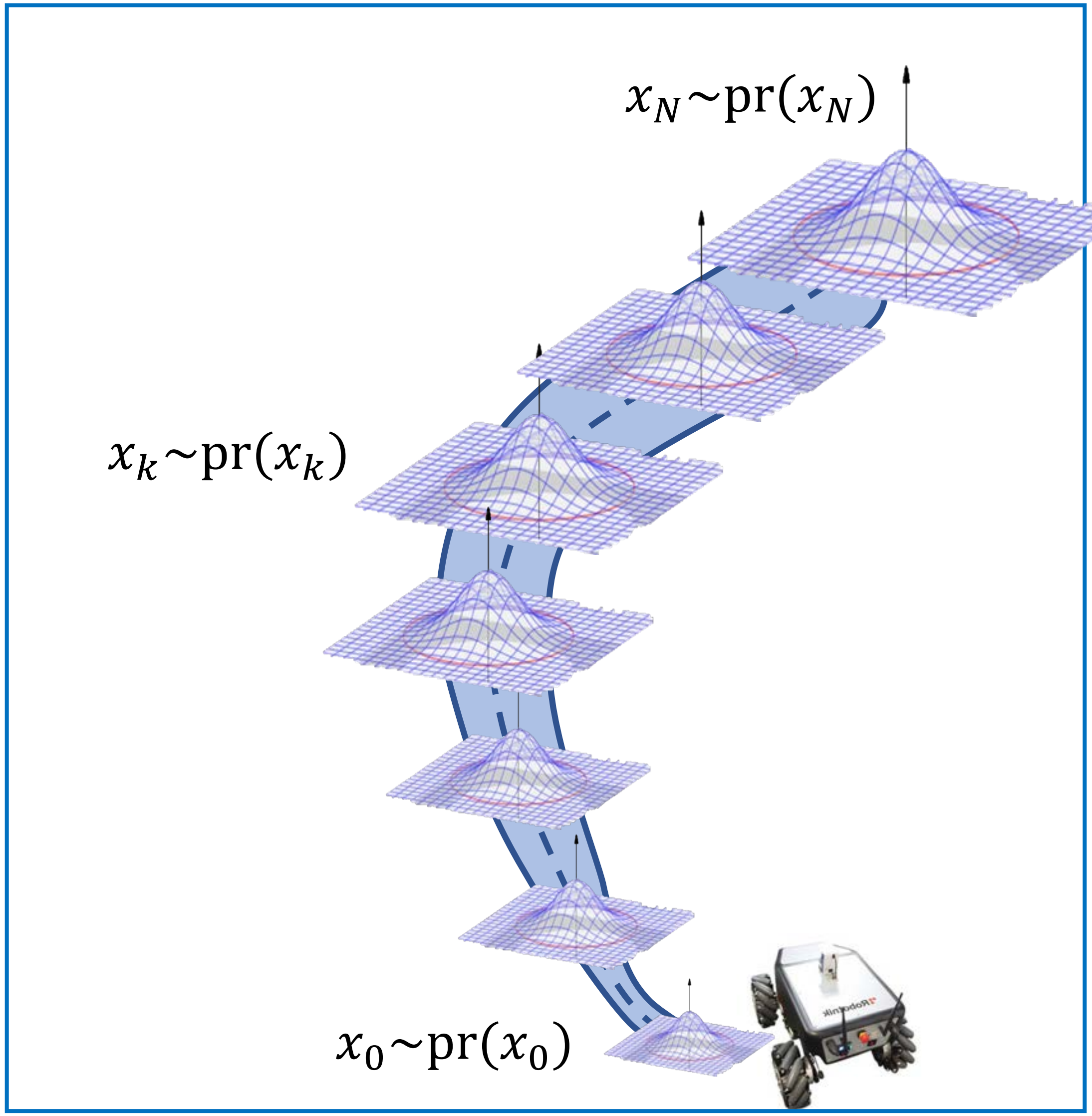
Discrete-Time Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states inputs Uncertainty ~ pr(ω_k):probability distribution

- For safety and control, we need to work with probability distribution s of the uncertainty along the planning horizon.

$$x_k \sim pr(x_k) \quad k = 0, \dots, N$$



Probabilistic Dynamical Systems and Probabilistic Safety Constraints

Continuous-Time Model

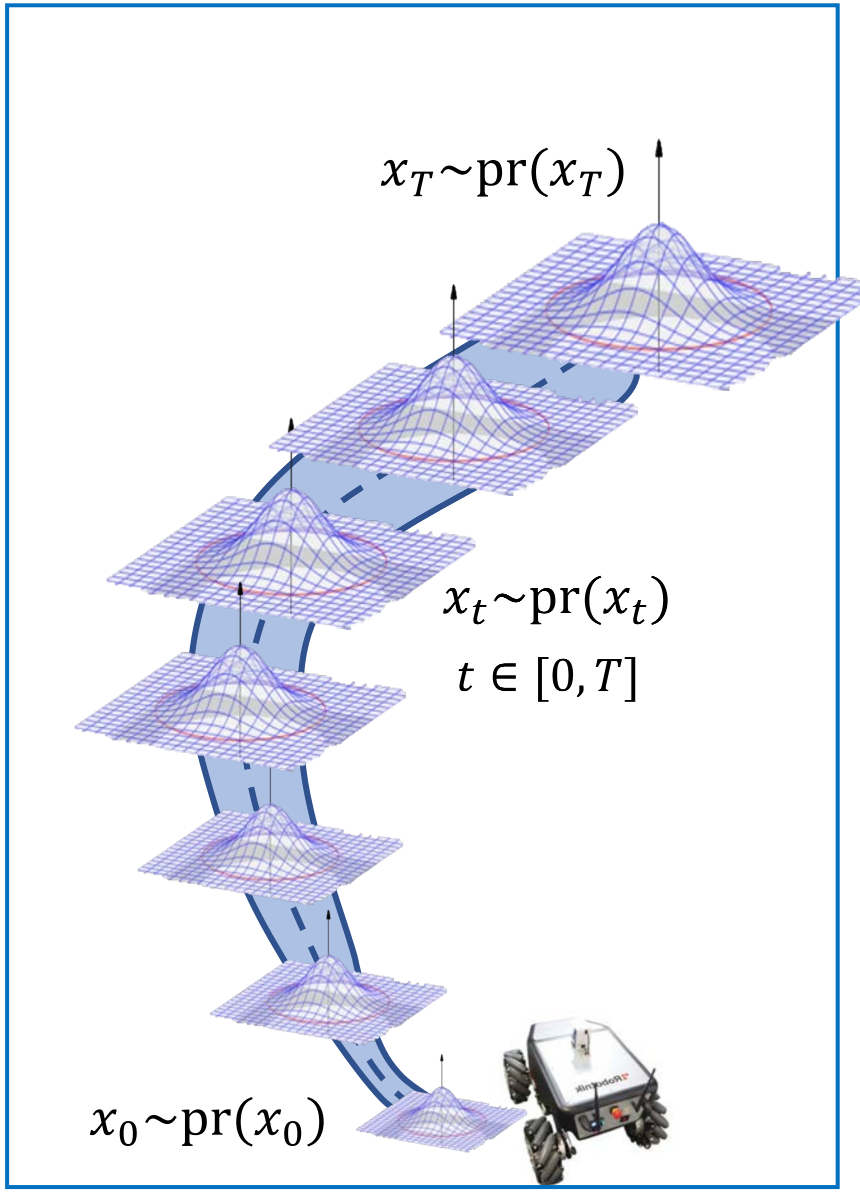
Ordinary Differential Equation (ODE)

$$\dot{x}(t) = f(x(t), u(t)) \qquad x_0 \sim pr(x_0)$$

- Due to probabilistic initial states, state of the system at each time t are also probabilistic.

- The initial measure is transported by the flow of the ODE.

$$x_t \sim pr(x_t) \quad t \in [0, T]$$



Probabilistic Dynamical Systems and Probabilistic Safety Constraints

Continuous-Time Model

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- Due to probabilistic initial states, state of the system at each time t are also probabilistic.

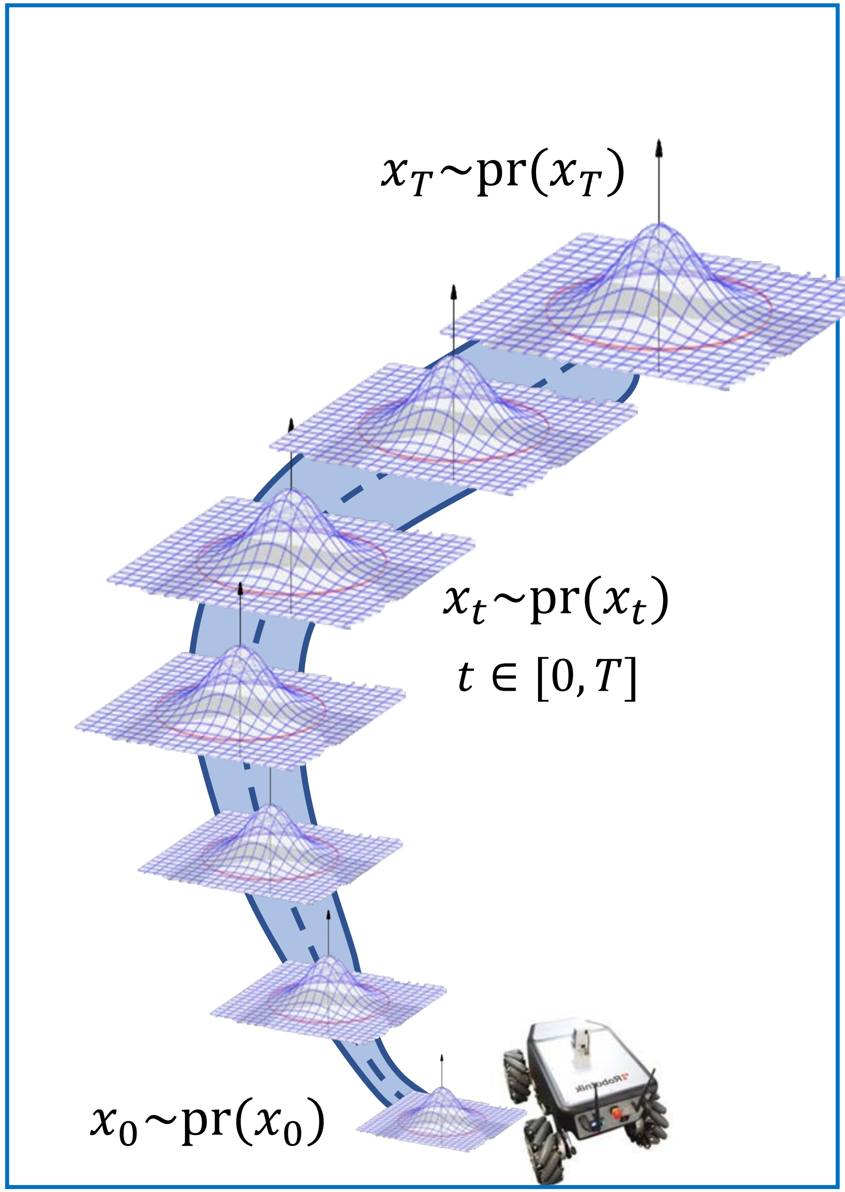
- The initial measure is transported by the flow of the ODE.

$$x_t \sim pr(x_t) \quad t \in [0, T]$$

- For safety and control,

• Instead of working with probability measures $x_t \sim pr(x_t)$ over planning horizon $t \in [0, T]$

• We work with 3 distributions:
  1) *Initial* distribution **2)** *Terminal* distribution,
  3) *Average Occupation Measure* that captures the information of the probabilistic trajectories



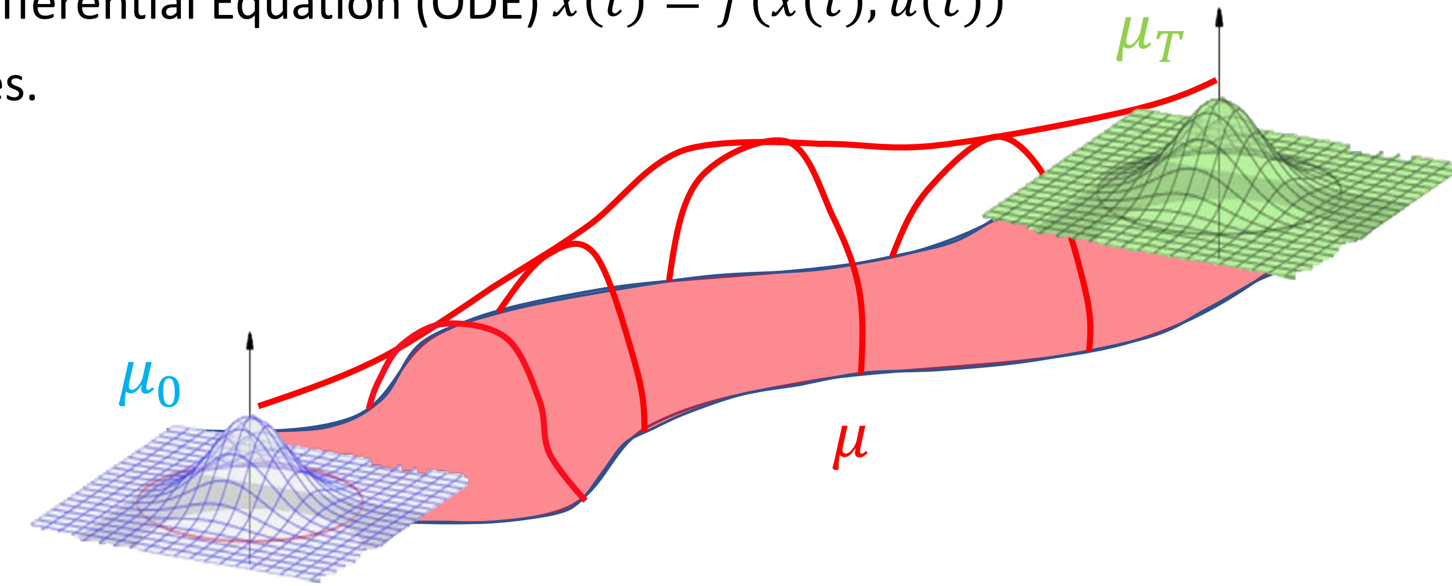
- We work with 3 distributions (measures):
  1) *Initial* distribution
  2) *Terminal* distributions,
  3) *Average Occupation Measure* that captures the information of the probabilistic trajectories

➢ **(Average)occupation measure** captures the information of dynamical systems in continuous-time.

➢ These measures satisfy **Linear** Partial Differential Equation (PDE).

- Instead of working with **Nonlinear** Ordinary Differential Equation (ODE) $\dot{x}(t) = f(x(t), u(t))$

  We work with **Linear** PDE in terms of measures.



- We can formulate control and planning problems of continuous-time dynamical systems as optimization problems with **differential constraints**.

Example: Optimal Control

$$\begin{array}{ll} \inf & \int_0^T l(t, x(t), u(t))dt \\ \text{s.t.} & \dot{x}(t) = f(t, x(t), u(t)), \\ & x(t) \in X, \;\; u(t) \in U, \;\; t \in [0, T], \\ & x(0) \in X_0, \quad x(T) \in X_T \end{array}$$

- Using notion of (average)**occupation Measure,** we can reformulate such optimizations in terms of measures (Linear Program) and their moments (Semidefinite Program).

Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

(Average)Occupation Measure
and
Liouville’s Equation

D. Henrion, M. Ganet-Schoeller, S. Bennani. ”Measures and LMI for space launcher robust control validation”, Proceedings of the IFAC Symposium on Robust Control Design, Aalborg, Denmark, June 2012.

D. Henrion ”Optimization on linear matrix inequalities for polynomial systems control”, Lecture notes used for a tutorial course given during the International Summer School of Automatic Control held at Grenoble, France, in September 2014.

Notations

Measure (Lecture 3: measure and moment based nonlinear optimization)

(Nonnegative) measure $\mu : \Sigma \rightarrow \mathbb{R}_+$

- In general (nonnegative) measure μ assigns real numbers to sets.(measures the size of the set)

$$\mu(A) = \int_A f(x)dx = \int_A d\mu = \int_A \mu(dx) = \int \mathbf{I}_A \mu(dx)$$

- A: Set in x domain
- $f(x)$: density function of μ
- $\mu(dx)$: To emphasize that measure is defined in x domain
- $\mathbf{I}_A$: Indicator function of set A

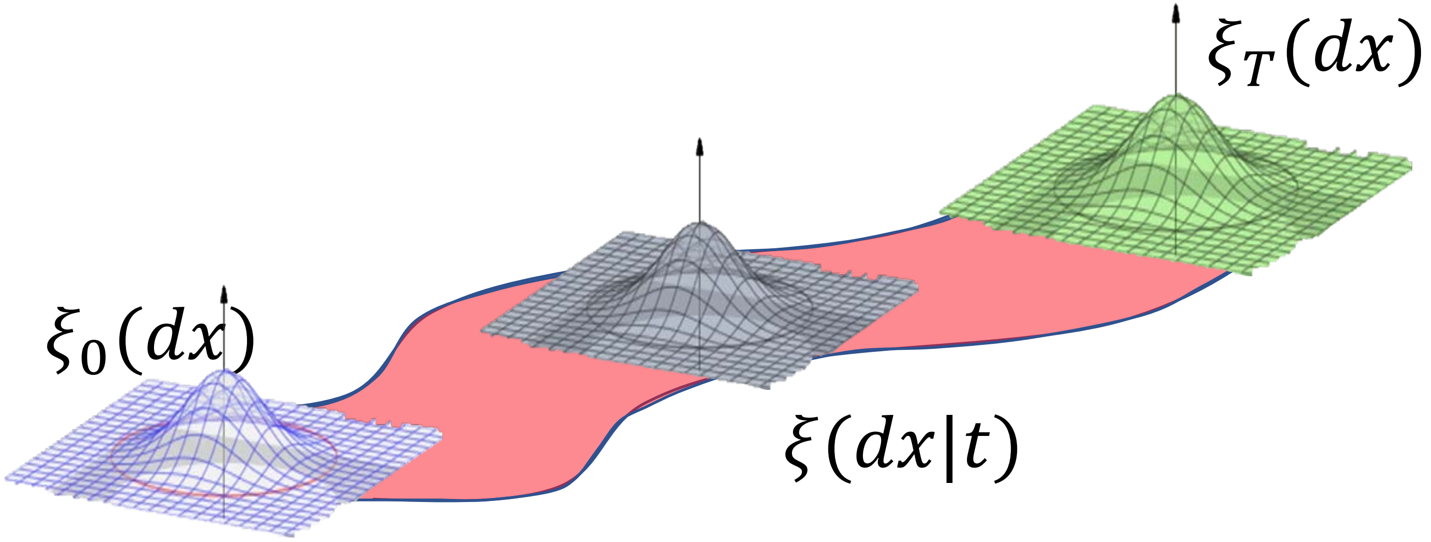
e.g., $x \sim \mu(dx)$ Probability measure of random variable in x domain

$\mu(A)$: probability that random variable is in set A

- **moment** of order α of a measure μ $\quad y_\alpha = \mathrm{E}_\mu[x^\alpha] = \int x^\alpha d\mu$

ODE: $\dot{x}(t) = f(t, x(t))$ $\quad t \in [0, T]$ $\quad x \in X$

- Initial states are random variable $x_0 \sim \xi_0(dx)$ (Probability measures)
- Due to random initial states, ODE has a **family of trajectories.** $x_t \sim \xi(dx|t)$ (probability measure of states for given t)
- Terminal states are random variable $x_T \sim \xi_T(dx)$ (Probability measures)



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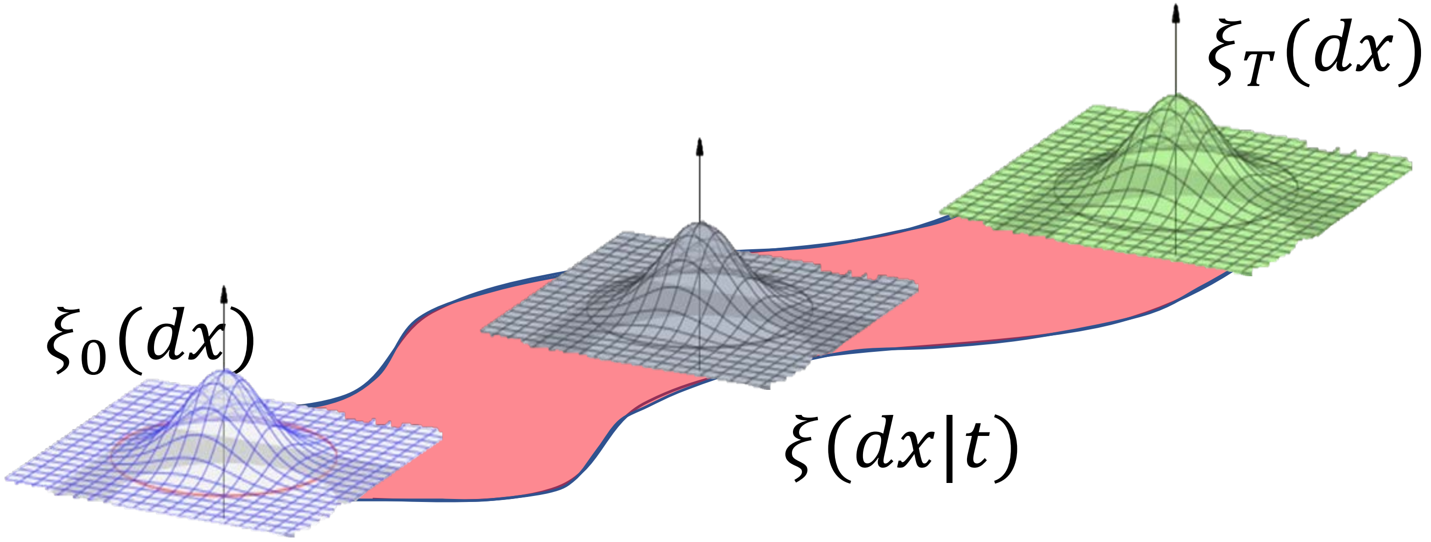
Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$ $t \in [0, T]$



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Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$ $t \in [0, T]$

- We add time to the description of probability measures
- We define measures whose **marginal** distributions are defined in 1) **state** space and 2) **time** domain

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Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$ ← Measures defined in **state** space

- ➢ We add time to the description of probability measures
- ➢ We define measures whose **marginal** distributions are defined in 1) **state** space and 2) **time** domain

e.g., $\mu(dt, dx) = \mu(dt)\, \mu(dx)$ ← Measure defined in **time and state** spaces

Marginal measure in time ← $\mu(dt)$

Marginal measure in states ← $\mu(dx)$

ODE: $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$

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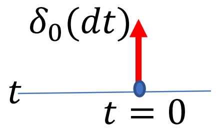
Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$

➢ We add time to the description of probability measures

Marginal measure in time

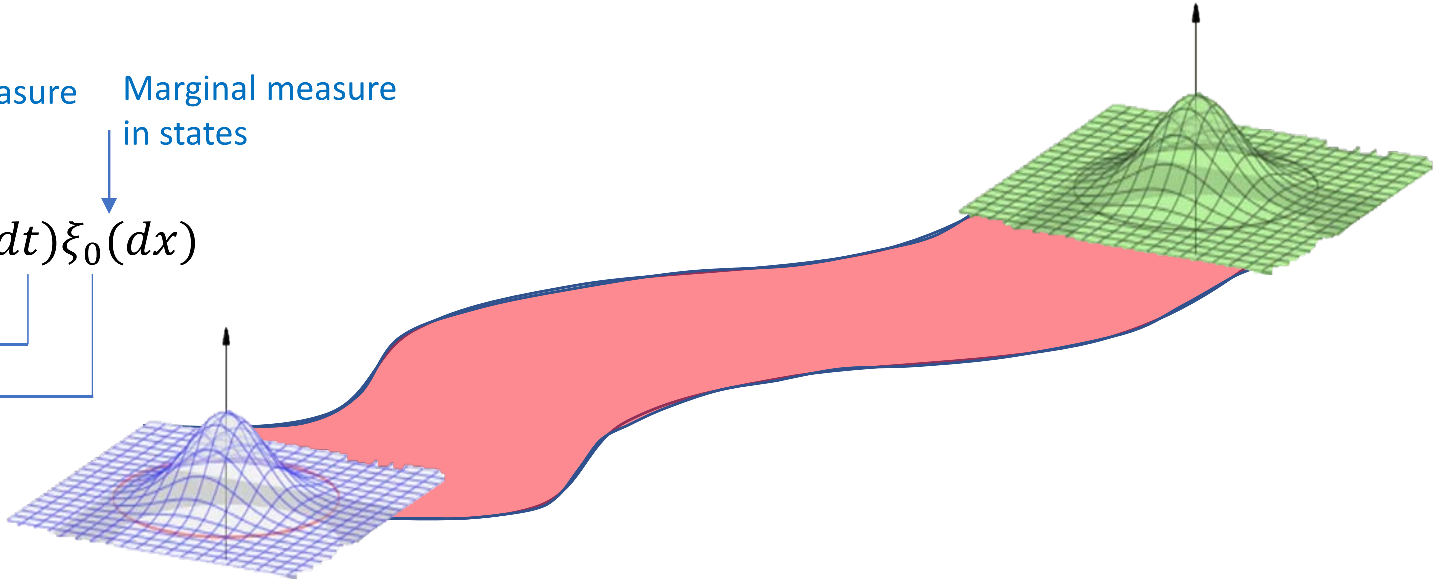
Marginal measure in states

Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$



Delta distribution of time t=0

Probability measure of states



ODE: $\dot{x}(t) = f(t, x(t)) \qquad t \in [0, T] \qquad x \in X$

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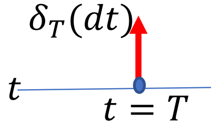
Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$

➢ We add time to the description of probability measures

Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$



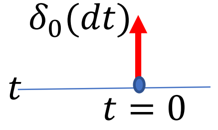
Delta distribution of time $t = T$

Probability measure of states

Marginal measure in time

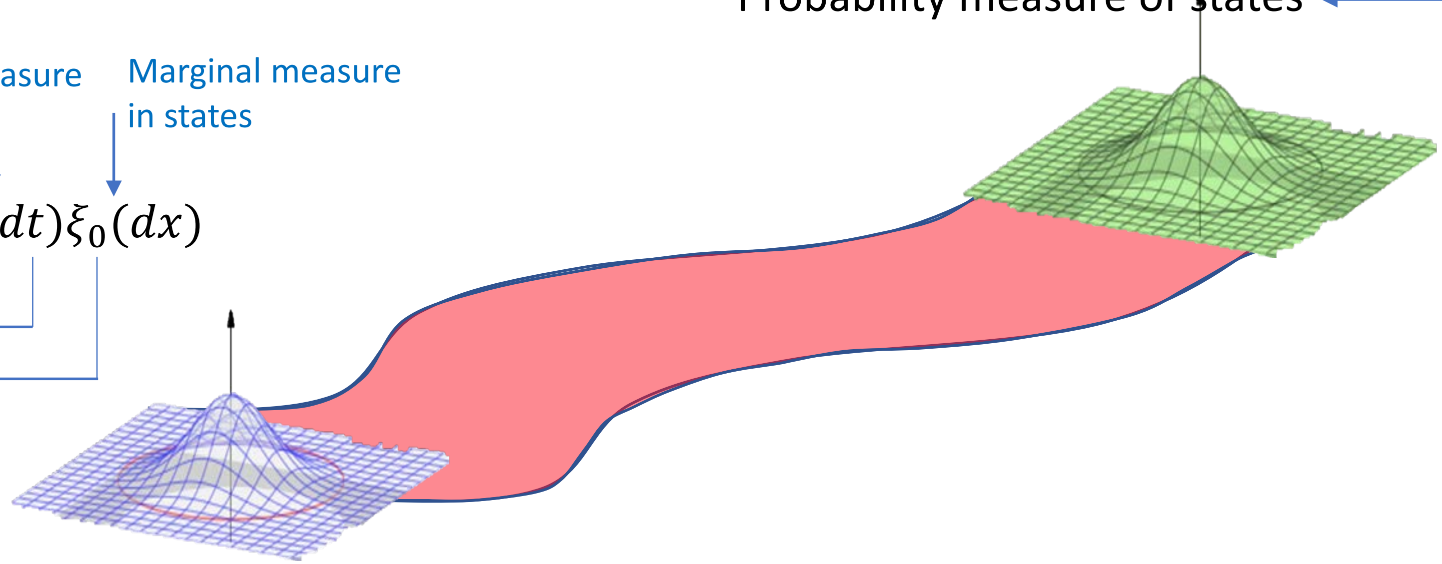
Marginal measure in states

Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$



Delta distribution of time t=0

Probability measure of states



ODE: $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$

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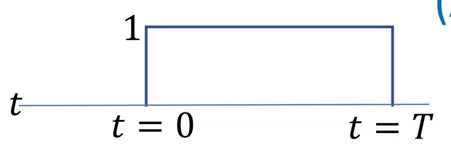
Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$

➢ We add time to the description of probability measures

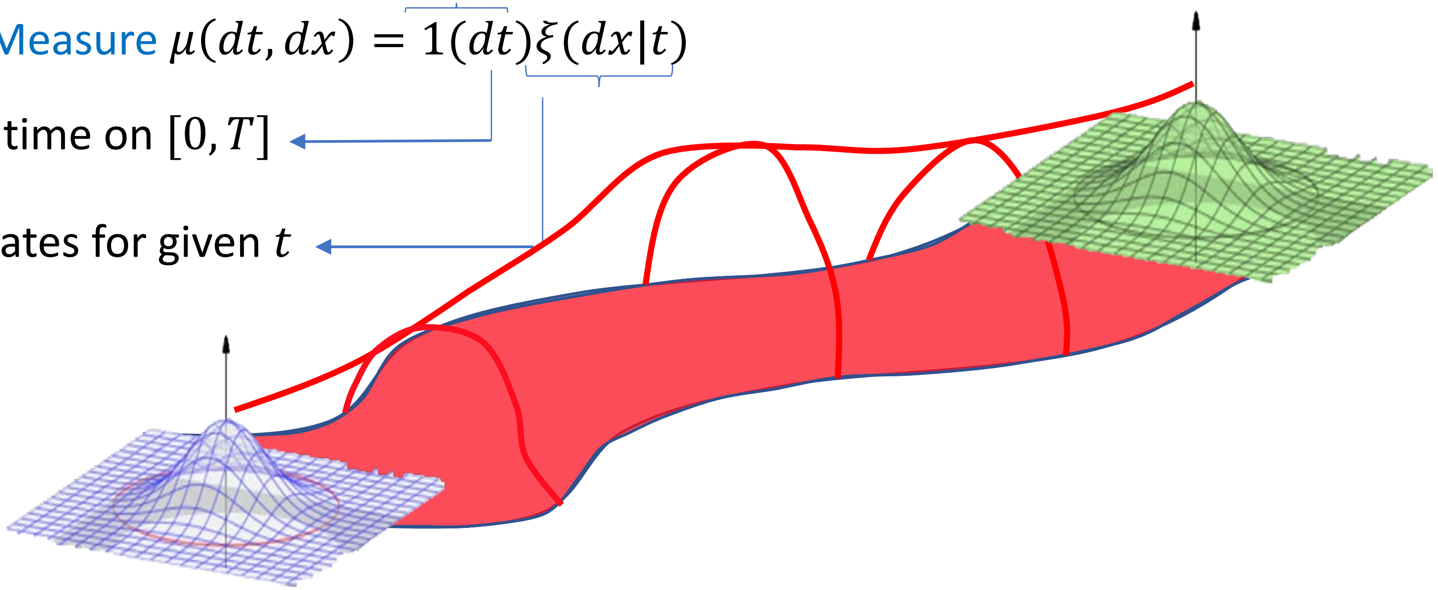
Marginal measure in time

(Average) Occupation Measure $\mu(dt, dx) = 1(dt)\xi(dx|t)$



Lebesgue Measure of time on $[0, T]$

probability measure of states for given t



Example:

- **ODE** $\dot{x}(t) = -x(t)$

- Initial state $x(0) = 1$

- Trajectory $x(t) = e^{-t}$ (solution of ODE for the given initial state)

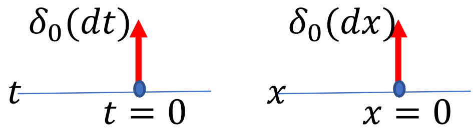
- $x(T = 0.693) = \frac{1}{2}$

Example:

- **ODE** $\dot{x}(t) = -x(t)$

- Initial state $x(0) = 1$ → Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\delta_1(dx)$

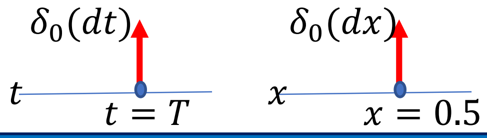
  - Marginal measure in time $t = 0$
  - Marginal measure in states
  - Probability measure of $x = 1$



- Trajectory $x(t) = e^{-t}$

- $x(T = 0.693) = \frac{1}{2}$ → Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\delta_{\frac{1}{2}}(dx)$

  - Marginal measure in time $t = T$
  - Marginal measure in states
  - Probability measure of $x = \frac{1}{2}$



Example:

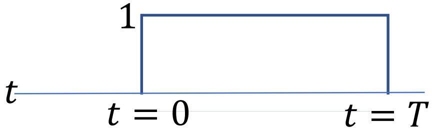
- **ODE** $\dot{x}(t) = -x(t)$

- Marginal measure in time $t = 0$
- Marginal measure in states
- Probability measure of $x = 1$

- Initial state $x(0) = 1$ → Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\delta_1(dx)$

- Marginal measure in time $t \in [0, T]$
- Conditional measure in states

- Trajectory $x(t) = e^{-t}$ → (Average)Occupation Measure $\mu(dt, dx) = 1(dt)\delta_{e^{-t}}(dx)$



Delta distributions along the trajectory $x = e^{-t}$

- (Average) Occupation Measure captures the information of trajectory

- $x(T = 0.693) = \frac{1}{2}$ → Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\delta_{\frac{1}{2}}(dx)$

- Marginal measure in time $t = T$
- Marginal measure in states
- Probability measure of $x = \frac{1}{2}$

Example:

- **ODE** $\dot{x}(t) = -x(t)$

- Initial state $x(0) = 1$
- Trajectory $x(t) = e^{-t}$
- $x(T = 0.693) = \frac{1}{2}$

- Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\delta_1(dx)$
- (Average) Occupation Measure $\mu(dt, dx) = 1(dt)\delta_{e^{-t}}(dx)$
- Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\delta_{\frac{1}{2}}(dx)$

- These 3 measure captures the information of dynamical system.

Example:

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- (Average) Occupation Measure $\mu(dt, dx) = 1(dt)\delta_{e^{-t}}(dx)$
- Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\delta_{\frac{1}{2}}(dx)$

- These 3 measure captures the information of dynamical system.
- In the case of **uncertain states**, measure of states are **non-delta** probability distributions.

ODE: $\dot{x}(t) = f(t, x(t)) \qquad t \in [0, T] \qquad x \in X$

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Probability measures of states $(\xi_0(dx), \xi(dx|t), \xi_T(dx))$

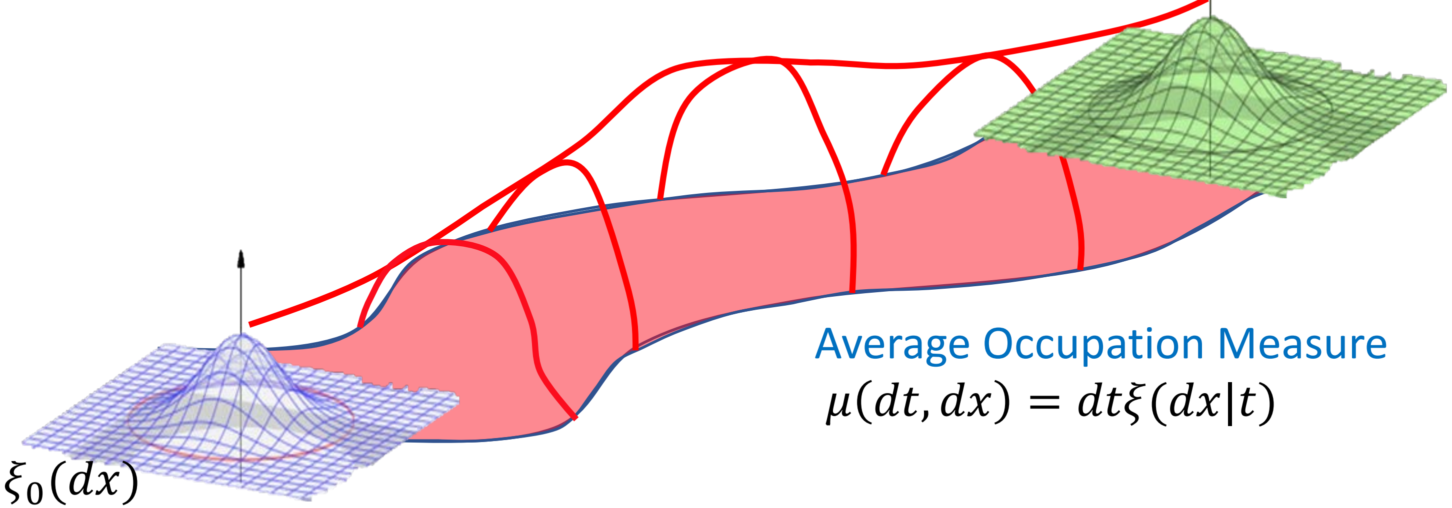
- Measures in time and state space

Measures $\left(\mu_0(dt, dx), \mu(dt, dx), \mu_T(dt, dx)\right)$

- These measures satisfy **Linear** Partial Differential Equation (PDE).

Terminal Measure

$\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$



Average Occupation Measure

$\mu(dt, dx) = dt\xi(dx|t)$

Initial Measure

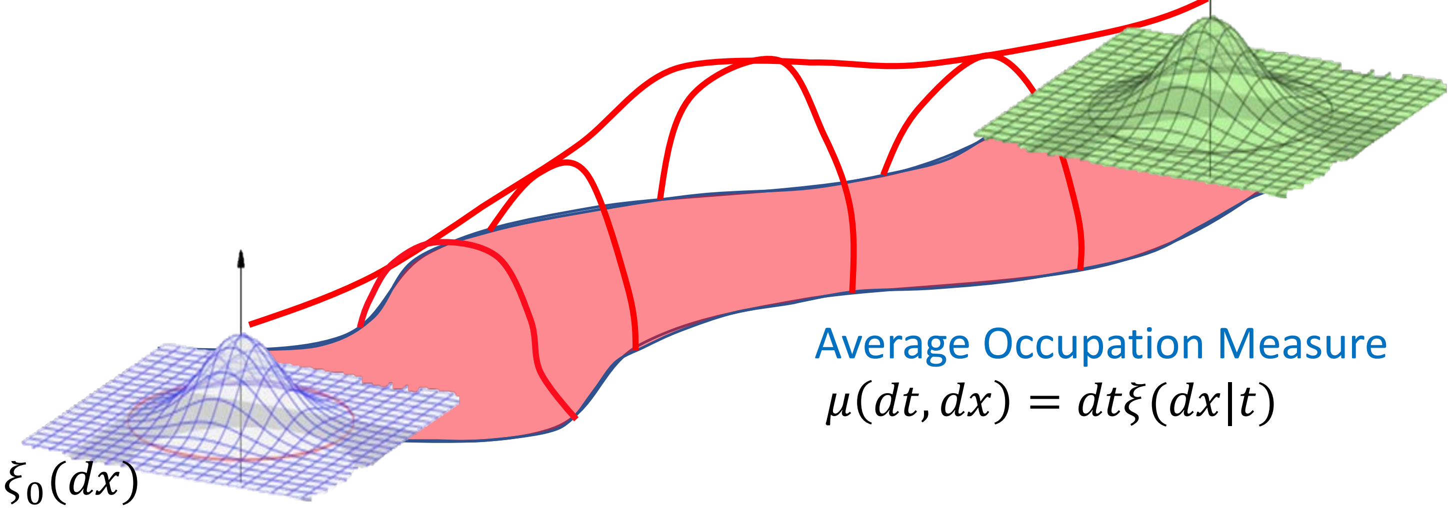
$\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$

ODE: $\dot{x}(t) = f(t, x(t)) \qquad t \in [0, T] \qquad x \in X$

Measures $\big(\mu_0(dt,dx), \mu(dt,dx), \mu_T(dt,dx)\big)$

Propagation of measures (PDE) $\dfrac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$ **Liouville's Equation**

- These measures satisfy **Linear** Partial Differential Equation (PDE).
- Infect, Liouville's equation captures the information of ODE (dynamical system)
- Hence, instead of working with **nonlinear ODE**, we can work with **linear PDE** in measure.



ODE: $\dot{x}(t) = f(t, x(t))$ $\quad t \in [0, T]$ $\quad x \in X$

Measures $\left(\mu_0(dt, dx), \mu(dt, dx), \mu_T(dt, dx)\right)$

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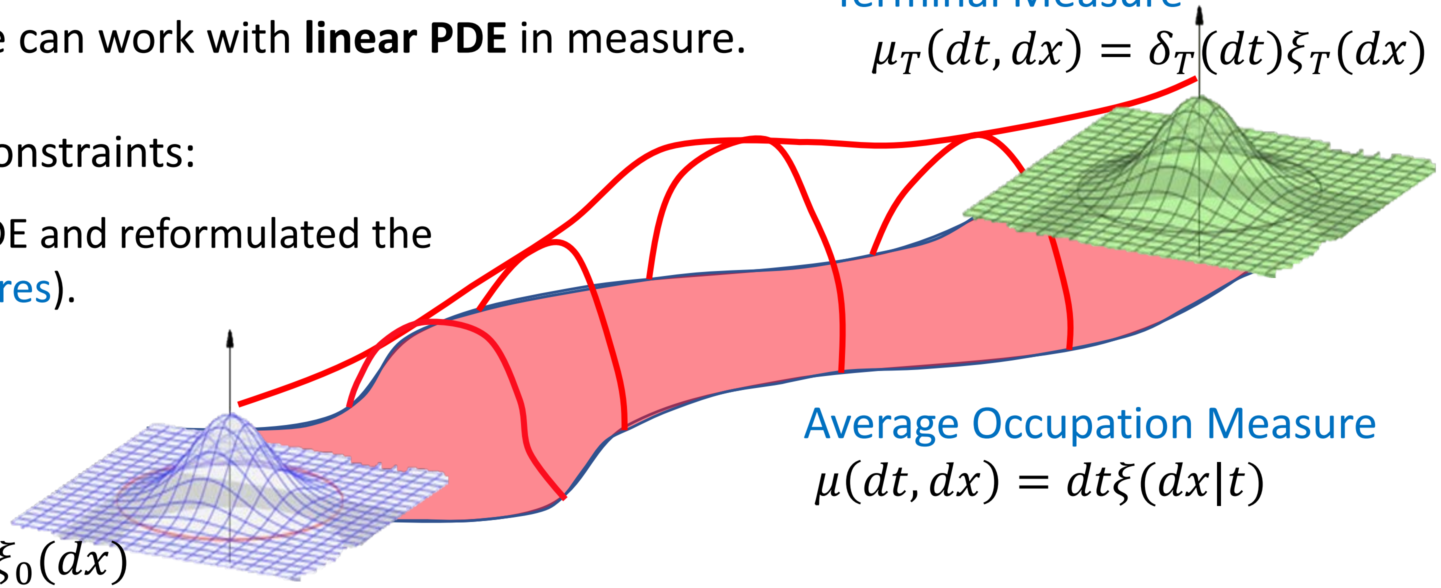
- Hence, instead of working with **nonlinear ODE**, we can work with **linear PDE** in measure.
- Give the nonlinear optimization with differential constraints:
  - We replace the differential constraints with linear PDE and reformulated the problem terms of measure (Linear Program in measures).



ODE: $\dot{x}(t) = f(t, x(t))$ $\quad t \in [0, T]$ $\quad x \in X$

Measures $\big(\mu_0(dt, dx), \mu(dt, dx), \mu_T(dt, dx)\big)$

Propagation of measures (PDE) $\dfrac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$ **Liouville's Equation**

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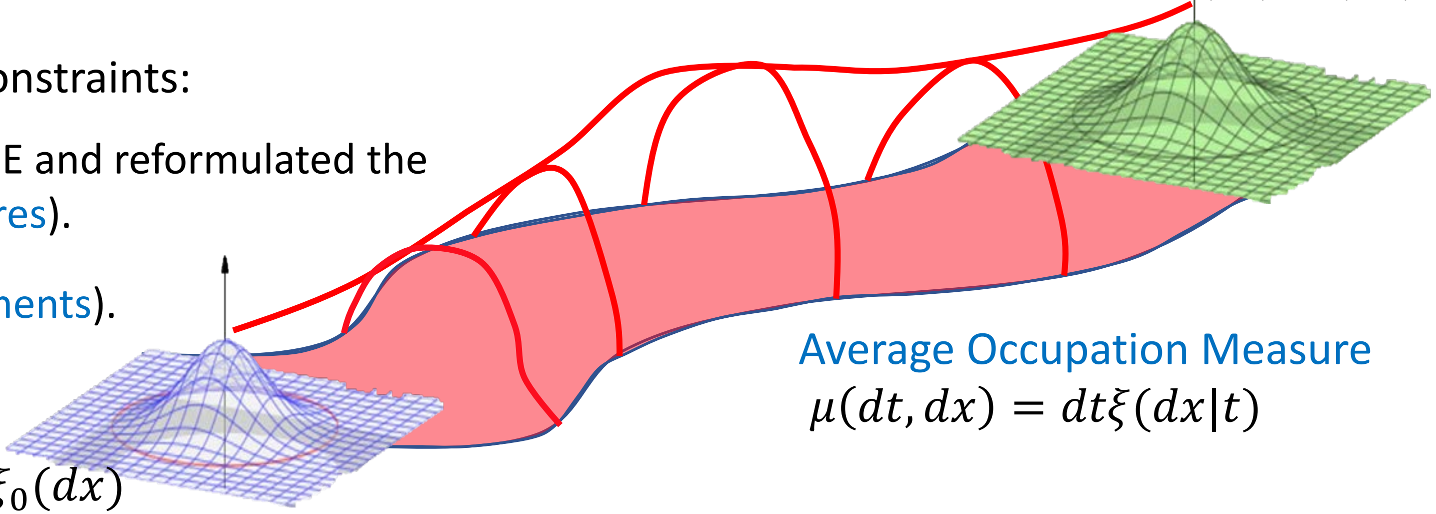
- Infect, Liouville's equation captures the information of ODE (dynamical system)
- Hence, instead of working with **nonlinear ODE**, we can work with **linear PDE** in measure.
- Give the nonlinear optimization with differential constraints:
  - We replace the differential constraints with linear PDE and reformulated the problem terms of measure (Linear Program in measures).
  - We work with the moments of measures (SDP in moments).



Terminal Measure
$\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$

Average Occupation Measure
$\mu(dt, dx) = dt\xi(dx|t)$

Initial Measure
$\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$

In the following, we will look at **(average) occupation measure** and **Liouville's Equation** in more details.

Occupation Measure-deterministic case

- Consider:

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

Occupation Measure-deterministic case

- Consider:

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

➢ Given an **initial condition x_0**, the **occupation measure** of a trajectory $x(t|x_0)$ is defined by

occupation measure: $\mu(S_t \times S_x \, |x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$ given sets $S_t \subset [0, T], S_x \subset X$

$S_t \subset [0, T]$ $S_x \subset X$ Indicator function of set S_x

- Occupation measure μ, measures the size of set $S_t \times S_x$ with respect to $\mathbf{I}_{S_x}(x(t|x_0))dt$

Occupation Measure-deterministic case

- Consider:

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$S_t \subset [0, T]$ $S_x \subset \mathrm{X}$ Indicator function of set S_x

- Occupation measure μ, measures the size of set $S_t \times S_x$ with respect to $\mathbf{I}_{S_x}(x(t|x_0))dt$

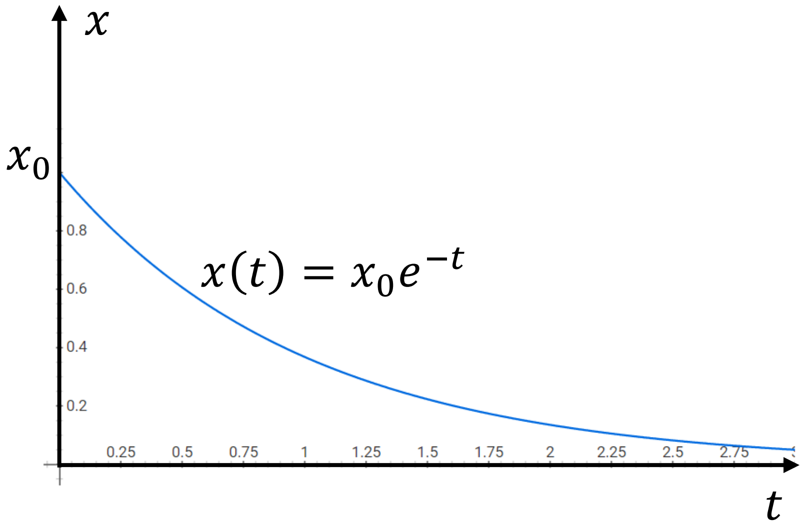
➢ **Geometric interpretation**

Occupation measure, measures the time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0))$ in a given set $S_t \times S_x$.

Example:

ODE $\dot{x}(t) = -x(t)$ $\quad$ $x(t) = x_0 e^{-t}$

$x(0) = x_0 \geq 0$



Example:

ODE $\dot{x}(t) = -x(t)$ $\qquad x(t) = x_0 e^{-t}$

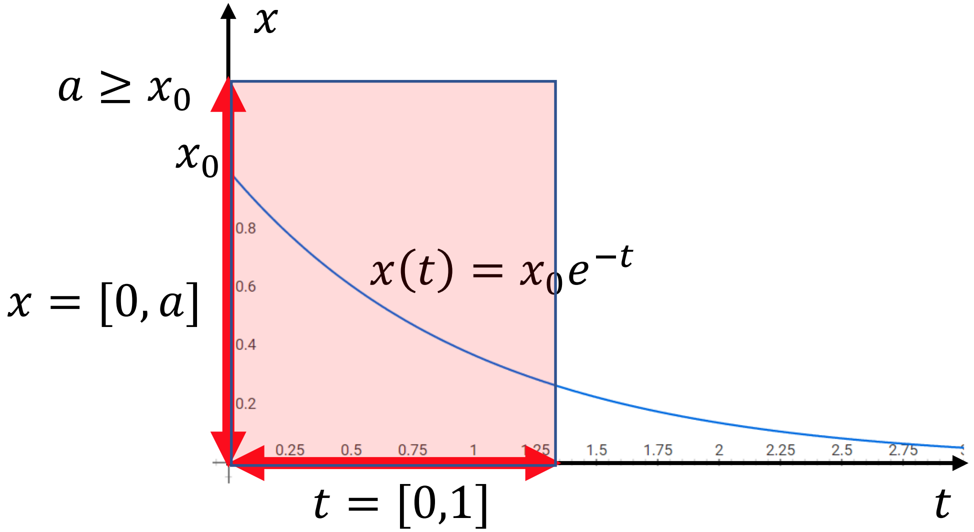
$x(0) = x_0 \geq 0$

occupation measure: $\mu(\, S_t \times S_x \;|x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$

- The time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0))$ in a given subset $S_t \times S_x$

$$\mu(\, \underbrace{[0,1] \times [0, a]}_{S_t \times S_x} \;|x_0) = 1$$

Where $a \geq x_0$



Example:

ODE $\dot{x}(t) = -x(t)$ $\quad x(t) = x_0 e^{-t}$

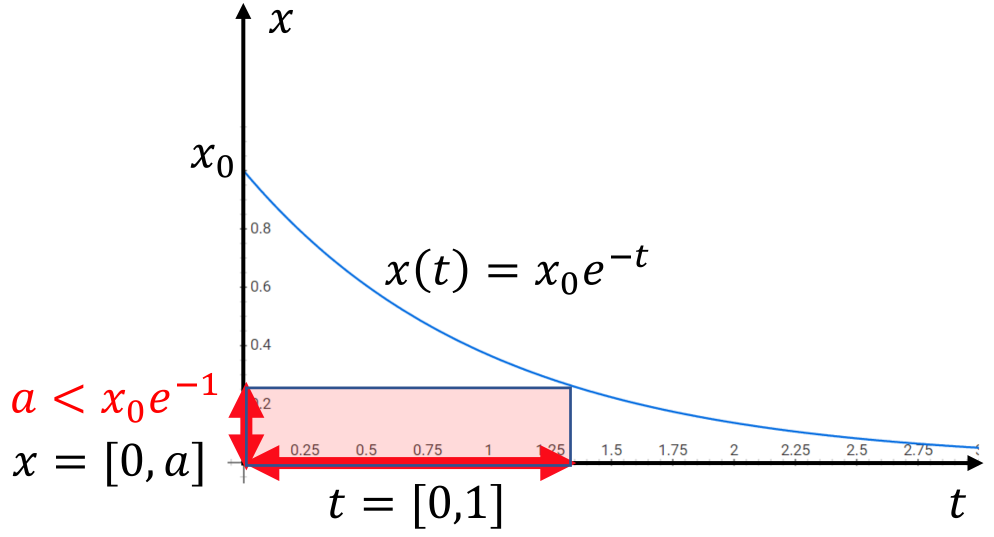
$x(0) = x_0 \geq 0$

occupation measure: $\mu(\, S_t \times S_x \;|x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$

- The time spent by the graph of the trajectory (t, $x(t|x_0)$) in a given subset $S_t \times S_x$

$$\mu(\underbrace{[0,1] \times [0,a]}_{S_t \times S_x} \;|x_0) = 0$$

Where $a < x_0 e^{-1}$



Example:

ODE $\dot{x}(t) = -x(t)$ $\qquad x(t) = x_0 e^{-t}$

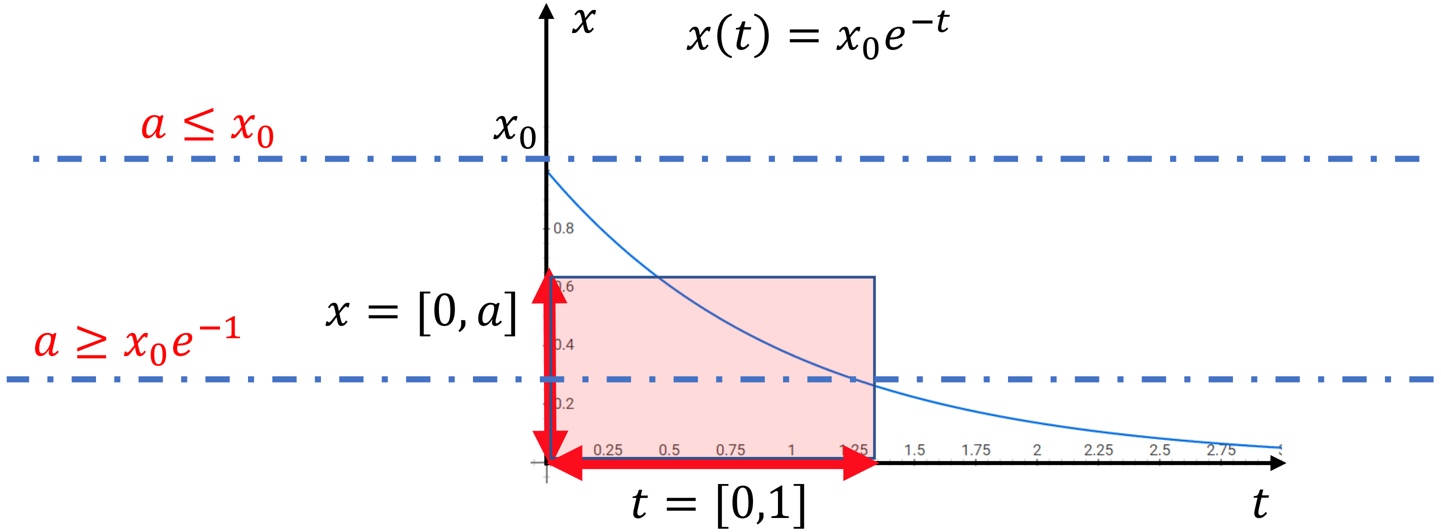
$x(0) = x_0 \geq 0$

occupation measure: $\mu(\, S_t \times S_x \;|x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$

- The time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0))$ in a given subset $S_t \times S_x$

$$\mu(\underbrace{[0,1] \times [0,a]}_{S_t \times S_x} |x_0) = 1 - ln\frac{x_0}{a}$$

Where $x_0 e^{-1} \leq a \leq x_0$



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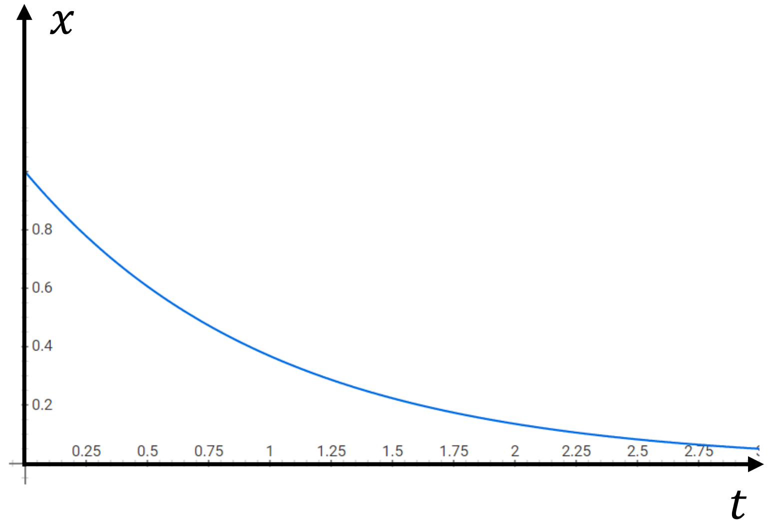
$$\mu(\underbrace{[0,1] \times [0,a]}_{S_t \times S_x} |x_0) = \int_{S_t} \mathbf{I}_{S_x}\big(x(t|x_0)\big)dt = \begin{cases} 1 & x_0 \leq a \\ 1 - ln\dfrac{x_0}{a} & a \leq x_0 \leq ae \\ 0 & x_0 > ae \end{cases}$$



Occupation Measure-deterministic case

- Consider:

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

➢ Given an **initial condition $\boldsymbol{x_0}$**, the **occupation measure** of a trajectory $x(t|x_0)$ is defined by

occupation measure: $\mu(S_t \times S_x \;|x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$ given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$

$S_t \subset [0, T]$ $S_x \subset \mathrm{X}$ Indicator function of set S_x

- Occupation measure μ, measures the size of set $S_t \times S_x$ with respect to $\mathbf{I}_{S_x}(x(t|x_0))dt$

➢ **Geometric interpretation:** Occupation measure, measures the time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0))$ in a given set $S_t \times S_x$.

Occupation Measure-deterministic case

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> **Analytic interpretation: Integration** with respect to **occupation measure** μ is equivalent to time-integration along a system trajectory, i.e.

Integral of a function $v(t, x)$ along the trajectory:

$$\int_0^T v(t, x(t|x_0))dt = \int_0^T \int_X v(t, x)\mu(dx, dt|x_0)$$

Occupation measure

Occupation Measure-deterministic case

- Consider:

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

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Information of the trajectory is captured by occupation measure

$$\int_0^T v(t, x(t|x_0))dt = \int_0^T \int_X v(t, x)\mu(dx, dt|x_0)$$

Occupation Measure-deterministic case

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

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- Now, we want to describe the time evolution of the function $v(t, x)$ along the trajectory of dynamical system.
- We will use the time-evolution to describe the time-evolution of the moments of measures.

Occupation Measure-deterministic case

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$ $x(t|x_0)$: Solution for given initial state

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$$\dot{x}(t) = f(x(t), t) \xrightarrow{\text{Solution at time } T} x(T) = x_0 + \int_0^T \dot{x}\, dt = x_0 + \int_0^T f(x(t), t)\, dt$$

Occupation Measure-deterministic case

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Given the function $v(t, x)$:

$$v(T, x(T|x_0)) = v(0, x_0) + \int_0^T \dot{v}(t, x(t|x_0))dt$$

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Occupation Measure-deterministic case

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Linear Operator:

$$\mathcal{L}v = \frac{\partial v}{\partial t} + \sum_{i=1}^n \frac{\partial v}{\partial x_i} f_i = \frac{\partial v}{\partial t} + (\nabla v)^T f$$

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Where $\mathcal{L}v = \frac{\partial v}{\partial t} + \sum_{i=1}^n \frac{\partial v}{\partial x_i} f_i = \frac{\partial v}{\partial t} + (\nabla v)^T f$

Occupation Measure-Probabilistic Case

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$

- **Initial Probability Measure:** x_0 is random variable $x_0 \sim \xi_0(dx)$
- Due to random initial states, ODE has a **family of trajectories.**

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Initial Probability measures: $\xi_0(dx)$

Terminal Probability measures: $\xi_T(S_x) = \int \mathbf{I}_{S_x}\big(x(T|x_0)\big)\xi_0(dx)$

↓

Probability that states at time $t = T$ are in set $S_x \in X$

$\xi_T(S_x) = \int_{S_x} (\text{probability distribution})\, dx = \int \mathbf{I}_{S_x}\big(x(T|x_0)\big)(\text{probability distribution})\, dx$

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Average Occupation Measure:
Given an initial probability measure of states ξ_0, the **average occupation measure** of the flow of trajectories is defined by

$$\mu(S_t \times S_x) = \int_X \underbrace{\mu(S_t \times S_x|x_0)}_{\text{Occupation measure}}\xi_0(dx)$$

given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$

Occupation measure
(spent time for single $x(t|x_0)$)

Occupation Measure-Probabilistic Case

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$

- **Initial Probability Measure:** x_0 is random variable $x_0 \sim \xi_0(dx)$
- Due to random initial states, ODE has a **family of trajectories.**

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$S_t \subset [0, T]$

Occupation measure

probability measure of states at time t

Occupation Measure-Probabilistic Case

ODE $\dot{x}(t) = f(t, x(t))$ $t \in [0, T]$ $x \in X$

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$S_t \subset [0, T]$

$\xi(S_x|t)$ → probability measure of states at time t

Average Occupation Measure → $\mu(dt, dx) = dt\,\xi(dx|t)$ → **probability measure of states for a given t**

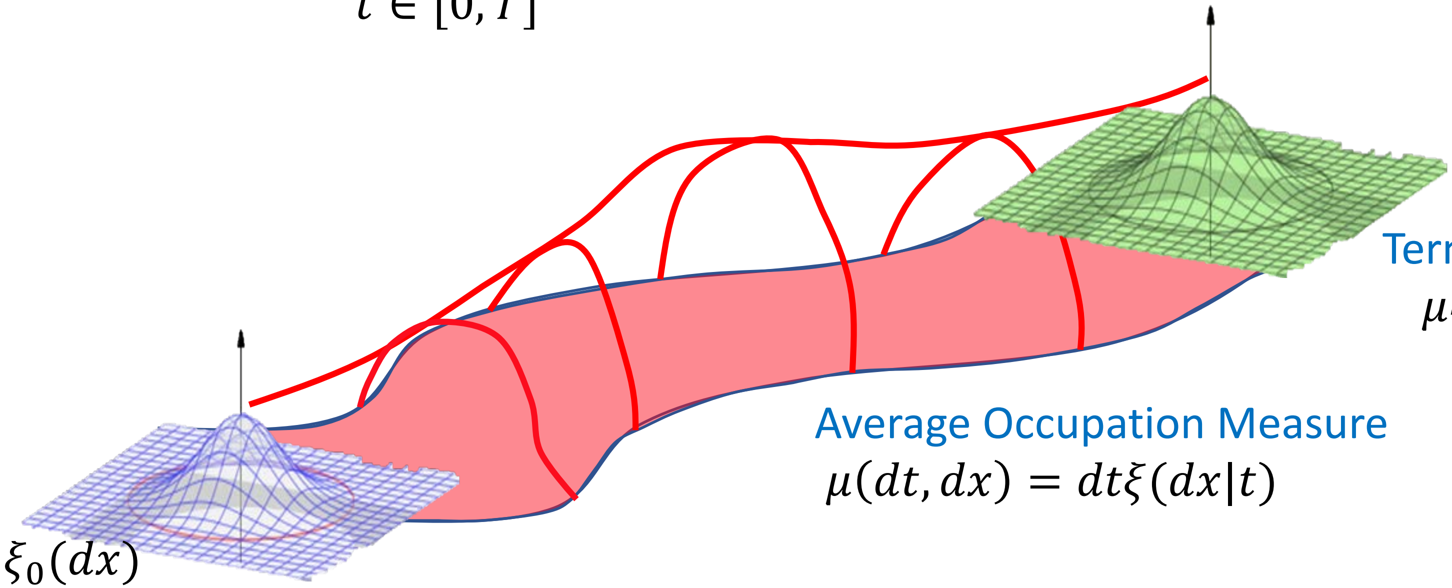
Occupation Measure-Probabilistic Case

ODE $\dot{x}(t) = f(t, x(t))$ $\quad t \in [0, T]$ $\quad x \in X$

- **Initial Probability Measure:** x_0 is random variable $x_0 \sim \xi_0(dx)$

- Initial Probability measures of states: $\xi_0(dx)$
- Terminal Probability measure of states: $\xi_T(dx)$
- Probability measure of states for given t: $\xi(dx|t)$ $\quad t \in [0, T]$

- **Initial Measure** $\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$
- **Terminal Measure** $\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$
- **Average Occupation Measure:** $\mu(dt, dx) = dt\xi(dx|t)$



Terminal Measure
$\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$

Average Occupation Measure
$\mu(dt, dx) = dt\xi(dx|t)$

Initial Measure
$\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$

- Now, we want to describe the time evolution of the function $v(t, x)$ in terms of Average occupation measure.

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Given x_0

$$v(T, x(T|x_0)) = v(0, x_0) + \int_0^T \int_X \mathcal{L}v(x(t|x_0)) \underbrace{\mu(dx, dt|x_0)}_{Occupation\ measure}$$

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Integrating with respect to ξ_0

$x_0 \sim \xi_0$ $$\int v(T,x)\underbrace{\xi_T(dx)}_{\text{Terminal Probability of state}} = \int v(0,x)\underbrace{d\xi_0(dx)}_{\text{Initial Probability of state}} + \int_0^T \int_X \mathcal{L}v(t,x)\underbrace{\mu(xd, dt)}_{\textit{Average Occupation measure}}$$

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Integrating with respect to ξ_0

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Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$ Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$

$x_0 \sim \xi_0$

$$\int v(t, x) d\mu_T = \int v(t, x) d\mu_0 + \int_0^T \int_X \mathcal{L}v(t, x) d\mu(x, t)$$

Average Occupation measure

Information of time is captured by μ_T and μ_0

- Now, we want to describe the time evolution of the function $v(t,x)$ in terms of Average occupation measure.

Given x_0 $$v(T, x(T|x_0)) = v(0, x_0) + \int_0^T \int_X \mathcal{L}v(x(t|x_0)) \underbrace{\mu(dx, dt|x_0)}_{Occupation\ measure}$$

Integrating with respect to ξ_0

$x_0 \sim \xi_0$ $$\int v(T, x)\underbrace{\xi_T(dx)}_{\text{Terminal Probability of state}} = \int v(0, x)\underbrace{d\xi_0(dx)}_{\text{Initial Probability of state}} + \int_0^T \int_X \mathcal{L}v(t, x)\underbrace{\mu(dx, dt)}_{Average\ Occupation\ measure}$$

Initial Measure $\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$ Terminal Measure $\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$

$x_0 \sim \xi_0$ $$\int v(t, x)\mu_T(dx, dt) = \int v(t, x)\mu_0(dx, dt) + \int_0^T \int_X \mathcal{L}v(t, x)d\mu(x, t)$$

- This describes the relation of **1)** initial measure $\mu_0(dx, dt)$, **2)** Terminal measure $\mu_T(dx, dt)$ **3)** average occupation measure $\mu(dx, dt)$
- We will use this equation to describe the relation of the moments (for polynomial $v(t, x)$)

To obtain the Liouville's Equation:

$x_0 \sim \xi_0$

$$\int v(t, x)\mu_T(dx, dt) = \int v(t, x)\mu_0(dx, dt) + \int_0^T \int_X \mathcal{L}v(t, x)d\mu(x, t)$$

Information of time is captured by

Average Occupation measure

Compact form

In terms of $\mathcal{L}$

$$< v, \mu_T > = < v, \mu_0 > + < \mathcal{L}v, \mu >$$

To obtain the Liouville's Equation:

$x_0 \sim \xi_0$

$$\int v(t,x)\mu_T(dx,dt) = \int v(t,x)\mu_0(dx,dt) + \int_0^T \int_X \mathcal{L}v(t,x)d\mu(x,t)$$

Information of time is captured by

Average Occupation measure

Compact form

In terms of $\mathcal{L}$

$$< v, \mu_T > = < v, \mu_0 > + < \mathcal{L}v, \mu >$$

We can represent in terms of adjoint operator

Linear Operator: $\mathcal{L}v = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i$

(Lecture 5 duality)

Adjoint linear operator $< v(t,x), \mathcal{L}^*\mu > = < \mathcal{L}v(t,x), \mu >$ ⇒ $\mathcal{L}^*\mu = -\frac{\partial \mu}{\partial t} - \sum_{i=1}^{n} \frac{\partial (f_i \mu)}{\partial x_i} = -\frac{\partial \mu}{\partial t} - div(f\mu)$

In terms of $\mathcal{L}^*$

$$< v, \mathcal{L}^*\mu > = < v, \mu_T > - < v, \mu_0 >$$

To obtain the Liouville's Equation:

$x_0 \sim \xi_0$

$$\int v(t,x)\mu_T(dx,dt) = \int v(t,x)\mu_0(dx,dt) + \int_0^T \int_X \mathcal{L}v(t,x)d\mu(x,t)$$

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In terms of $\mathcal{L}^*$ $$< v, \mathcal{L}^*\mu > = < v, \mu_T > - < v, \mu_0 >$$

This is required to hold for all functions v, we obtain a linear PDE on measure as $\mathcal{L}^*\mu = \mu_T - \mu_0$

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In terms of $\mathcal{L}$ $\quad < v, \mu_T > = < v, \mu_0 > + < \mathcal{L}v, \mu >$

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Linear Operator: $\mathcal{L}v = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i$

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$$\frac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$$

Liouville's Equation

Nonlinear ODE: $\dot{x}(t) = f(t, x(t)) \quad t \in [0, T] \quad x \in X$

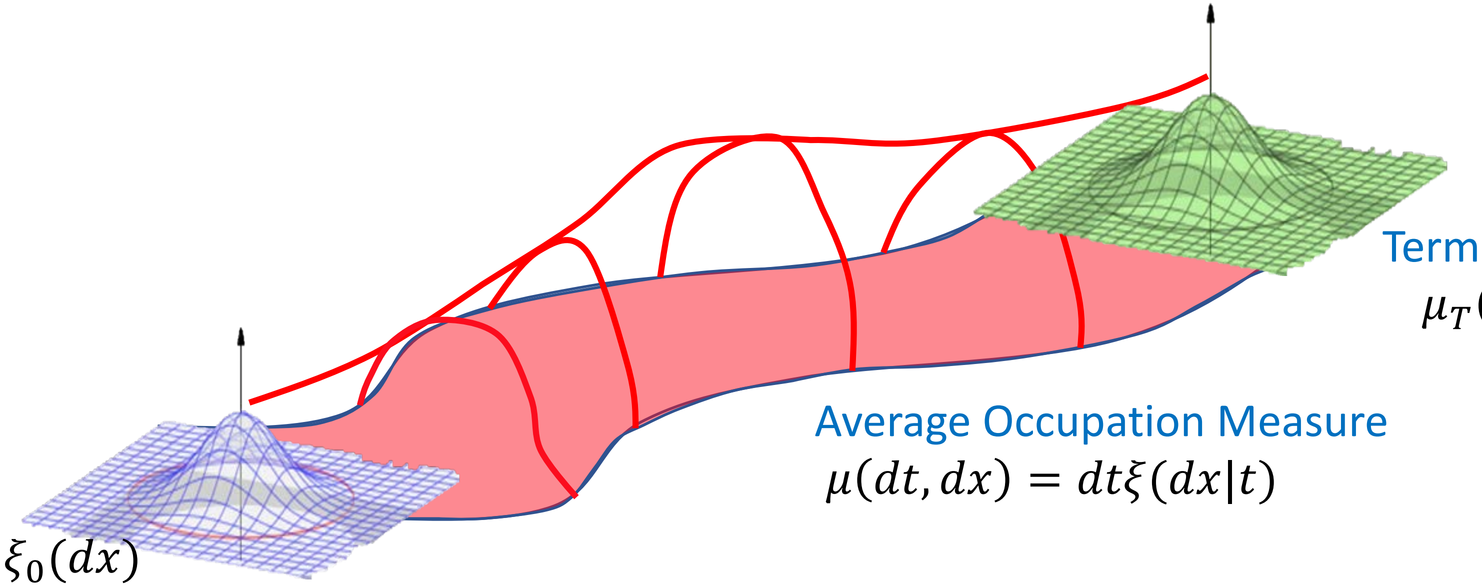
- Initial Probability measure of states: $\xi_0(dx)$

Linear PDE:

(Liouville's Equation) $\dfrac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$ Measures is time and state space $\big(\mu_0(dt, dx), \mu(dt, dx), \mu_T(dt, dx)\big)$

To describe the moments we will use: $$\int v(t,x)\mu_T(dt,dx) = \int v(t,x)\mu_0(dt,dx) + \int_0^T \int_X \mathcal{L}v(t,x)\mu(dt,dx)$$

(Integral form of Liouville's Equation)



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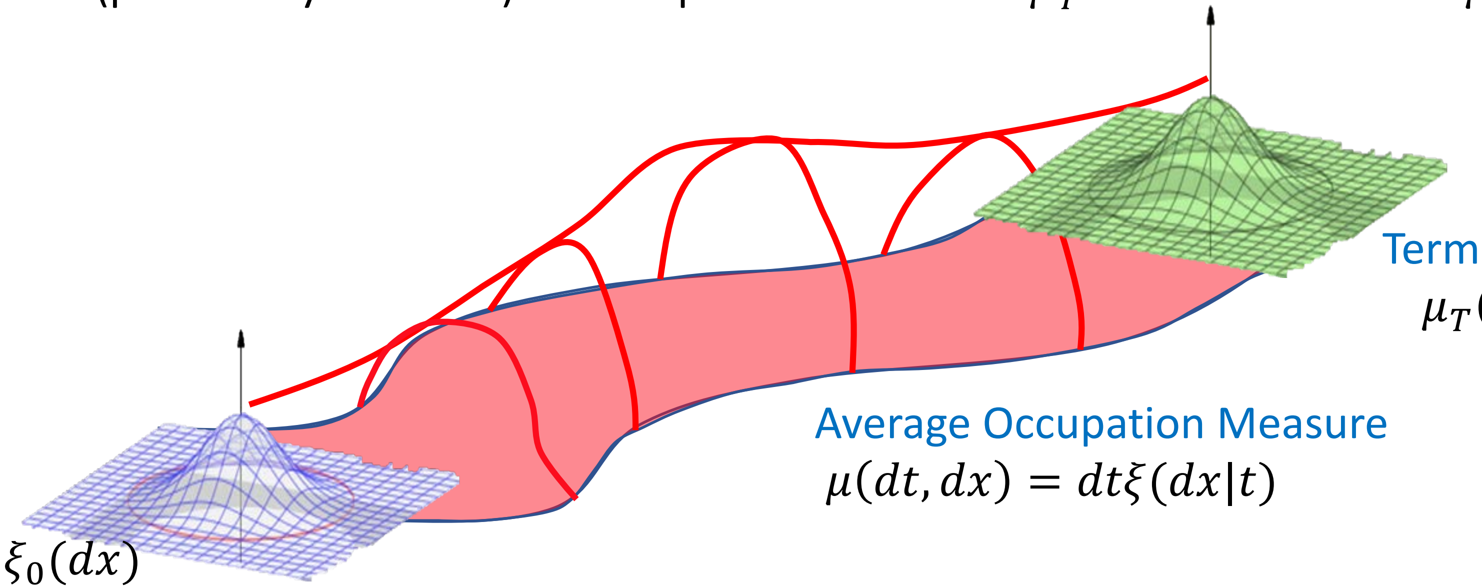
To describe the moments we will use: $\int v(t, x)\mu_T(dt, dx) = \int v(t, x)\mu_0(dt, dx) + \int_0^T \int_X \mathcal{L}v(t, x)\mu(dt, dx)$

(Integral form of Liouville's Equation)

- The mass of μ_0 is one (probability measure). This implies that mass of μ_T is one and mass of μ is equal to T.



Terminal Measure
$\mu_T(dt, dx) = \delta_T(dt)\xi_T(dx)$

Average Occupation Measure
$\mu(dt, dx) = dt\xi(dx|t)$

Initial Measure
$\mu_0(dt, dx) = \delta_0(dt)\xi_0(dx)$

Moments Time-Evolution:

We assume that all the functions are polynomials

$v(t,x)$:polynomial $f(t,x)$:polynomial $\Longrightarrow$ $\frac{\partial v}{\partial t}$ and $(\nabla v)^T f = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i$: polynomials

$\dot{x}(t) = f(t, x(t))$

- Moments of $\mu_0(dx,dt)$: $y_1 = \int t^{\alpha_1} x^{\alpha_2} \mu_0(dx,dt)$
- Moments $\mu(dx,dt)$: $y_2 = \int t^{\alpha_1} x^{\alpha_2} \mu(dx,dt)$
- Moments $\mu_T(dx,dt)$: $y_3 = \int t^{\alpha_1} x^{\alpha_2} \mu_T(dx,dt)$

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- We choose functions $v(t,x)$ which are monomials of the form $t^{\alpha_1} x^{\alpha_2}$, $(\alpha_1,\alpha_2)_j, j = 1, \dots m$

$$\int v(t,x)\mu_T(dt,dx) = \int v(t,x)\mu_0(dt,dx) + \int_0^T \int_X \mathcal{L}v(t,x)\mu(dt,dx)$$

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$$\underbrace{\int v_j(t,x)\mu_T(dt,dx)}_{\text{Moments of } \mu_T(dx,dt)} = \underbrace{\int v_j(t,x)\mu_0(dt,dx)}_{\text{Moments of } \mu_0(dx,dt)} + \underbrace{\int \mathcal{L} v_j(t,x)\mu(dt,dx)}_{\text{Moments of } \mu(dx,dt)} \qquad j = 1, \dots m$$

Moments Time-Evolution:

We assume that all the functions are polynomials

$v(t,x)$:polynomial $f(t,x)$:polynomial $\Longrightarrow$ $\frac{\partial v}{\partial t}$ and $(\nabla v)^T f = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i$: polynomials

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$$\underbrace{\int v_j(t,x)\mu_T(dt,dx) = \int v_j(t,x)\mu_0(dt,dx) + \int \mathcal{L} v_j(t,x)\mu(dt,dx)}$$ $j = 1, \dots m$

Linear sum of the moments: $\sum_{i=1}^{3} \sum_{\alpha} a_{ij\alpha} y_{i\alpha} = b_j$ $i = 1, \dots, 3$ $j = 1, \dots m$

Nonlinear ODE: $\dot{x}(t) = f(t, x(t)) \quad t \in [0, T] \quad x \in X$

- Initial Probability measure of states: $\xi_0(dx)$

➢ Information of the nonlinear ODE in measure:

Linear PDE: $\frac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$ Measures is time and state space $\left(\mu_0(dt, dx), \mu(dt, dx), \mu_T(dt, dx)\right)$

➢ Information of the nonlinear ODE in moments:

$$\sum_{i=1}^{3} \sum_{\alpha} a_{ij\alpha} y_{i\alpha} = b_j \qquad i = 1, \dots, 3 \qquad j = 1, \dots m$$

Obtained by $\int v_j(t,x)\mu_T(dt,dx) = \int v_j(t,x)\mu_0(dt,dx) + \int \mathcal{L}v_j(t,x)\mu(dt,dx)$, $v_j(t,x) = t^{\alpha_1}\, x^{\alpha_2} \quad (\alpha_1, \alpha_2)_j,$ $\quad j = 1, \dots, m$

Dealing with uncertainty

- We can incorporate real parametric uncertainty in the dynamics.
- Each uncertain parameter must be introduced as an additional state of the system.

$$\dot{x}(t) = f(t, x(t), \omega)$$

New states: $[x, \omega]$ →

$$\dot{x}(t) = f(t, x(t), \omega)$$
$$\dot{\omega}(t) = 0$$

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New states: $[x, \omega]$

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$$\dot{\omega}(t) = 0$$

- Unknow parameter
- $\omega \sim$Probability distribution
- It is fixed in time.

- Occupation Measure and Liouville's Equation

- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

1) Reformulate the problem as nonlinear optimization with differential constraints

2) Replace the differential constraints with linear PDE and reformulated the problem terms of measure (Linear Program in measures).

3) Use the moment representation of the measure (SDP in moments).

Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

Trajectory Optimization

D. Henrion, M. Ganet-Schoeller, S. Bennani. ”Measures and LMI for space launcher robust control validation”, Proceedings of the IFAC Symposium on Robust Control Design, Aalborg, Denmark, June 2012.

D. Henrion ”Optimization on linear matrix inequalities for polynomial systems control”, Lecture notes used for a tutorial course given during the International Summer School of Automatic Control held at Grenoble, France, in September 2014.

Consider the following dynamic optimization problem with polynomial differential constraints

$$\begin{aligned} \inf_{x(t)} \quad & \int_0^T l(t, x(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(t, x(t)), \;\; x(t) \in X, \;\; t \in [0, T] \\ & x(0) \in X_0, \;\; x(T) \in X_T \end{aligned}$$

State trajectory $x(t)$ constrained in a compact basic semialgebraic set

$$X = \{x \in \mathbb{R}^n \; : \; p_k(x) \geq 0, \; k = 1, \ldots, n_X\}$$

Initial and terminal states are constrained in compact basic semialgebraic sets

$$X_0 = \{x \in \mathbb{R}^n \; : \; p_{0k}(x) \geq 0, \; k = 1, \ldots, n_0\} \subset X$$

$$X_T = \{x \in \mathbb{R}^n \; : \; p_{Tk}(x) \geq 0, \; k = 1, \ldots, n_T\} \subset X$$

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\end{aligned}$$

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- The final time T is either given, or free, in which case it becomes a decision variable, jointly with $x(t)$.

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- The final time T is either given, or free, in which case it becomes a decision variable, jointly with $x(t)$.

We look for trajectory $x(t)$ starting in X_0, ending in X_T , and staying in X that minimizes the given cost.

Nonlinear Dynamic Optimization:

$$\begin{aligned} \inf_{x(t)} \quad & \int_0^T l(t, x(t)) dt \\ \text{s.t.} \quad & \dot{x}(t) = f(t, x(t)), \;\; x(t) \in X, \;\; t \in [0, T] \\ & x(0) \in X_0, \;\; x(T) \in X_T \end{aligned}$$

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- We encode the state trajectory $x(t)$ in an occupation measure μ and we come up with an infinite-dimensional LP problem:

$$\dot{x}(t) = f(t, x(t)) \quad \Rightarrow \quad \frac{\partial \mu}{\partial t} + div(f\mu) = \mu_0 - \mu_T$$

Nonlinear Dynamic Optimization:

$$\inf_{x(t)} \int_0^T l(t, x(t))dt$$
$$\text{s.t.} \quad \dot{x}(t) = f(t, x(t)), \; x(t) \in X, \; t \in [0, T]$$
$$x(0) \in X_0, \; x(T) \in X_T$$

$$X = \{x \in \mathbb{R}^n : p_k(x) \geq 0, \; k = 1, \dots, n_X\}$$
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Objective function

$$\min \int_0^T l(t, x(t))dt \quad \Rightarrow \quad \min \; \mathrm{E}\left[\int_0^T l(t, x(t))dt\right] = \int \int_0^T l(t, x(t))dt \;\; \xi(dx|t)$$

Lecture 3: moment based nonlinear optimization

$$= \int \int_0^T l(t, x(t)) \, \mu(dx, dt)$$

$$= < l, \mu >$$

$\mu(dt, dx) = dt\xi(dx|t)$

Average occupation measure

Nonlinear Dynamic Optimization:

$$\begin{aligned} \inf_{x(t)} \quad & \int_0^T l(t, x(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(t, x(t)), \;\; x(t) \in X, \;\; t \in [0, T] \\ & x(0) \in X_0, \;\; x(T) \in X_T \end{aligned}$$

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Infinite-dimensional LP problem:

$$\begin{aligned} \inf \quad & \langle l, \mu \rangle \\ \text{s.t.} \quad & \frac{\partial \mu}{\partial t} + \operatorname{div} f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

Initial measure: $\mu_0 \in \mathscr{M}_+(\{0\} \times X_0)$

Terminal measure: $\mu_T \in \mathscr{M}_+(\{T\} \times X_T)$

Average Occupation Measure: $\mu \in \mathscr{M}_+([0, T] \times X)$

- μ_0 , μ_T and T can be free, or given.

Nonlinear Dynamic Optimization:

$$\begin{aligned} \inf_{x(t)} \quad & \int_0^T l(t, x(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(t, x(t)), \ \ x(t) \in X, \ \ t \in [0, T] \\ & x(0) \in X_0, \ \ x(T) \in X_T \end{aligned}$$

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Initial measure: $\mu_0 \in \mathscr{M}_+(\{0\} \times X_0)$

Terminal measure: $\mu_T \in \mathscr{M}_+(\{T\} \times X_T)$

Average Occupation Measure: $\mu \in \mathscr{M}_+([0, T] \times X)$

$$\begin{aligned} \inf \quad & \langle l, \mu \rangle \\ \text{s.t.} \quad & \frac{\partial \mu}{\partial t} + \text{div } f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

- If terminal time T is free and function l in objective function and the dynamics f do not depend explicitly on time t, Then it can be shown without loss of generality that in measure-LP measures do not depend explicitly on time either. The terminal time is equal to the mass of the occupation measure $T = \mu(X)$

Infinite-dimensional LP problem:

$$\begin{aligned} \inf \quad & \langle l, \mu \rangle \\ \text{s.t.} \quad & \text{div } f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

$(\mu, \ \mu_0, \ \mu_T) \in \mathscr{M}_+(X) \times \mathscr{M}_+(X_0) \times \mathscr{M}_+(X_T)$

Example:

- **ODE** $\dot{x}(t) = -x(t)$

We want to find trajectories minimizing the state energy $\int_0^T x^2(t)dt$.

$$\begin{aligned} &\inf \\ &\text{s.t.} \quad x(t) \in X, \ \ t \in [0, T] \\ &\quad x(0) \in X_0, \ \ x(T) \in X_T \end{aligned}$$

$$X_0 := \{x \in \mathbb{R} \,:\, p_0(x) := \frac{1}{4} - \left(x - \frac{3}{2}\right)^2 \geq 0\},$$

$$X_T := \{x \in \mathbb{R} \,:\, p_T(x) := \frac{1}{4} - x^2 \geq 0\}$$

$$X := \{x \in \mathbb{R} \,:\, p(x) := 4 - x^2 \geq 0\}$$

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$$X := \{x \in \mathbb{R} \, : \, p(x) := 4 - x^2 \geq 0\}$$

Variables of LP in measures:

- Initial Measure $\mu_0(dt,dx) = \delta_0(dt)\xi_0(dx)$ supported on $\mu_0 \in \mathcal{M}_+(\{0\} \times X_0)$
- Terminal Measure $\mu_T(dt,dx) = \delta_T(dt)\xi_T(dx)$ supported on $\mu_T \in \mathcal{M}_+(\{T\} \times X_T)$
- Average occupation measure $\mu(dt,dx) = dtd\xi(dx|t)$ supported on $\mu \in \mathcal{M}_+([0,T] \times X)$

Example:

We want to find trajectories minimizing the state energy $\int_0^T x^2(t)dt$.

$$\begin{array}{ll} \inf & \int_0^T x^2(t)dt \\ \text{s.t.} & \dot{x} = -x \qquad x(t) \in X,\ t \in [0,T] \\ & x(0) \in X_0,\ x(T) \in X_T \end{array}$$

Infinite-dimensional LP problem:

(1)

$$\begin{array}{ll} \inf & \langle l, \mu \rangle \\ \text{s.t.} & \frac{\partial \mu}{\partial t} + \text{div } f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array} \Rightarrow \begin{array}{ll} \inf & \langle x^2, \mu \rangle \\ \text{s.t.} & \frac{\partial \mu}{\partial t} - \frac{\partial (x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array}$$

w.r.t. terminal time T and nonnegative measures μ, μ_0, μ_T supported on $[0,T] \times X$, $\{0\} \times X_0$, $\{T\} \times X_T$.

Free final Time T:

(2)

$$\begin{array}{ll} \inf & \langle x^2, \mu \rangle \\ \text{s.t.} & -\frac{\partial (x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array}$$

μ, μ_0, μ_T supported on X, X_0, X_T.

Example:

- This problem can be solved analytically, with optimal trajectory

  $x(t) = e^{-t}$ leaving X_0 at $x(0) = 1$ and reaching X_T at $x(T) = \frac{1}{2}$ for $T = log2 \approx 0.6931$

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  $x(t) = e^{-t}$ leaving X_0 at $x(0) = 1$ and reaching X_T at $x(T) = \frac{1}{2}$ for $T = log2 \approx 0.6931$

(1)

$$\begin{aligned} \inf \quad & \langle x^2, \mu \rangle \\ \text{s.t.} \quad & \frac{\partial \mu}{\partial t} - \frac{\partial (x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

- So the optimal measures solving the LP are

$$\mu(dt, dx) = \underbrace{dt\ \delta_{e^{-t}}(dx)}_{x(t) = e^{-t}}, \quad \mu_0(dt, dx) = \underbrace{\delta_0(dt)\ \delta_1(dx)}_{x(0) = 1}, \quad \mu_T(dt, dx) = \underbrace{\delta_{\log 2}(dt)\ \delta_{\frac{1}{2}}(dx)}_{x(T = log2) = \frac{1}{2}}$$

(2)

$$\begin{aligned} \inf \quad & \langle x^2, \mu \rangle \\ \text{s.t.} \quad & -\frac{\partial (x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

- So the optimal measures solving the LP are $\mu(dx) = \int_0^T \delta_{e^{-t}}(dx)dt, \quad \mu_0(dx) = \delta_1(dx), \quad \mu_T(dx) = \delta_{\frac{1}{2}}(dx).$

Infinite-dimensional LP problem:

$$\begin{aligned} \inf \quad & \langle l, \mu \rangle \\ \text{s.t.} \quad & \tfrac{\partial \mu}{\partial t} + \operatorname{div} f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{aligned}$$

Initial measure: $\mu_0 \in \mathscr{M}_+(\{0\} \times X_0)$

Terminal measure: $\mu_T \in \mathscr{M}_+(\{T\} \times X_T)$

Occupation Measure: $\mu \in \mathscr{M}_+([0, T] \times X)$

To obtain finite SDP, we will work with finite number of moments:

- Moments of measure: $\sum_{i=1}^{3} \sum_{\alpha} a_{ij\alpha} y_{i\alpha} = b_j \qquad i = 1, \dots, 3 \qquad j = 1, \dots, m$
- Moments should also satisfy Moment and Localizing Matrices

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Moment Formulation

$$\begin{aligned} \inf \quad & \textstyle\sum_{i=1}^n \sum_\alpha c_{i\alpha} y_{i\alpha} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^n \sum_\alpha a_{ij\alpha} y_{i\alpha} = b_j, \quad j = 1, \ldots, m \\ & y_i \text{ has a representing measure} \end{aligned}$$

$n = 3$

$$X_i := \{x \in \mathbb{R}^n \ : \ p_{ik}(x) \geq 0, \ k = 1, \ldots, n_i\}$$

Moment SDP:

$$\begin{aligned} \inf \quad & \textstyle\sum_{i=1}^n \sum_\alpha c_{i\alpha} y_{i\alpha} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^n \sum_\alpha a_{ij\alpha} y_{i\alpha} = b_j, \quad j = 1, \ldots, m \\ & M(y_i) \succeq 0, \ M(p_{ik}\, y_i) \succeq 0, \ i = 1, \ldots, n, \ k = 1, \ldots, n_i. \end{aligned}$$

$n = 3$

Example:

We want to find trajectories minimizing the state energy $\int_0^T x^2(t)dt.$

$$\begin{array}{ll} \inf & \int_0^T x^2(t)dt \\ \text{s.t.} & \dot{x} = -x \qquad x(t) \in X,\ t \in [0,T] \\ & x(0) \in X_0,\ x(T) \in X_T \end{array}$$

Measure in LP for free final time T:

$$\begin{array}{ll} \inf & \langle x^2, \mu \rangle \\ \text{s.t.} & -\frac{\partial(x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array}$$

μ, μ_0, μ_T supported on X, X_0, X_T.

(Integral form of Liouv $(\text{grad}\, v)' f)\ \mu = \int v\mu_T - \int v\mu_0$ $\quad v = x^\alpha$

$$\begin{array}{ll} \inf & \int x^2 \mu(dx) \\ \text{s.t.} & -\alpha \int x^\alpha \mu(dx) = \int x^\alpha \mu_T(dx) - \int x^\alpha \mu_0(dx), \quad \alpha = 0, 1, 2, \ldots \\ & \int \mu_0(dx) = 1 \end{array}$$

Example:

$$
\begin{aligned}
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& \int \mu_0(dx) = 1
\end{aligned}
$$

$$
\begin{aligned}
\inf \quad & y_2 \\
\text{s.t.} \quad & -\alpha y_\alpha = y_{T\alpha} - y_{0\alpha}, \;\; \alpha = 0, 1, 2 \ldots \\
& y_{00} = 1 \\
& y \textit{ has a representing measure } \mu \in \mathscr{M}_+(X) \\
& y_0 \textit{ has a representing measure } \mu_0 \in \mathscr{M}_+(X_0) \\
& y_T \textit{ has a representing measure } \mu_T \in \mathscr{M}_+(X_T)
\end{aligned}
$$

Moment SDP:

$$
\begin{aligned}
\inf \quad & y_2 \\
\text{s.t.} \quad & -\alpha y_\alpha = y_{T\alpha} - y_{0\alpha}, \;\; \alpha = 0, 1, \ldots, 2r \\
& y_{00} = 1 \\
& M_r(y) \succeq 0, \; M_{r-1}(p\, y) \succeq 0 \\
& M_r(y_0) \succeq 0, \; M_{r-1}(p_0\, y_0) \succeq 0 \\
& M_r(y_T) \succeq 0, \; M_{r-1}(p_T\, y_T) \succeq 0
\end{aligned}
$$

Moment and Localizing Matrices

Example:

- This problem can be solved analytically, with optimal trajectory

$x(t) = e^{-t}$ leaving X_0 at $x(0) = 1$ and reaching X_T at $x(T) = \frac{1}{2}$ for $T = log2 \approx 0.6931$

$$\begin{array}{ll} \inf & \langle x^2, \mu \rangle \\ \text{s.t.} & -\frac{\partial(x\mu)}{\partial x} = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array}$$

- So the **optimal measures** solving the LP are $\mu(dx) = \int_0^T \delta_{e^{-t}}(dx)dt, \quad \mu_0(dx) = \delta_1(dx), \quad \mu_T(dx) = \delta_{\frac{1}{2}}(dx).$

Optimal moments:

Initial moments : Moments of δ_1 → $y_{0\alpha} = \int x^\alpha \mu_0(dx)$

Terminal moments :Moments of $\delta_{\frac{1}{2}}$ → $y_{T\alpha} = \int x^\alpha \mu_T(dx)$

Moments of $\mu(dx) = \int_0^T \delta_{e^{-t}}(dx)dt,$ → $y_\alpha = \int x^\alpha \mu(dx) = \int_0^{\log 2} e^{-\alpha t} dt = \frac{1-2^{-\alpha}}{\alpha}, \quad \alpha = 1, 2, \ldots$

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- The moment matrices of the initial and terminal measures both have rank one

$$x(0) = y_{0_{\alpha=1}} = 1 \qquad\qquad x(T) = y_{T_{\alpha=1}} = \frac{1}{2}$$

- To recover the trajectory $x(t)$ we need to look at Dual problem in polynomials.

Extension to piecewise polynomial dynamics

$$\begin{array}{ll} \inf & f_0(x(T)) + \int_0^T l(t,x(t))dt \\ \text{s.t.} & \dot{x}(t) = f_j(t,x(t)), \;\; x(t) \in X_j, \;\; j = 1,\ldots,N, \;\; t \in [0,T] \\ & x(0) \in X_0, \;\; x(T) \in X_T \end{array}$$

- We assume that the state-space partitioning sets X_j are disjoint.
- We can then extend the measure LP framework to several measures μ_j, one supported on each cell X_j so that the global (average)occupation measure is $\mu = \sum_{j=1}^{N} \mu_j.$

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- We assume that the state-space partitioning sets X_j are disjoint.
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➢ Measure LP:

$$\begin{aligned} \inf \quad & \langle f_0, \mu_T \rangle + \textstyle\sum_{j=1}^{N} \langle l, \mu_j \rangle \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{N} \left(\frac{\partial \mu_j}{\partial t} + \operatorname{div} f_j \mu_j \right) + \mu_T = \mu_0 \\ & \langle 1, \mu_0 \rangle = 1. \end{aligned}$$

(Liouville's Equation)

➢ Moment SDP

(Integral form of Liouville's Equation)

$$\begin{aligned} \inf_\mu \quad & \int f_0(x(T))\, d\mu_T(x) + \textstyle\sum_k \int l(t, x(t)) d\mu_k(t, x) \\ \text{s.t.} \quad & \textstyle\sum_k \int \frac{\partial v(t,x)}{\partial t}\, d\mu_k(t, x) \; + \\ & \textstyle\sum_k \int Dv(t, x) \cdot f_k(t, x)\, d\mu_k(t, x) \; = \\ & \int v(T, x)\, d\mu_T(x) - \int v(0, x)\, d\mu_0(x) \end{aligned}$$

⇒

$$\begin{aligned} \inf_y \quad & \textstyle\sum_k \sum_\alpha c_{k\alpha} y_{k\alpha} \\ \text{s.t.} \quad & \textstyle\sum_k \sum_\alpha a_{ki\alpha} y_{k\alpha} = b_i, \quad \forall i \end{aligned}$$

⇒

$$\begin{aligned} \inf_y \quad & c^T y \\ \text{s.t.} \quad & Ay = b \\ & M_d(y) \succeq 0 \\ & M_d(g_{kj}, y) \succeq 0, \quad \forall j, k \end{aligned}$$

Example: one-degree-of-freedom model of a launcher attitude control system in orbital phase

$$I\ddot{\theta}(t) = u(t) \qquad x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

where I is a given constant inertia, $\theta(t)$ is the angle and $u(t)$ is the torque control

- The torque control is given by $u(x(t)) = \mathrm{sat}(F'\mathrm{dz}(x_r(t) - x(t)))$

  where $x_r(t)$ is the given reference signal,
  F is a given state feedback,
  sat is a saturation function $\mathrm{sat}(y) = y$ if $|y| \leq L$ $\quad \mathrm{sat}(y) = L\,\mathrm{sign}(y)$ otherwise
  dz is a dead-zone function such that $\mathrm{dz}(x) = 0$ if $|x_i| \leq D_i$ $\quad \mathrm{dz}(x) = 1$ otherwise $\quad i = 1, 2$

  Thresholds $L > 0, D_1 > 0, D_2 > 0$ are given.

Trajectory Optimization:

$$\begin{aligned} \inf \quad & f_0(x(T)) + \int_0^T l(t, x(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f_j(t, x(t)), \ \ x(t) \in X_j, \ \ j = 1, \ldots, N, \ \ t \in [0, T] \\ & x(0) \in X_0, \ \ x(T) \in X_T \end{aligned}$$

T=50

$$I\ddot{\theta}(t) = u(t) \qquad x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \qquad \dot{x}(t) = \begin{bmatrix} x_1(t) \\ \frac{u(x(t))}{I} \end{bmatrix} \qquad u(x(t)) = \mathrm{sat}(F'\mathrm{dz}(x_r(t) - x(t)))$$

Due to saturation function $\mathrm{sat}(y) = y$ if $|y| \leq L$ $\mathrm{sat}(y) = L\,\mathrm{sign}(y)$ otherwise

We have 3 partition of state-space:

Linear regime: $X_1 = \{x \in \mathbb{R}^2 \,:\, |F'x| \leq L\}, \quad f_1(x) = \begin{bmatrix} x_1 \\ -F'x \end{bmatrix}$

Upper saturation $X_2 = \{x \in \mathbb{R}^2 \,:\, F'x \geq L\}, \quad f_2(x) = \begin{bmatrix} x_1 \\ L \end{bmatrix}$

Lower saturation $X_3 = \{x \in \mathbb{R}^2 \,:\, F'x \leq -L\} \quad f_3(x) = \begin{bmatrix} x_1 \\ -L \end{bmatrix}$

The system state $x(t)$ reaches a given subset $X_T = \{(x_1, x_2) : x^T x \leq \epsilon\}$ ⇨ The objective function of the optimization: $x(T)^T x(T)$

GloptiPoly:

```
I = 27500; % inertia
kp = 2475; kd = 19800; % controller gains
L = 380; % input saturation level
thetamax = 5*pi/180; omegamax = 0.4*pi/180; % bounds on initial conditions
T = 50; % final time

d = input('order of relaxation ='); d = 2*d;

% states
mpol('x1',2); % linear regime
mpol('x2',2); % upper sat
mpol('x3',2); % lower sat
mpol('x0',2); % initial
mpol('xT',2);  % terminal

% time
mpol('t1', 1); % time for linear regime
mpol('t2', 1); % time for upper saturation
mpol('t3', 1); % time for lower saturation

% measures
m1 = meas([x1', t1]); % linear regime
m2 = meas([x2', t2]); % upper sat regime
m3 = meas([x3', t3]); % lower sat regime
m0 = meas(x0);  % initial
mT = meas(xT); % terminal
```

```
% dynamics on normalized time range [0,1]
% saturation input y normalized in [-1,1]
K = -[kp kd]/L;
y1 = K*x1; f1 = T*[x1(2); L*y1/I]; % linear regime
y2 = K*x2; f2 = T*[x2(2); L/I]; % upper sat
y3 = K*x3; f3 = T*[x3(2); -L/I]; % lower set

% test functions for each measure = monomials
g1 = mmon([x1', t1],d);
g2 = mmon([x2', t2],d);
g3 = mmon([x3', t3],d);

% unknown moments of initial measure
p = genpow(4,d); p = p(:,2:end); % powers
y0 = ones(size(p,1),1)*[x0' 0];
y0 = mom(prod((y0.^p)')');

% unknown moments of terminal measure
p = genpow(4,d); p = p(:,2:end); % powers
yt = ones(size(p,1),1)*[xT' 1];
yT = mom(prod((yt.^p)')');
```

```
% input LMI moment problem
cost = mom(xT'*xT);
Ay = mom(diff(g1,x1)*f1)+mom(diff(g1,t1))...
        + mom(diff(g2,x2)*f2) + mom(diff(g2,t2))...
        + mom(diff(g3,x3)*f3) + mom(diff(g3,t3)); % dynamics

% trajectory constraints
X = [y1^2<=1; y2>=1; y3<=-1];

% initial constraints
X0 = [x0(1)^2<=thetamax^2, x0(2)^2<=omegamax^2];

% bounds on trajectory
B = [x1'*x1<=1; x2'*x2<=1; x3'*x3<=1];

% bounds on time - scaled to one
Tlim = [t1 >= 0, t1 <= 1, t2 >= 0, t2 <= 1, t3 >= 0, t3 <= 1];

% input LMI moment problem
P = msdp(max(cost), ...
        mass(m1)+mass(m2)+mass(m3)==1, ...
        mass(m0)==1, ...
        Ay==yT-y0, ...
        X, X0, B, Tlim);

% solve LMI moment problem
[status,obj] = msol(P)
```

- For more examples and codes

  https://homepages.laas.fr/henrion/papers/safev.pdf

  D. Henrion, M. Ganet-Schoeller, S. Bennani. ”Measures and LMI for space launcher robust control validation”, Proceedings of the IFAC Symposium on Robust Control Design, Aalborg, Denmark, June 2012.

Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

Optimal Control

D. Henrion, J. B. Lasserre, C. Savorgnan, Nonlinear optimal control synthesis via occupation measures, Proc. IEEE Conf. Decision and Control, 2008.

J. B. Lasserre, D. Henrion, C. Prieur, E. Tr´elat, Nonlinear optimal control via occupation measures and LMI relaxations, SIAM J. Control Opt., 47(4):1643-1666, 2008.

POCP - Matlab package for solving polynomial optimal control problems. Can be freely downloaded and used. Developed by Didier Henrion, Jean-Bernard Lasserre and Carlo Savorgnan. http://homepages.laas.fr/henrion/software/pocp/

Optimal control problem:

$$
\begin{aligned}
\inf \quad & \int_0^T l(t, x(t), u(t))dt \\
\text{s.t.} \quad & \dot{x}(t) = f(t, x(t), u(t)), \\
& x(t) \in X, \;\; u(t) \in U, \;\; t \in [0, T], \\
& x(0) \in X_0, \quad x(T) \in X_T
\end{aligned}
$$

Optimization with respect to a control law u over $t \in [0, T]$

Occupation Measure:

- Given an **initial condition $\boldsymbol{x_0}$**, the **occupation measure** of a trajectory $x(t|x_0)$ is defined by

occupation measure: $$\mu(S_t \times S_x \; |x_0) = \int_{S_t} \mathbf{I}_{S_x}(x(t|x_0))dt$$ given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$

➢ **Geometric interpretation:** measures the time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0))$ in a given set $S_t \times S_x$

Controlled Occupation Measure:

- Given an **initial condition $\boldsymbol{x_0}$**, and a **control law $\boldsymbol{u(t)}$**, the **controlled occupation measure** of a trajectory $x(t|x_0, u)$ is defined by

Controlled occupation measure: $$\mu(S_t \times S_x \times S_u|x_0, u) = \int_{S_t} \mathbf{I}_{S_x \times S_u}(x(t|x_0, u))dt$$ given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$, $S_u \subset \mathrm{U}$

➢ **Geometric interpretation** : measures the time spent by the graph of the trajectory $(\mathrm{t}, x(t|x_0, u), u(t))$ in a given set $S_t \times S_x \times S_u$.

Average Occupation Measure:

Given an initial probability measure of states ξ_0, the **average occupation measure** of the flow of trajectories is defined by

$$\mu(S_t \times S_x) = \int_X \underbrace{\mu(S_t \times S_x | x_0)}_{\text{Occupation measure}} \xi_0(dx)$$

Occupation measure
(spent time for single $x(t|x_0)$)

given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$

Average Controlled Occupation Measure:

Given an initial measure ξ_0, and control $u(t)$, the average occupation measure of the flow of trajectories is defined by

$$\mu(S_t \times S_x \times S_u | u) = \int_X \mu(\, S_t \times S_x \times S_u | x_0, u)\xi_0(dx_0)$$

given sets $S_t \subset [0, T], S_x \subset \mathrm{X}$
$S_u \subset \mathrm{U}$

- Average Controlled Occupation Measure, initial measure, terminal measure, i.e. μ, μ_0, μ_T, are **linked** by a **linear PDE.**

➢ Average Controlled Occupation Measure, initial measure, terminal measure, i.e. μ, μ_0, μ_T, are **linked** by a **linear PDE.**

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \mu_0 - \mu_T$$

Controlled Liouville Equation

- The difference with the uncontrolled Liouville equation is that both μ and f now also depend on the control variable u.
- An occupation measure satisfying Controlled Liouville Equation encodes **state trajectories** but also **control trajectories.**

LP in measure:

$$\begin{array}{ll} \inf & \langle l, \mu \rangle \\ \text{s.t.} & \frac{\partial \mu}{\partial t} + \operatorname{div} f\mu = \mu_0 - \mu_T \\ & \langle 1, \mu_0 \rangle = 1 \end{array}$$

$$\text{measures } (\mu,\ \mu_0,\ \mu_T) \in \mathscr{M}_+([0,T] \times X \times U) \times \mathscr{M}_+(\{0\} \times X_0)) \times \mathscr{M}_+(\{T\} \times X_T)$$

Moment SDP: moment representation of the measures.

Moment of Measures: $y_\alpha = \int t^{\alpha_1} x^{\alpha_2} u^{\alpha_3} \mu(dx, dt, du)$

Relaxed control

We consider following (disintegrated) form for Average Controlled Occupation Measure:

$$\mu(dt, dx, du) = dt\, \xi(dx \mid t)\, \omega(du \mid t, x)$$

the three components are as follows

- dt is the time marginal,(the Lebesgue measure of time)
- $\xi(dx|t)$ is the distribution of state for given time t
- $\omega(du|t,x)$ is the distribution of the control conditional on t and x (probability measure on U for each $t \in [0,T]$)

➢ instead of a control law u, we have a relaxed control, a probability measure

$$\omega \in \mathscr{M}_+(U), \quad \int \omega = 1$$

parametrized in time $t \in [0,T]$ and space $x \in X$. (Young measures)

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parametrized in time $t \in [0,T]$ and space $x \in X$. (Young measures)

Instead of working with $\dot{x}(t) = f(x(t), u(t), t)$ ⇨ We work with $\dot{x}(t) = \int_U f(x(t), u(t), t)\omega(du|t,x)$

- The set of trajectories modeled by the controlled Liouville equation is **larger** than the set of trajectories of the original control system.

Example:

$$\begin{array}{ll} \inf & \int_0^T (x^2(t) + u^2(t))dt \\ \text{s.t.} & \dot{x}(t) = u(t), \quad t \in [0, T] \\ & x(0) = 1, \quad x(T) = 0 \end{array}$$

Corresponding autonomous measure LP:

$$\begin{array}{ll} \inf & \langle x^2 + u^2, \mu \rangle \\ \text{s.t.} & \frac{\partial(u\mu)}{\partial x} = \delta_1 - \delta_0 \end{array}$$

- In terms of moments

$$\begin{array}{ll} \inf & \int (x^2 + u^2)\mu(dx, du) \\ \text{s.t.} & \alpha \int x^{\alpha-1} u\mu(dx, du) = -1, \quad \alpha = 0, 1, 2 \ldots \end{array}$$

$$\begin{array}{ll} \inf & y_{20} + y_{02} \\ \text{s.t.} & y_{01} = 2y_{11} = 3y_{21} = \cdots = -1 \\ & M(y) \geq 0 \end{array}$$

$$y_\alpha = \int x^{\alpha_1} u^{\alpha_2} \mu(dx, du), \quad \alpha = 0, 1, 2 \ldots$$

- Moment SDP

$$\begin{array}{ll} \inf & y_{20} + y_{02} \\ \text{s.t.} & y_{01} = 2y_{11} = -1 \\ & M_1(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix} \geq 0 \end{array}$$

$\Rightarrow$

$$M_1(y^*) = \begin{pmatrix} 3.66 & 1.00 & -1.00 \\ 1.00 & 0.500 & -0.500 \\ -1.00 & -0.500 & 0.500 \end{pmatrix}$$

Example:

$$\begin{array}{ll} \inf & \int_0^T (x^2(t) + u^2(t))dt \\ \text{s.t.} & \dot{x}(t) = u(t), \quad t \in [0, T] \\ & x(0) = 1, \quad x(T) = 0 \end{array}$$

This example can be solved analytically

Optimal solution: $u(t) = -x(t)$ $\qquad$ $x(t) = e^{-t}$

Optimal occupation measure $\qquad \mu(dx, du) = \int_0^\infty \delta_{e^{-t}}(dx)\delta_{-e^{-t}}(du)dt$

With moments: $\qquad y_\alpha = (-1)^{\alpha_2} \int_0^\infty e^{-(\alpha_1+\alpha_2)t} dt$

$$y_{10} = 1, y_{01} = -1, y_{20} = \frac{1}{2}, y_{11} = -\frac{1}{2}, y_{02} = \frac{1}{2}$$

Solution obtained by solving moment SDP $\qquad M_1(y^*) = \begin{pmatrix} 3.66 & 1.00 & -1.00 \\ 1.00 & 0.500 & -0.500 \\ -1.00 & -0.500 & 0.500 \end{pmatrix}$

Optimal control recovery

- To recover the optimal control, or the optimal state trajectory from the moments,

we can use **the dual problem**, which is a relaxation of the Hamilton-Jacobi-Bellman PDE of optimal control.

Optimal control recovery

- To recover the optimal control, or the optimal state trajectory from the moments,

we can use **the dual problem**, which is a relaxation of the Hamilton-Jacobi-Bellman PDE of optimal control.

- Using the duality of moments and SOS polynomials (lecture 5) and defining adjoint linear operator, dual reads as:

$$\max_{\substack{\varphi\in\mathbb{R}[t,x]_r, s\in\Sigma[t,x,u]_k, q\in\Sigma[x]_r \\ s_j\in\Sigma[t,x,u]_{k-d_{T_j}}, q_j\in\Sigma[x]_{r-d_{F_j}}}} \quad \varphi(0,x(0))$$

$$\frac{\partial\varphi(x,t)}{\partial t}+\nabla_x\varphi(x,t)f(t,x,u)+h(t,x,u)=s(t,x,u)+\sum_{j=1}^{n_T} g_{T_j}(t,x,u)s_j(t,x,u)$$

$$\varphi(x,T)-H(x)=-q(x)-\sum_{j=1}^{n_F} g_{F_j}(x)q_j(x).$$

In terms of the parameters of the set initial , final, and state sets

$X_0 = \{x : g_{I_j}(x)\le 0,\ j=1,\dots,n_I\}$ $\quad X_T = \{x : g_{F_j}(x)\le 0,\ j=1,\dots,n_F\}$ $\quad X = \{(t,x,u) : g_{T_j}(t,x,u)\le 0,\ j=1,\dots,n_T\}$

and cost-function of optimal control: $\int_0^T h(t,x(t),u(t))dt + H(x(T))$

- Constraints are polynomial nonnegativity conditions.

Optimal control recovery

$$\max_{\substack{\varphi\in\mathbb{R}[t,x]_r, s\in\Sigma[t,x,u]_k, q\in\Sigma[x]_r \\ s_j\in\Sigma[t,x,u]_{k-d_{T_j}}, q_j\in\Sigma[x]_{r-d_{F_j}}}} \quad \varphi(0,x(0))$$

$$\frac{\partial\varphi(x,t)}{\partial t} + \nabla_x\varphi(x,t)f(t,x,u) + h(t,x,u) = s(t,x,u) + \sum_{j=1}^{n_T} g_{T_j}(t,x,u)s_j(t,x,u)$$

$$\varphi(x,T) - H(x) = -q(x) - \sum_{j=1}^{n_F} g_{F_j}(x)q_j(x).$$

In terms of the parameters of the set initial , Terminal, and Trajectory sets

$X_0 = \{x : g_{I_j}(x) \le 0,\ j = 1,\ldots,n_I\}$ $\quad X_T = \{x : g_{F_j}(x) \le 0,\ j = 1,\ldots,n_F\}$ $\quad X = \{(t,x,u) : g_{T_j}(t,x,u) \le 0,\ j = 1,\ldots,n_T\}$

and cost-function of optimal control: $\int_0^T h(t,x(t),u(t))dt + H(x(T))$

- Every feasible solution φ is such that:

1) $\frac{\partial\varphi(x,t)}{\partial t} + \nabla_x\varphi(x,t)f(t,x,u) + h(t,x,u) \ge 0 \quad \forall\ (t,x,u) \in X$ (Trajectory Set)

2) $H(x) - \varphi(T,x) \ge 0 \quad \forall\ x \in X_T$ (Terminal Set)

- Polynomial $\varphi(t, x)$ is polynomial subsolution of the Hamilton-Jacobi-Bellman equation which approximates the value function along all the optimal trajectories.

- Therefore, given an optimal solution $\varphi(t, x)$ of the SOS optimization, control law $u(x(t))$ is a global minimizer of

$$\min_{u \in U(t,x)} \left[\frac{\partial \varphi(x,t)}{\partial t} + \nabla_x \varphi(x,t) f(t,x,u) + h(t,x,u) \right]$$

POCP - Matlab package for solving polynomial optimal control problems

http://homepages.laas.fr/henrion/software/pocp/

D. Henrion, J. B. Lasserre, C. Savorgnan, Nonlinear optimal control synthesis via occupation measures, Proc. IEEE Conf. Decision and Control, 2008.

J. B. Lasserre, D. Henrion, C. Prieur, E. Tr´elat, Nonlinear optimal control via occupation measures and LMI relaxations, SIAM J. Control Opt., 47(4):1643-1666, 2008.

POCP - Matlab package for solving polynomial optimal control problems. Can be freely downloaded and used. Developed by Didier Henrion, Jean-Bernard Lasserre and Carlo Savorgnan. http://homepages.laas.fr/henrion/software/pocp/

Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

Region Of Attraction Set

M. Korda, D. Henrion, C. N. Jones ,"Region of attraction approximations for polynomial dynamical systems", Conference on Geometry and Algebra of Linear Matrix Inequalities, GeoLMI 2013 , http://homepages.laas.fr/henrion/geolmi13/korda.pdf

D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, IEEE Transactions on Automatic Control, Vol. 59, No. 2, pp. 297-312, 2014.

M. Korda, D. Henrion, C. N. Jones. Controller design and region of attraction estimation for nonlinear dynamical systems , Proceedings at the IFAC World Congress on Automatic Control, Cape Town, South Africa, August 2014.

M. Korda, D. Henrion, C. N. Jones. Inner approximations of the region of attraction for polynomial dynamical systems, Proceedings of the IFAC Symposium on Nonlinear Control Systems, Toulouse, France, September 2013.

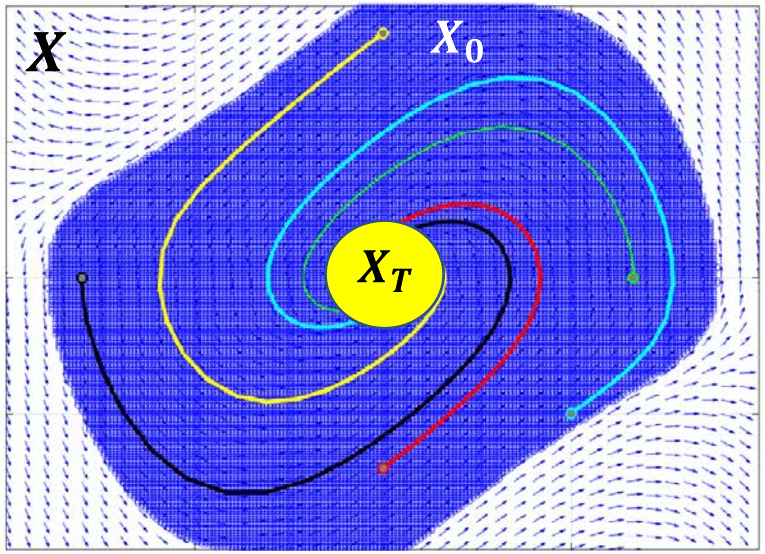
System: $\dot{x}(t) = f(t, x(t), \omega)$ $t \in [0, T]$ $x \in X$

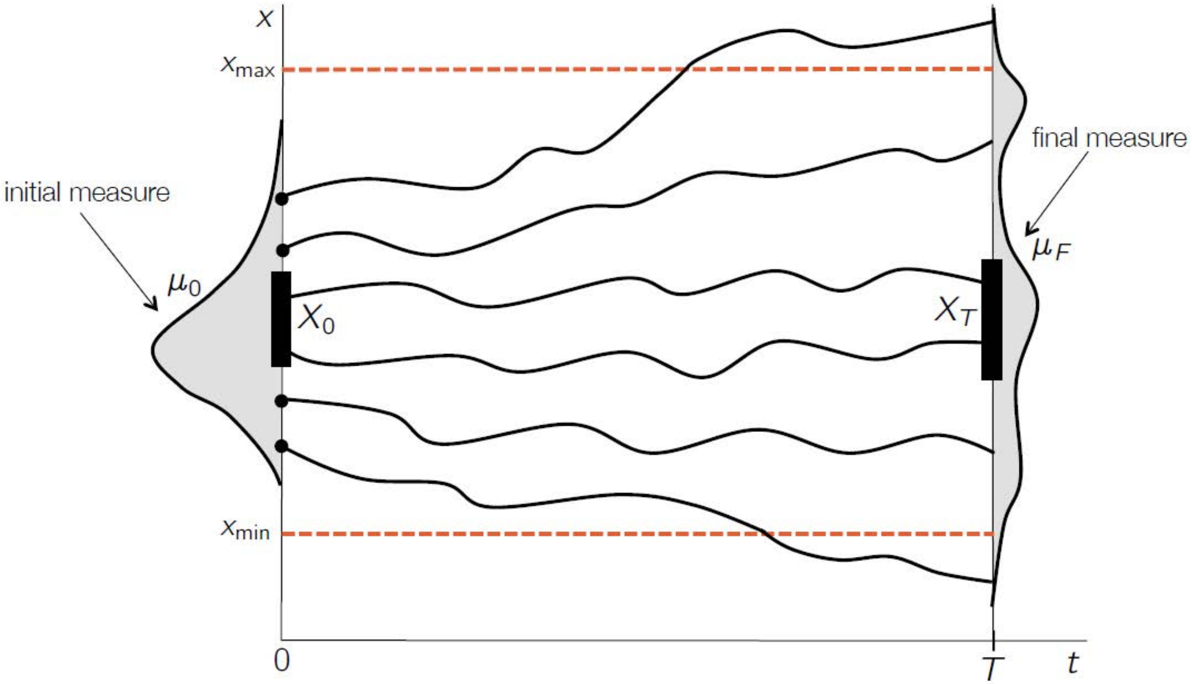
Given Terminal Set X_T

Region Of Attraction Set

$$\mathcal{X}(x_0) := \left\{ x(\cdot) \;:\; \exists\, u(\cdot) \in \mathcal{U} \text{ s.t. } \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. } x(0) = x_0,\; x(t) \in X,\; x(T) \in X_T,\; \forall\, t \in [0, T] \right\},$$

- ROA is the set of all initial conditions for which there exists an admissible trajectory, i.e., the set of all initial conditions that can be steered to the target set in an admissible way.





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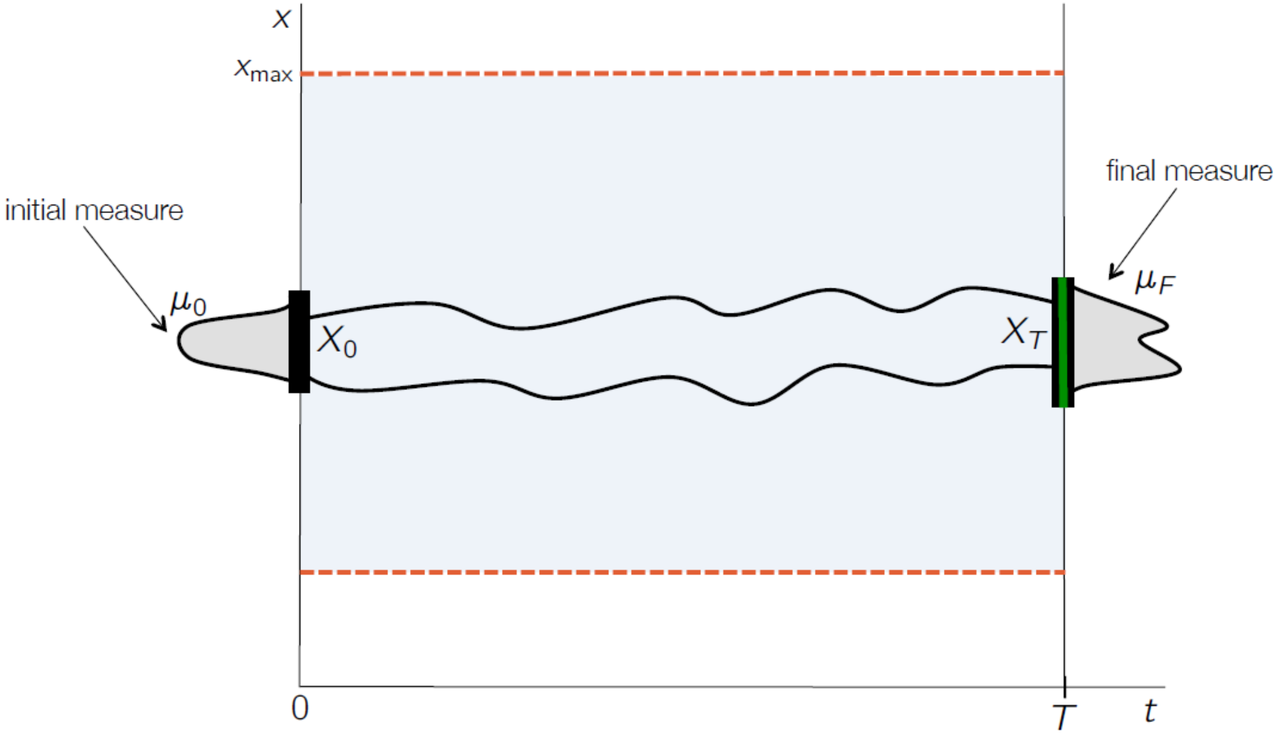

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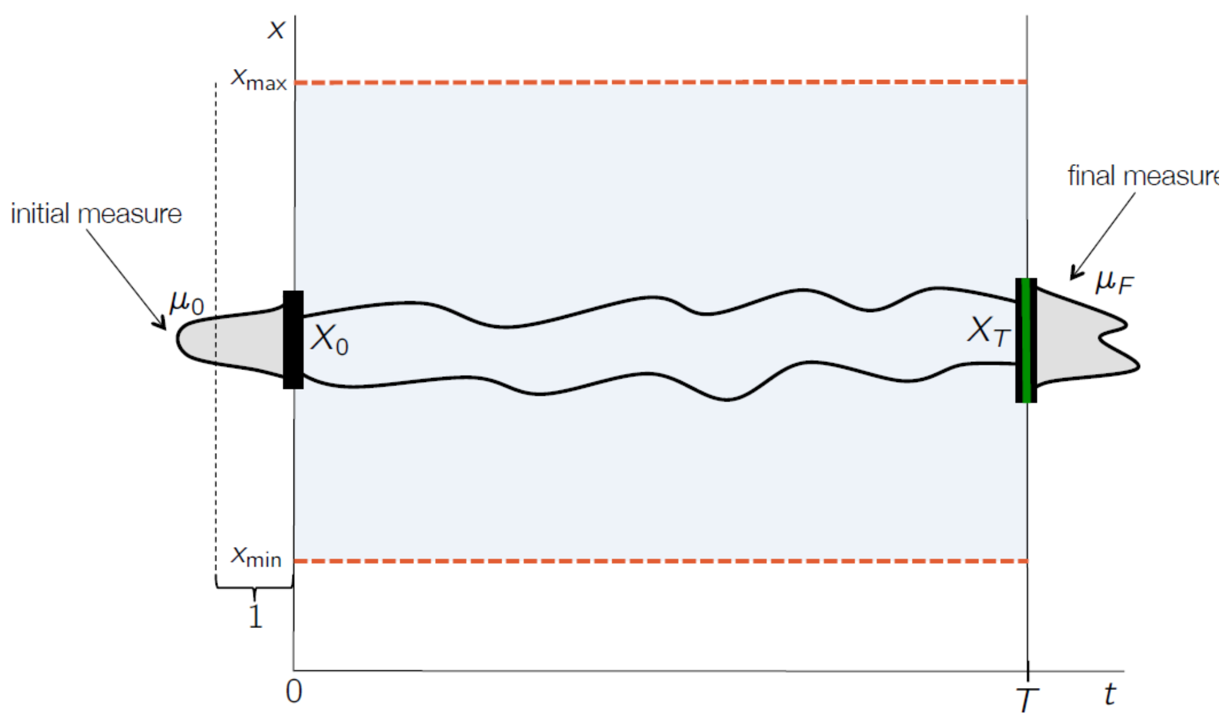
- ROA set is characterized with the support set of initial measure.
- Look for initial measure that can be steered to the target set.
- Initial and terminal measures are linked through Liouville’s Equation.

M. Korda, D. Henrion, C. N. Jones ,”Region of attraction approximations for polynomial dynamical systems”, Conference on Geometry and Algebra of Linear Matrix Inequalities, GeoLMI 2013 , http://homepages.laas.fr/henrion/geolmi13/korda.pdf



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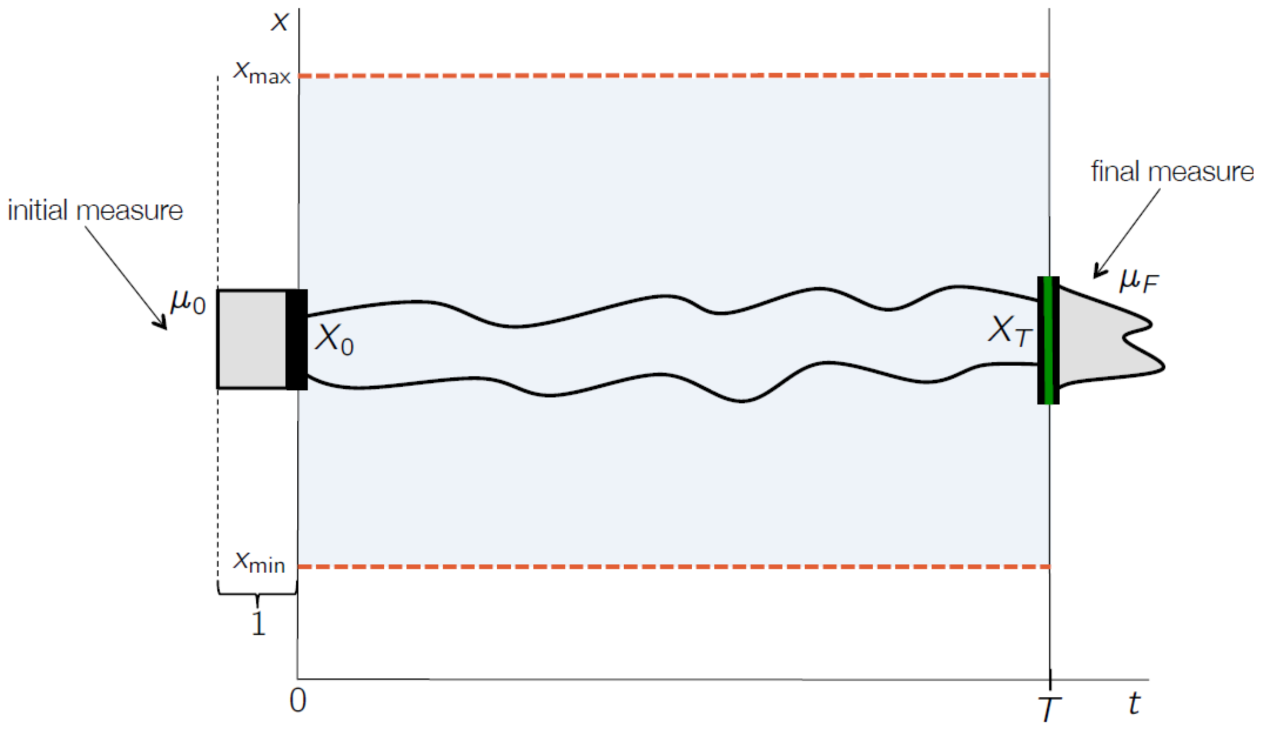

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- To obtain largest ROA set, maximize the volume of initial measure.

$$\max \mu_0(X) = \int_X d\mu_0$$

- Optimal initial measure is the Lebesgue measure over the ROA set.

M. Korda, D. Henrion, C. N. Jones ,"Region of attraction approximations for polynomial dynamical systems", Conference on Geometry and Algebra of Linear Matrix Inequalities, GeoLMI 2013 , http://homepages.laas.fr/henrion/geolmi13/korda.pdf

LP in measure

$$
\begin{aligned}
\sup \quad & \mu_0(X) && (1)\\
\text{s.t.} \quad & \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu && (2)\\
& \mu_0 + \hat{\mu}_0 = \lambda && (3)\\
& \mu \geq 0,\ \mu_0 \geq 0,\ \mu_T \geq 0,\ \hat{\mu}_0 \geq 0 \\
& \operatorname{spt} \mu \subset [0, T] \times X \times U && (4)\\
& \operatorname{spt} \mu_0 \subset X,\ \operatorname{spt} \mu_T \subset X_T \\
& \operatorname{spt} \hat{\mu}_0 \subset X.
\end{aligned}
$$

(1) We model ROA with the support of initial measure μ_0 max $\mu_0(X) = \int_X d\mu_0$

(2) Liouville's Equation captures the information of dynamical system.

(3) To ensures that the optimal value is the Lebesgue measure $\lambda \geq \mu_0$ —Slack measure→ $\mu_0 + \hat{\mu}_0 = \lambda$

(4) Support set of measures

D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, IEEE Transactions on Automatic Control, Vol. 59, No. 2, pp. 297-312, 2014.

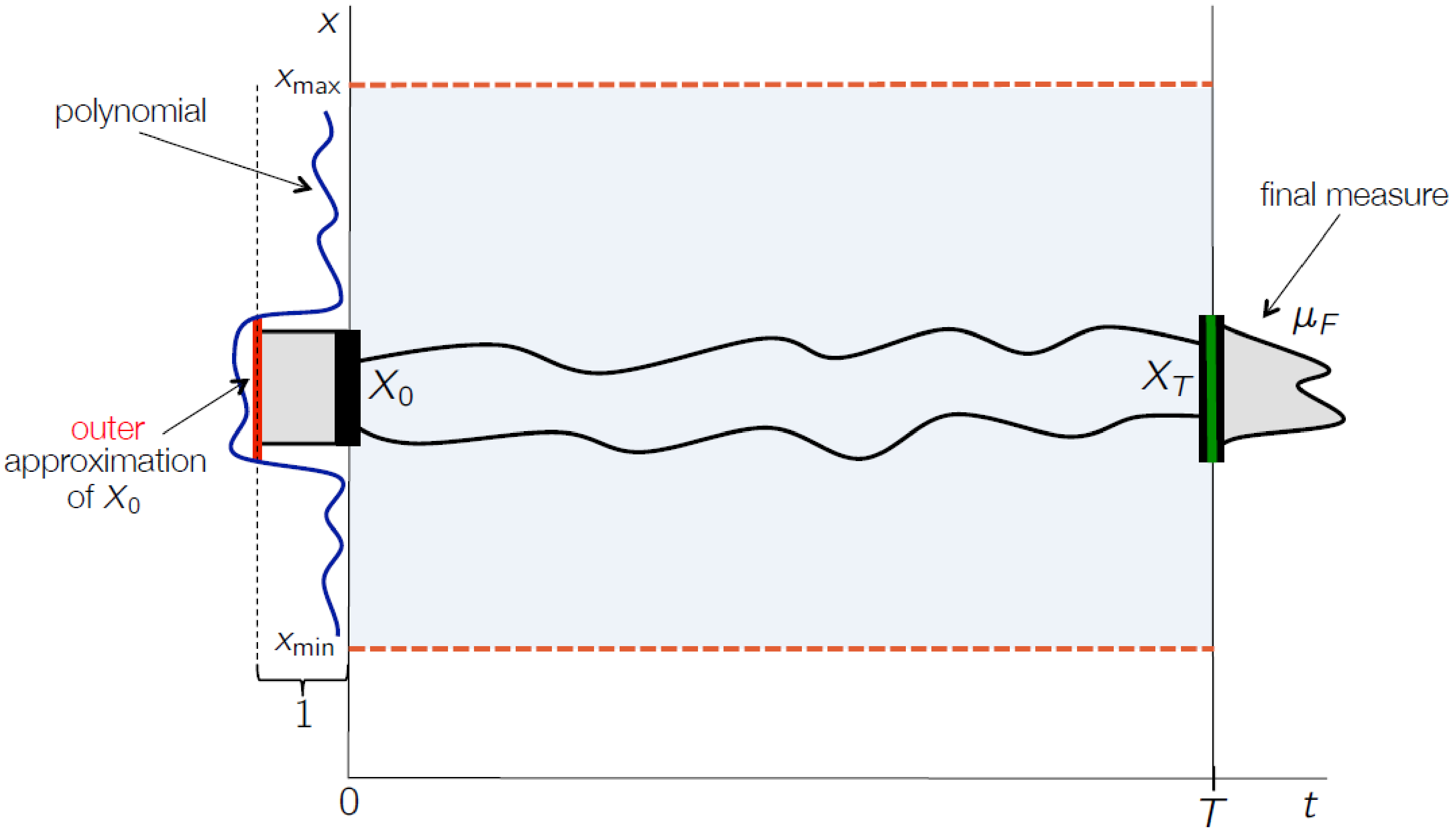
LP in measure

$$
\begin{array}{ll}
\sup & \mu_0(X) \\
\text{s.t.} & \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu \\
& \mu_0 + \hat{\mu}_0 = \lambda \\
& \mu \geq 0,\ \mu_0 \geq 0,\ \mu_T \geq 0,\ \hat{\mu}_0 \geq 0 \\
& \operatorname{spt} \mu \subset [0,T] \times X \times U \\
& \operatorname{spt} \mu_0 \subset X,\ \ \operatorname{spt} \mu_T \subset X_T \\
& \operatorname{spt} \hat{\mu}_0 \subset X.
\end{array}
$$

Dual Optimization (SOS Optimization)

$$
\begin{array}{lll}
\inf & \int_X w(x)\, d\lambda(x) & \\
\text{s.t.} & \mathcal{L}v(t,x,u) \leq 0, & \forall\,(t,x,u) \in [0,T] \times X \times U \\
& w(x) \geq v(0,x) + 1, & \forall\, x \in X \\
& v(T,x) \geq 0, & \forall\, x \in X_T \\
& w(x) \geq 0, & \forall\, x \in X,
\end{array}
$$

- $ROA \subset \{x\colon\ \omega(x) - 1 \geq 0\}$



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- $ROA \subset \{x\colon \omega(x) - 1 \geq 0\}$

Dual

$$\begin{aligned} \inf \; & \int_X w(x)\, d\lambda(x) \\ \text{s.t.} \; & \mathcal{L}v(t,x,u) \leq 0, && \forall\, (t,x,u) \in [0,T] \times X \times U && (1) \\ & w(x) \geq v(0,x) + 1, && \forall\, x \in X && (2) \\ & v(T,x) \geq 0, && \forall\, x \in X_T && (3) \\ & w(x) \geq 0, && \forall\, x \in X, \end{aligned}$$

Interpretation: similar to barrier function based safety verification(Lecture 8, page 29)

(1): v is decreasing along trajectories of the system.

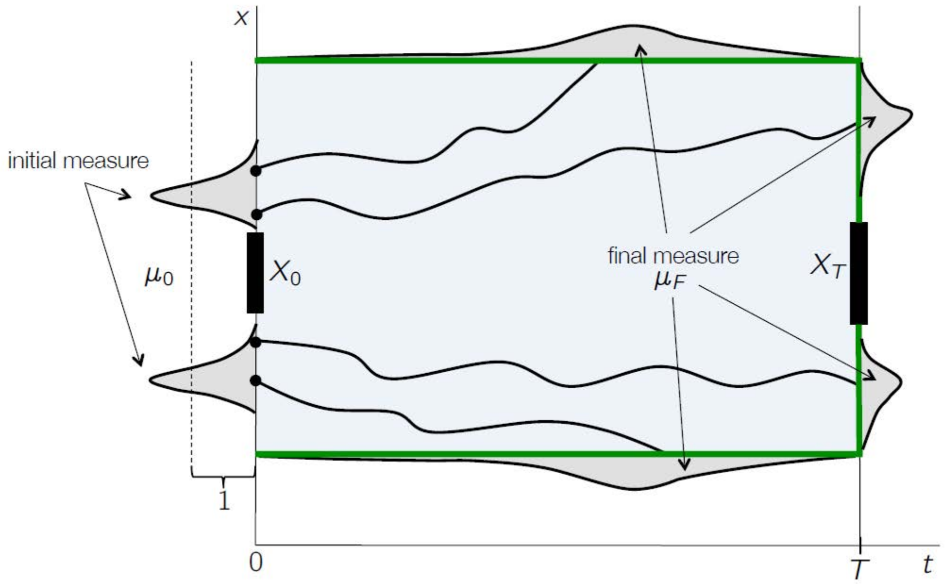
(3): $v(T,x) \geq 0$ on X_T.

(1) and (3): $\{x\colon\ v(0,x) < 0\}$ is an inner approximation to the set of points that **cannot reach** the target set.

- Hence, $\{x\colon\ v(0,x) \geq 0\}$ is an outer approximation to the set of points **reach** the target set.

    - $ROA \subset \{x\colon v(0,x) \geq 0\} = \{x\colon\ \omega(x) - 1 \geq 0\}$ (2)

Inner approximation



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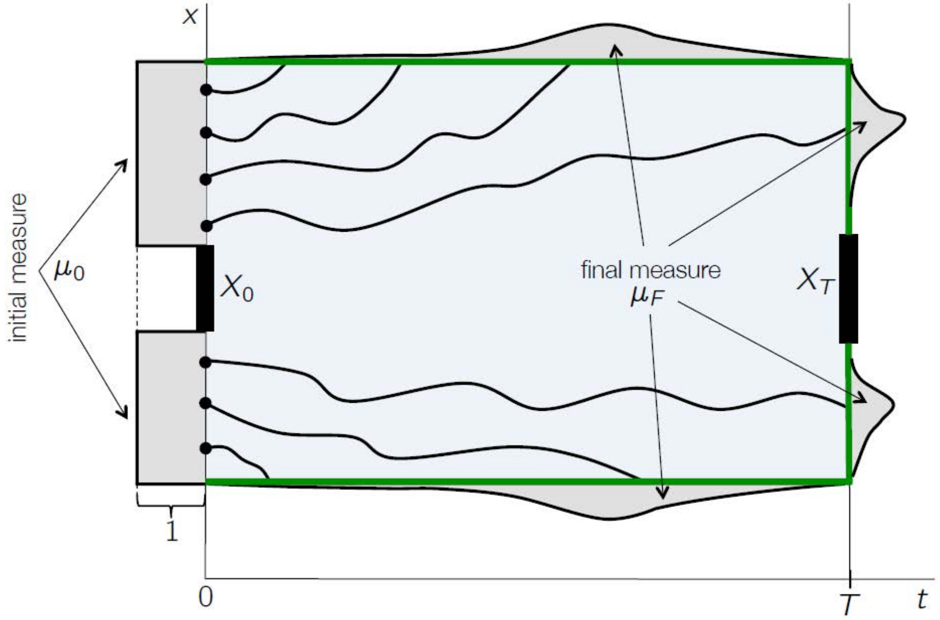

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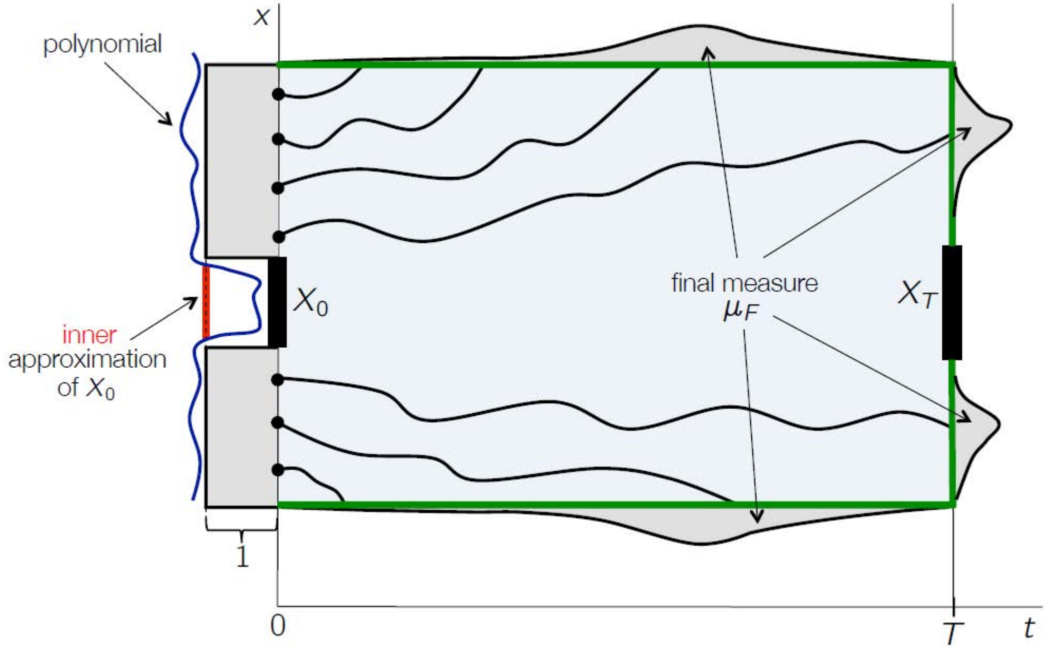
- One can apply the same methodology to find the outer approximation to the target set, i.e., $\{x\colon\ \omega(x) - 1 \geq 0\}$

- Inner approximation of ROA: $\{x\colon\ \omega(x) - 1 < 0\}$



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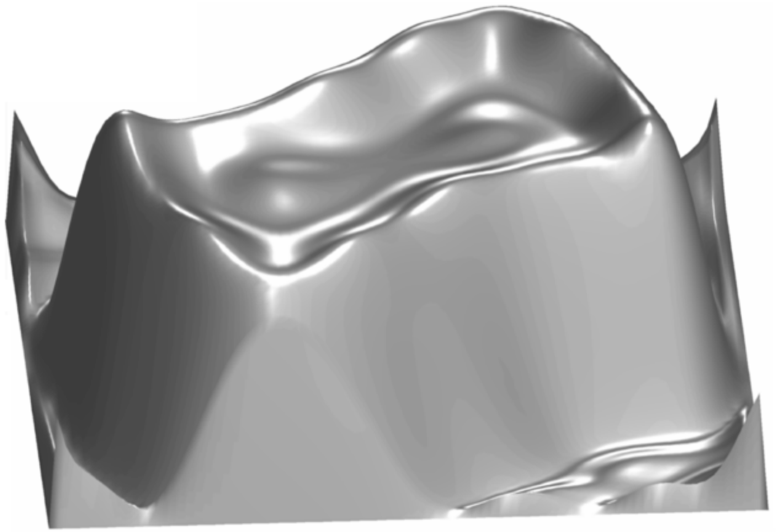
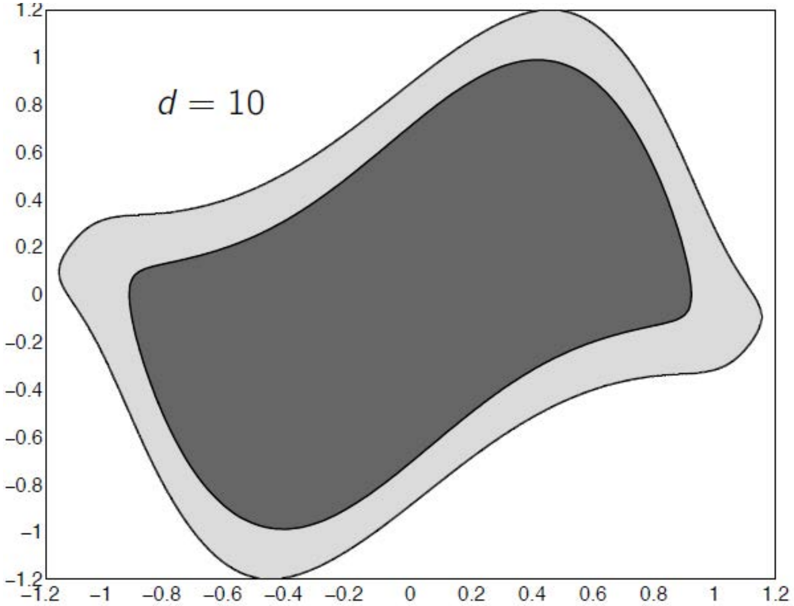
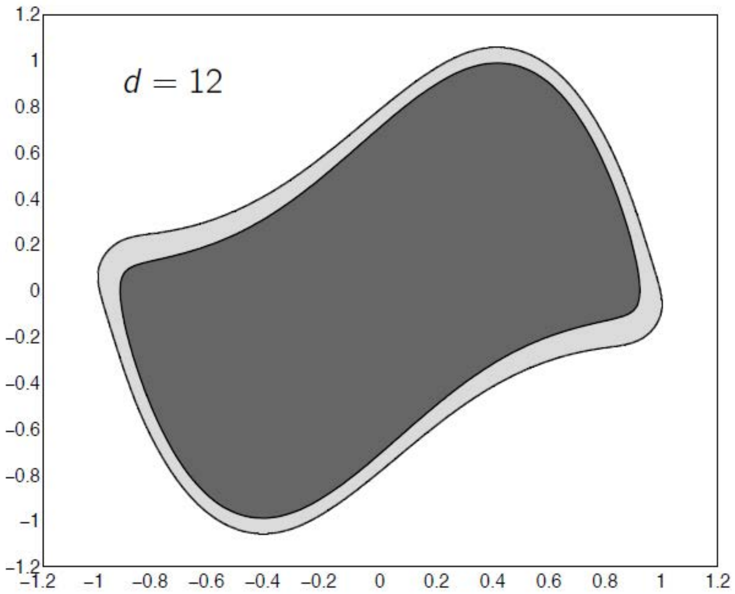
Backward Van der Pol oscillator

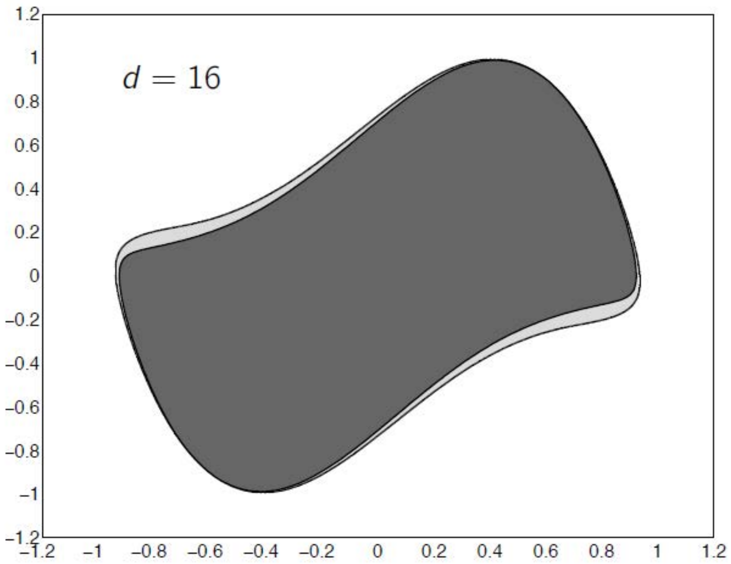
$$\dot{x}_1 = -2x_2$$
$$\dot{x}_2 = 0.8x_1 + 10(x_1^2 - 0.21)x_2$$
$$X = [-1.2, -1.2]^2$$
$$X_T = \{x \mid \|x\|_2 \leq 0.01\},\ T = 100$$



ROA Code: https://homepages.laas.fr/henrion/software/







D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, IEEE Transactions on Automatic Control, Vol. 59, No. 2, pp. 297-312, 2014.

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ M(x)^{-1}N(x,u) \end{bmatrix} \in \mathbb{R}^4,$$

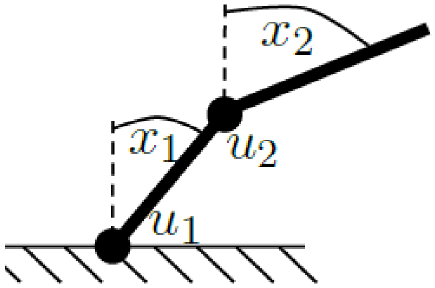
$$M(x) = \begin{bmatrix} 3+\cos(x_2) & 1+\cos(x_2) \\ 1+\cos(x_2) & 1 \end{bmatrix}$$

$$N(x,u) = \begin{bmatrix} g\sin(x_1+x_2) - a_1x_3 + a_2\sin(x_1) + x_4\sin(x_2)(2x_3+x_4) + u_1 \\ -\sin(x_2)x_3^2 - a_1x_4 + g\sin(x_1+x_2) + u_2 \end{bmatrix}$$

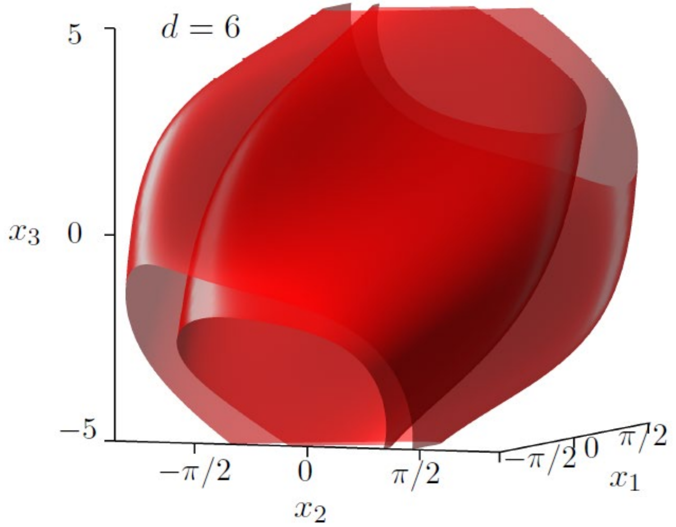
$$U = [-10, 10] \times [-10, 10]$$

$$X = [-\pi/2, \pi/2] \times [-\pi, \pi] \times [-5, 5] \times [-5, 5]$$

$$X_T = \{ x \mid : x \leq \epsilon \}$$



Acrobot – sketch



ROA Code: https://homepages.laas.fr/henrion/software/

D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, IEEE Transactions on Automatic Control, Vol. 59, No. 2, pp. 297-312, 2014.

Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

Nonlinear Feedback Control
and
Backward Reachable Set

Anirudha Majumdar, Ram Vasudevan, Mark M. Tobenkin, and Russ Tedrake. Convex optimization of nonlinear feedback controllers via occupation measures. International Journal of Robotics Research (IJRR), 33(9):1209-1230, August 2014

Anirudha Majumdar, Ram Vasudevan, Mark M. Tobenkin, and Russ Tedrake. Convex optimization of nonlinear feedback controllers via occupation measures. In Proceedings of Robotics: Science and Systems (RSS), 2013

- Control-affine system with feedback control $\dot{x}(t) = f\left(t, x(t)\right) + g\left(t, x(t)\right) u(t, x)$

- Input constraint $u(t, x) \in U = [a_1, b_1] \times \ldots \times [a_m, b_m]$

- Bounding set, and target set as
$$X = \{x \in \mathbb{R}^n \mid h_{X_i}(x) \geq 0, \forall i = \{1, \ldots, n_X\}\},$$
$$X_T = \{x \in \mathbb{R}^n \mid h_{T_i}(x) \geq 0, \forall i = \{1, \ldots, n_T\}\},$$

- Given a finite final time $T > 0$, let the **backwards reachable set** (BRS) for a particular control policy u be defined as:

$$\mathcal{X}(u) = \left\{x_0 \in \mathbb{R}^n \mid \dot{x}(t) = f\big(t, x(t)\big) + g\big(t, x(t)\big) u\big(t, x(t)\big) \text{ a.e. } t \in [0, T],\ x(0) = x_0,\ x(T) \in X_T,\ x(t) \in X\ \forall t \in [0, T]\right\}$$

$\chi(u)$ is the set of initial conditions for trajectories of dynamical system that remain in the bounding set and arrive in the target set at the final time when control law u is applied.

➢ Find a controller u that maximizes the volume of the BRS, i.e., $\max\ \lambda(\chi(u))$ $\qquad \lambda(\chi(u)) = \int_{\chi(u)} dx$

λ → Lebesgue measure

- Control-affine system with feedback control $\dot{x}(t) = f\left(t, x(t)\right) + g\left(t, x(t)\right) u(t, x)$

- We maximize the volume of the BRS using control input u.

- Control-affine system with feedback control $\dot{x}(t) = f\left(t, x(t)\right) + g\left(t, x(t)\right) u(t, x)$

- We maximize the volume of the BRS using control input u.

- Occupation measure formulation

- Instead of working with controlled average occupation measure $\mu(dt, dx, du)$, we work with average occupation measure $\mu(dt, dx)$

- We decompose $\int_{A \times B} u_j(t, x) d\mu(t, x)$ inside the Liouville's Equation in terms of (nonnegative) measures

$\sigma^+, \sigma^- \in \left(\mathcal{M}\left([0, T] \times X\right)\right)^m$

$$\int_{A \times B} u_j(t, x) d\mu(t, x) = \int_{A \times B} d[\sigma^+]_j(t, x) - \int_{A \times B} d[\sigma^-]_j(t, x)$$

where $\sigma^+ - \sigma^-$ is a signed measure. (This will let us to extract the solution)

To obtain BRS, we solve measure-LP:

$$
\begin{aligned}
\sup \quad & \mu_0(X) && \text{(1)} \\
\text{s.t.} \quad & \mathcal{L}_f'\mu + \mathcal{L}_g'(\sigma^+ - \sigma^-) = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0, && \text{(2)} \\
& [\sigma^+]_j + [\sigma^-]_j + [\hat{\sigma}]_j = \mu && \text{(3)} \quad \forall j \in \{1, \ldots, m\}, \\
& \mu_0 + \hat{\mu}_0 = \lambda, \\
& [\sigma^+]_j, [\sigma^-]_j, [\hat{\sigma}]_j \geq 0 && \forall j \in \{1, \ldots, m\}, \\
& \mu, \mu_0, \mu_T, \hat{\mu}_0 \geq 0,
\end{aligned}
$$

(1): We model BRS with the support of initial measure μ_0 max $\mu_0(X) = \int_X d\mu_0$ → max $volume$(BRS)

(2): Liouville's Equation captures the information of dynamical system.

(3): To ensures that we are able to extract a **bounded** control law:

$\mu \geq [\sigma^+]_j + [\sigma^-]_j$ —Slack measure→ $[\sigma^+]_j + [\sigma^-]_j + [\hat{\sigma}]_j = \mu$

To obtain BRS, we solve measure-LP:

$$
\begin{array}{llll}
\sup & \mu_0(X) & (1) & \\
\text{s.t.} & \mathcal{L}'_f \mu + \mathcal{L}'_g(\sigma^+ - \sigma^-) = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0, & (2) & \\
& [\sigma^+]_j + [\sigma^-]_j + [\hat{\sigma}]_j = \mu & (3) & \forall j \in \{1, \ldots, m\}, \\
& \mu_0 + \hat{\mu}_0 = \lambda, & (4) & \\
& [\sigma^+]_j, [\sigma^-]_j, [\hat{\sigma}]_j \geq 0 & & \forall j \in \{1, \ldots, m\}, \\
& \mu, \mu_0, \mu_T, \hat{\mu}_0 \geq 0, & &
\end{array}
$$

(4): To ensures that the optimal value is the Lebesgue measure

volume of BRS in terms of Lebesgue measure: $\lambda(\chi(u))$

We model BRS with the support of initial measure μ_0: $\mu_0(X)$

$\lambda \geq \mu_0$ —Slack measure→ $\mu_0 + \hat{\mu}_0 = \lambda$

- Supports:

$$(\sigma^+, \sigma^-, \hat{\sigma}, \mu, \mu_0, \hat{\mu}_0, \mu_T) \in (\mathcal{M}([0,T] \times X))^m \times (\mathcal{M}([0,T] \times X))^m \times (\mathcal{M}([0,T] \times X))^m \times \mathcal{M}([0,T] \times X) \times \mathcal{M}(X) \times \mathcal{M}(X) \times \mathcal{M}(X_T)$$

- To extract Polynomial u from the moments $y^*_{k,\sigma^+}, y^*_{k,\sigma^-}$, and $y^*_{k,\mu}$

$$\int_{A\times B} u_j(t,x)d\mu(t,x) = \int_{A\times B} d[\sigma^+]_j(t,x) - \int_{A\times B} d[\sigma^-]_j(t,x)$$

$$\int_{[0,T]\times X} t^{\alpha_0}x^{\alpha}[u_k]_j(t,x)\, d\mu(t,x) = \int_{[0,T]\times X} t^{\alpha_0}x^{\alpha}d[\sigma^+ - \sigma^-]_j(t,x),$$

Coefficient of polynomial feedback

- Direct calculation shows the linear system of equations

$$M_k(y^*_{k,\mu})\mathrm{vec}([u_k]_j) = y^*_{k,[\sigma^+]_j} - y^*_{k,[\sigma^-]_j}$$

The dual optimization (SOS optimization)allows us to obtain approximations of the BRS

$$\begin{aligned}
\inf \quad & \int_X w(x)d\lambda(x) & \\
\text{s.t.} \quad & \mathcal{L}_f v + \Sigma_{i=1}^m [p]_i \leq 0, & \\
& [p]_i \geq 0, \quad [p]_i \geq |[\mathcal{L}_g v]_i| & \forall i \in \{1, \dots, m\}, \\
& w \geq 0, & \\
& w(x) \geq v(0, x) + 1 & \forall x \in X, \\
& v(T, x) \geq 0 & \forall x \in X_T
\end{aligned}$$

$BRS \subset \{x \mid w(x) \geq 1\}$ $w(x)$ is upper bound approximation of the indictor function of the BRS set

Dual Optimization

$$
\begin{aligned}
\inf \quad & \int_X w(x) d\lambda(x) \\
\text{s.t.} \quad & \left.\begin{aligned} & \mathcal{L}_f v + \Sigma_{i=1}^m [p]_i \leq 0, \\ & [p]_i \geq 0, \quad [p]_i \geq |[\mathcal{L}_g v]_i| \end{aligned}\right\} \text{(1)} && \forall i \in \{1, \ldots, m\}, \\
& w \geq 0, \\
& w(x) \geq v(0, x) + 1 \quad \text{(2)} && \forall x \in X, \\
& v(T, x) \geq 0 \quad \text{(3)} && \forall x \in X_T
\end{aligned}
$$

Interpretation: similar to barrier function based safety verification(Lecture 8, page 29)

(1): v decrease along trajectories of the system for any valid control input.

(3): $v(T, x) \geq 0$ on X_T.

(1) and (3): $\{x: \ v(0, x) < 0\}$ is an inner approximation to the set of points that **cannot reach** the target set.

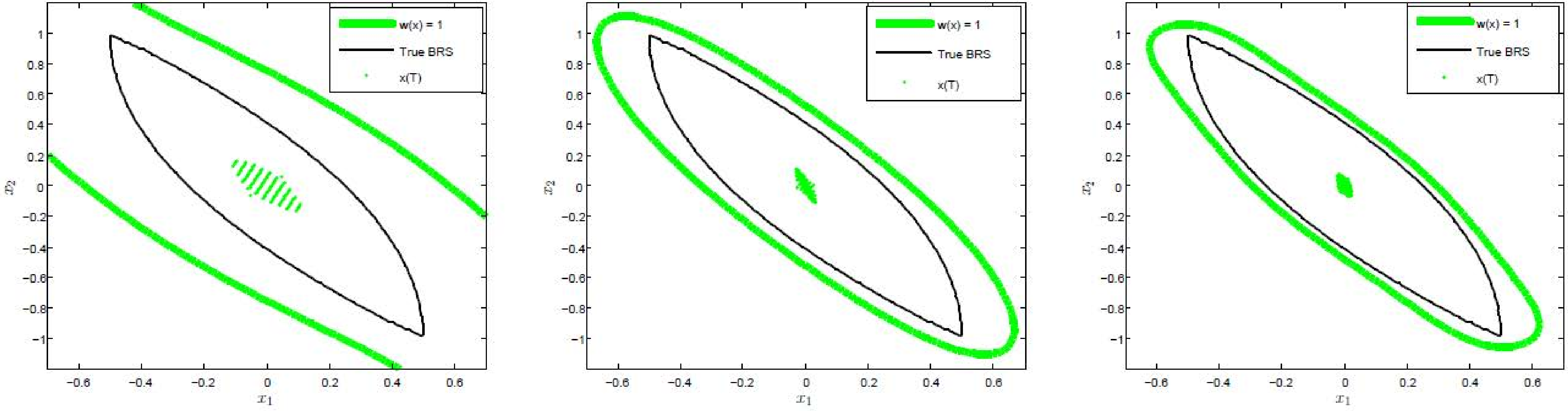
- Hence, $\{x: \ v(0, x) \geq 0\}$ is an outer approximation to the set of points **reach** the target set.

  - $ROA \subset \{x: v(0, x) \geq 0\} = \{x: \ \omega(x) - 1 \geq 0\}$ (2)

Example 1:

$$\dot{x}_1 = x_2, \qquad U = [-1, 1]$$
$$\dot{x}_2 = u, \qquad X_T = \{0\} \quad T = 1$$

Outer approximation of BRS:



(a) $k = 2$ (b) $k = 3$ (c) $k = 4$

Obtained feedback control input $u_2(t, x) = -1.541x_1 - 4.046x_1t - 1.099x_2 - 3.677x_2t.$

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Example 2: Vehicle Control

$$\dot{a} = v\cos(\theta),$$
$$\dot{b} = v\sin(\theta),$$
$$\dot{\theta} = \omega,$$

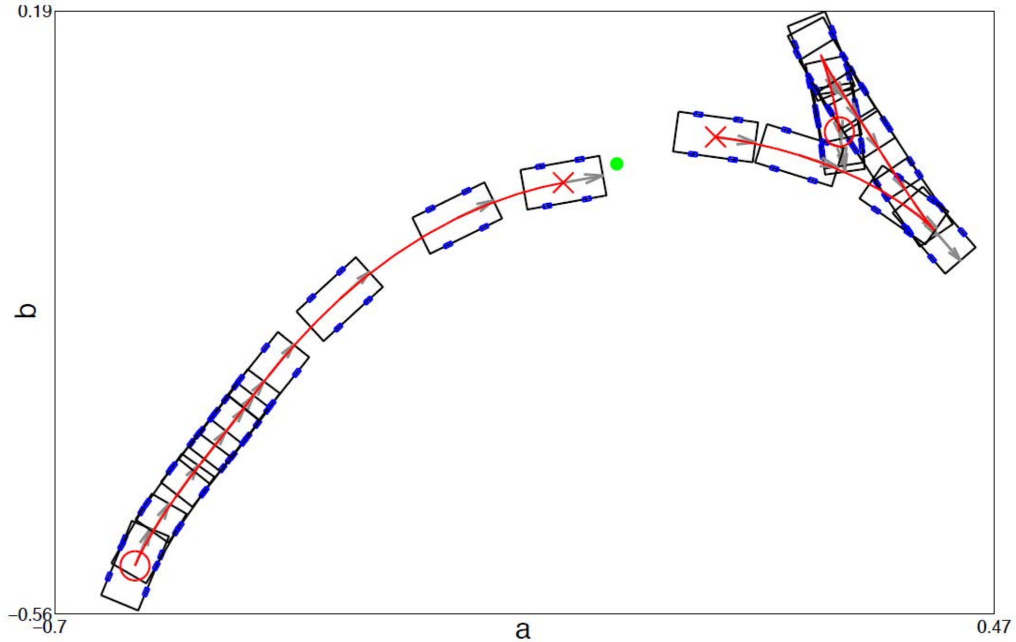
Polynomial dynamics

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = u_2,$$
$$\dot{x}_3 = x_1u_2 - x_2u_1.$$

initial conditions in $X = \{x \mid \|x\|^2 \leq 4\}$

$X_T = \{x \mid \|x\|^2 \leq 0.1^2\}$

$u_1, u_2 \in [-1, 1]$



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Topics:

- Occupation Measure and Liouville's Equation
- Trajectory Optimization
- Optimal Control
- Region of Attraction Set
- Nonlinear Feedback Control and Backward Reachable Set

- D. Henrion, M. Ganet-Schoeller, S. Bennani. ”Measures and LMI for space launcher robust control validation”, Proceedings of the IFAC Symposium on Robust Control Design, Aalborg, Denmark, June 2012.
- D. Henrion ”Optimization on linear matrix inequalities for polynomial systems control”, Lecture notes used for a tutorial course given during the International Summer School of Automatic Control held at Grenoble, France, in September 2014.
- D. Henrion, J. B. Lasserre, C. Savorgnan, Nonlinear optimal control synthesis via occupation measures, Proc. IEEE Conf. Decision and Control, 2008.
- J. B. Lasserre, D. Henrion, C. Prieur, E. Tr´elat, Nonlinear optimal control via occupation measures and LMI relaxations, SIAM J. Control Opt., 47(4):1643-1666, 2008.
- POCP - Matlab package for solving polynomial optimal control problems. Can be freely downloaded and used. Developed by Didier Henrion, Jean-Bernard Lasserre and Carlo Savorgnan. http://homepages.laas.fr/henrion/software/pocp/
- M. Korda, D. Henrion, C. N. Jones ,”Region of attraction approximations for polynomial dynamical systems”, Conference on Geometry and Algebra of Linear Matrix Inequalities, GeoLMI 2013 , http://homepages.laas.fr/henrion/geolmi13/korda.pdf
- D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, IEEE Transactions on Automatic Control, Vol. 59, No. 2, pp. 297-312, 2014.
- M. Korda, D. Henrion, C. N. Jones. Controller design and region of attraction estimation for nonlinear dynamical systems., Proceedings at the IFAC World Congress on Automatic Control, Cape Town, South Africa, August 2014.
- M. Korda, D. Henrion, C. N. Jones. Inner approximations of the region of attraction for polynomial dynamical systems, Proceedings of the IFAC Symposium on Nonlinear Control Systems, Toulouse, France, September 2013.
- Anirudha Majumdar, Ram Vasudevan, Mark M. Tobenkin, and Russ Tedrake. Convex optimization of nonlinear feedback controllers via occupation measures. International Journal of Robotics Research (IJRR), 33(9):1209-1230, August 2014
- Anirudha Majumdar, Ram Vasudevan, Mark M. Tobenkin, and Russ Tedrake. Convex optimization of nonlinear feedback controllers via occupation measures. In Proceedings of Robotics: Science and Systems (RSS), 2013