Lecture 14

Sum-of-Squares Optimization Based Robust Planning for Uncertain Nonlinear Systems

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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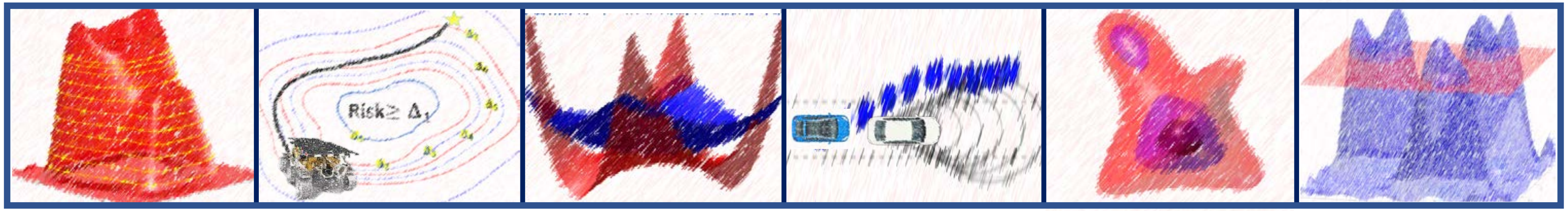


- In this lecture, we will mainly use

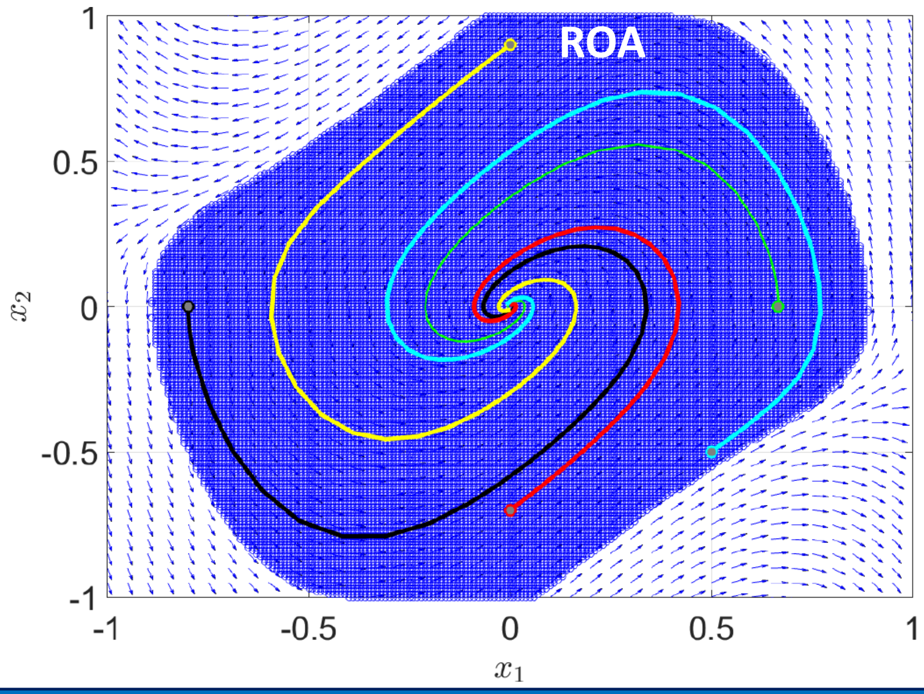
  1) Lyapunov based reasoning and 2) SOS optimization

  for safety and control of uncertain nonlinear systems.

- For safety we will construct i) Region of attraction set, ii) Invariant Set, and iii) Funnel

➢ Region Of Attraction Set (ROA)

Set of all initial states of the dynamical system whose trajectories converge to the equilibrium point of the dynamical system



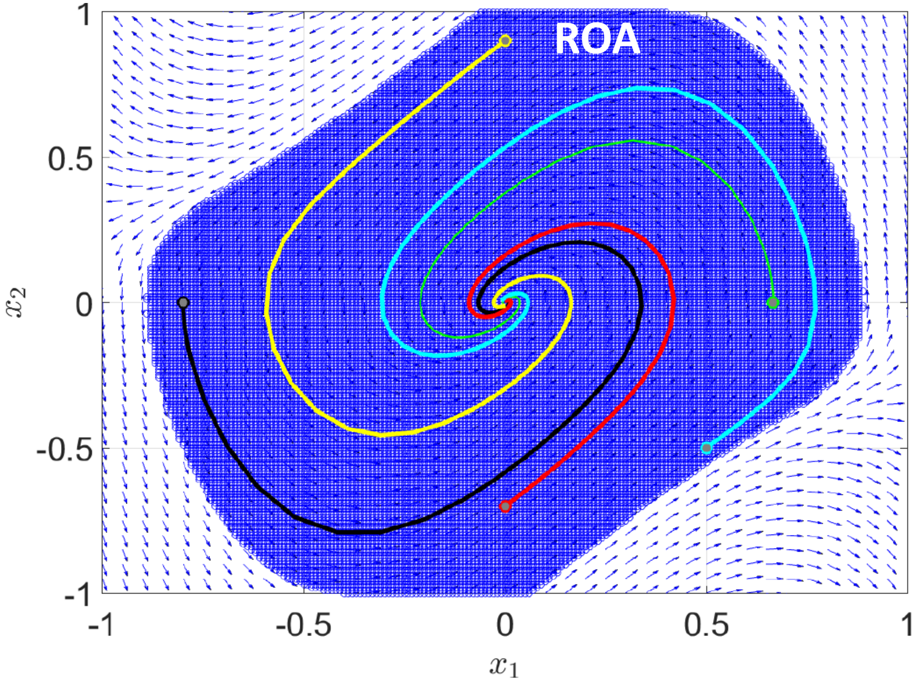
➢ Region Of Attraction Set (ROA)

Set of all initial states of the dynamical system whose trajectories converge to the equilibrium point of the dynamical system

Example:

- Equilibrium point: desired pose of the robot
- States of the closed-loop system due to external disturbances deviate from the desired pose.

  As long as disturbed states are inside the ROA set, they will converge to the desired pose.



- **Region Of Attraction Set (ROA)**

Set of all initial states of the dynamical system whose trajectories converge to the equilibrium point of the dynamical system

- In lecture 12, we looked at chance constrained Backward reachable sets.
- In lecture 13, we used occupation measure to compute ROA and Backward reachable sets.
- In this lecture, we leverage on Lyapunov stability and SOS optimization to compute ROA sets

Invariant Set

Set of all initial states of the dynamical system whose trajectories remains inside the set despite all disturbances.

Example:

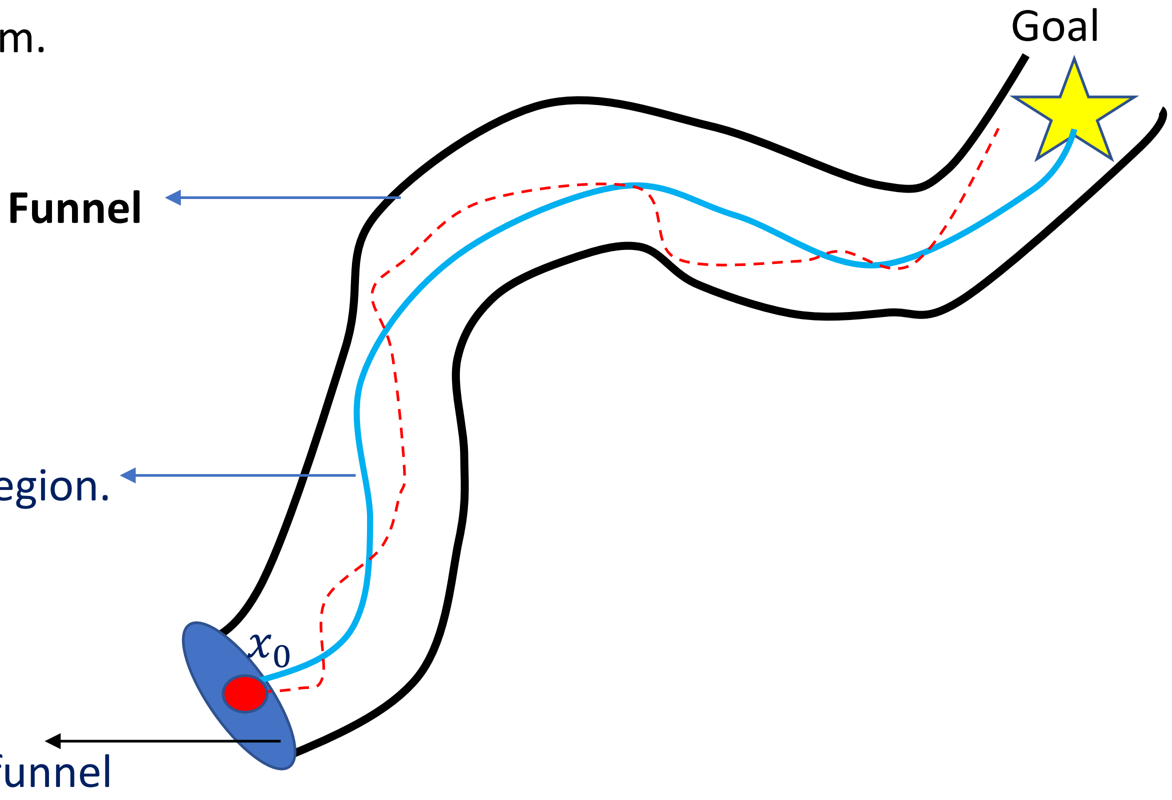
- If **obstacle free** region becomes **invariant set** of closed-loop robot system.

  Then robot remain safe despite all disturbances.

➢ **Funnel**

Sequence of invariant sets along the trajectory of the system.

Example:



- As along as, states of the disturbed closed-loop system remains inside the funnel, they will reach to the goal region.
- As long as, initial states are inside inlet if the funnel, they will reach to the goal region

- **Funnel**

Sequence of invariant sets along the trajectory of the system.

- In lecture 11, we looked at probabilistic funnels.
- In this lecture, we will look at robust funnels.

Topics:

- SOS optimization for Robust Control
- Lyapunov Stability and SOS optimization
- Barrier Function based Safety and SOS optimization
- Region of attraction Set Estimation and Design
- Invariant Set Estimation and Deign
- Funnel Based Robust Control
- Reachable Sets
- Constrained Volume Optimization

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  - They result SOS optimization involving bilinear constraints
  - We need Iterative algorithms.

- Constrained Volume Optimization →
  - It generates convex SOS optimization

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Sum-of Squares Constraints (review)

Nonnegativity Constrain $p(x) \geq 0, \quad \forall x \in \chi$

SOS Relaxation:

Let $\chi = \{x \in \mathbb{R}^n : g_{x\,i}(x) \geq 0, i = 1, ..., n_x\}$

$$p(x) = \sigma_0(x) + \textstyle\sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x) \qquad \sigma_i(x) \in SOS_{2d_i}, \; i = 0, ..., n_x$$

$$p(x) - \textstyle\sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x) \in SOS \qquad \sigma_i(x) \in SOS_{2d_i}, \; i = 1, ..., n_x$$

Yalmip: $\mathrm{SOS}(p(x) - \sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x))$ $\quad \mathrm{SOS}(\sigma_i(x)) \quad i = 1, ..., n_x$

Sum-of Squares Constraints (review)

Nonnegativity Constrain $p(x) \geq 0, \quad \forall x \in \chi = \{x \in \mathbb{R}^n : g_{x_i}(x) = 0, i = 1, \ldots, n_x\}$

Sum-of Squares Constraints (review)

Nonnegativity Constrain $p(x) \geq 0, \quad \forall x \in \chi = \{x \in \mathbb{R}^n : g_{x\,i}(x) = 0, i = 1, \ldots, n_x\}$

SOS Relaxation:

$$p(x) = \sigma_0(x) + \sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x) \qquad \sigma_0(x) \in SOS$$

$$p(x) - \sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x) \in SOS$$

Yalmip: $\mathrm{SOS}(p(x) - \sum_{i=1}^{n_x} \sigma_i(x) g_{x\,i}(x))$

Topics:

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Example: Robust Trajectory Optimization

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$ to derive the robot to the goal region in the presence of uncertainties.

$$\underset{u_k|_{k=0}^{N-1}}{\text{minimize}} \quad \sum_{k=0}^{N-1} u^2(k)$$

$$\text{subject to} \quad x_0 = x_0^*, x_N \in x_N^*$$

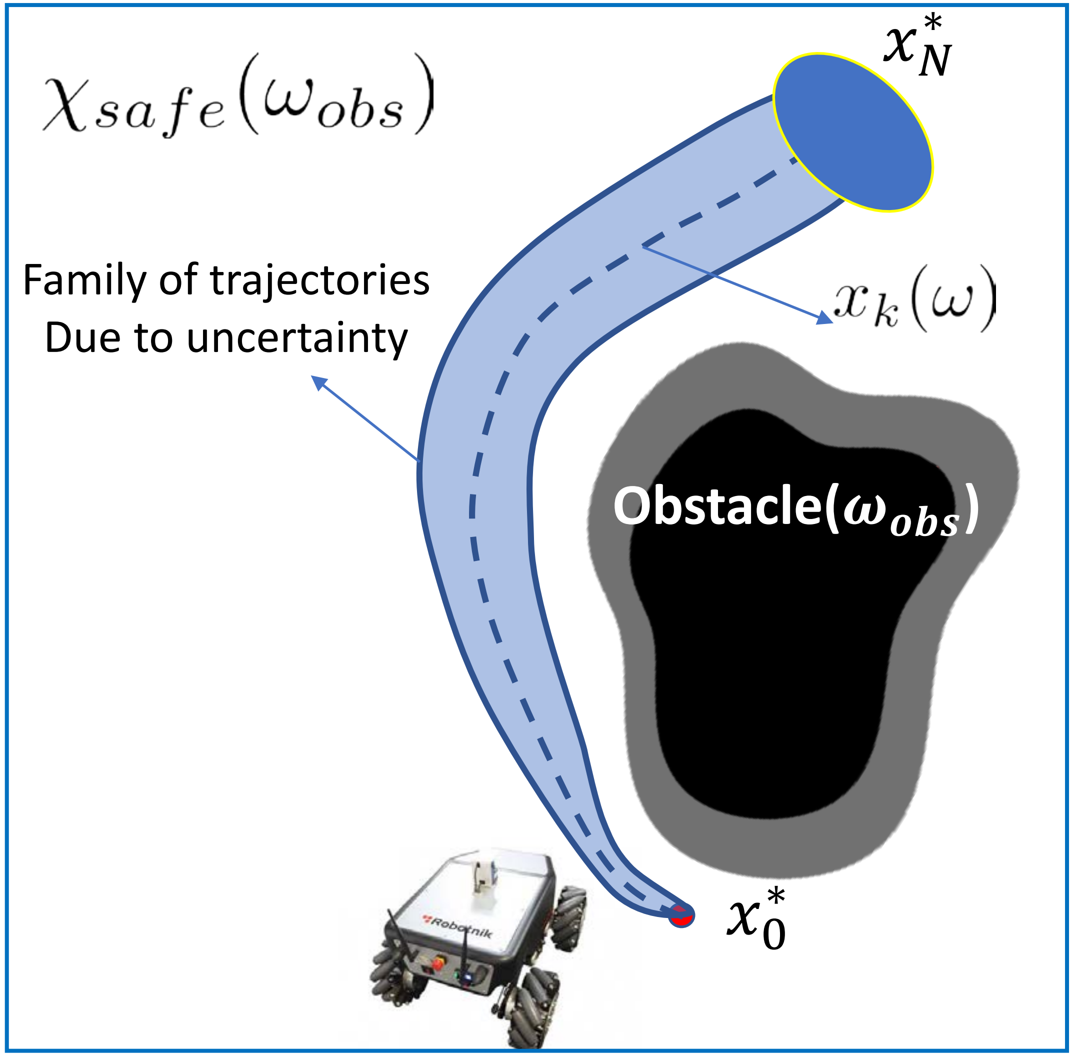
$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$$x_k \in \chi_{safe}(\omega_{obs})$$

$$\forall \omega_{x_k} \in \Omega_x, \ \forall \omega_{obs} \in \Omega_{obs}$$

$$u_k \in \mathcal{U}$$

Robust safety constrain



Inner Approximation of Robust Set

Robust Set: $\chi_R = \{x \in \chi : g(x,\omega) \leq 0, \ \forall \omega \in \Omega\}$

- x → Design parameter
- $g(x,\omega)$ → Safety Constraint
- ω → uncertainty

Inner approximation: $\chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$

SOS Program:

$$\begin{aligned} \underset{p_d(x)}{\text{minimize}} \quad & \int_{\mathbf{B}} p_d(x)dx \\ \text{subject to} \quad & p_d(x) - g(x,\omega) \geq 0 \ \ \forall x \in \chi, \ \forall \omega \in \Omega \end{aligned}$$

→ SOS Constraints

- We can obtain the set of control inputs that satisfies the safety constraints for all valise of uncertainty.

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $(x_1(0), x_2(0), x_3(0)) \in X_0$

Uncertain Parameter $\omega(k) \in \Omega_k = [\underline{\omega_k}, \overline{\omega_k}]$

- Source of uncertainty at time k:

  Coupled uncertainty $(x_1(k), x_2(k), x_3(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, x_3, \omega) = 0.1^2 - x_1^2 - x_2^2 - x_3^2 - \omega^2 \geq 0\}$

- We want to find a set of control inputs at time k that steer states $(x_1(k+1), x_2(k+1), x_3(k+1))$ to the neighborhood of the given way-point $(0,0,0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1-0}{0.2}\right)^2 - \left(\frac{x_2-0}{0.2}\right)^2 - \left(\frac{x_3-0.9}{0.4}\right)^3 \geq 0$, for all possible values of uncertainty $(x_1(k), x_2(k), x_3(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, x_3, \omega) = 0.1^2 - x_1^2 - x_2^2 - x_3^2 - \omega^2 \geq 0\}$.

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

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- Source of uncertainty at time k:

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$$U_R = \left\{ u(k) : \; 1 - \left(\frac{x_1(k+1)}{0.2}\right)^2 - \left(\frac{x_2(k+1)}{0.2}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0, \; \forall x_k, \omega_k \in \Omega_x \right\}$$

Example: Control of Uncertain Nonlinear System

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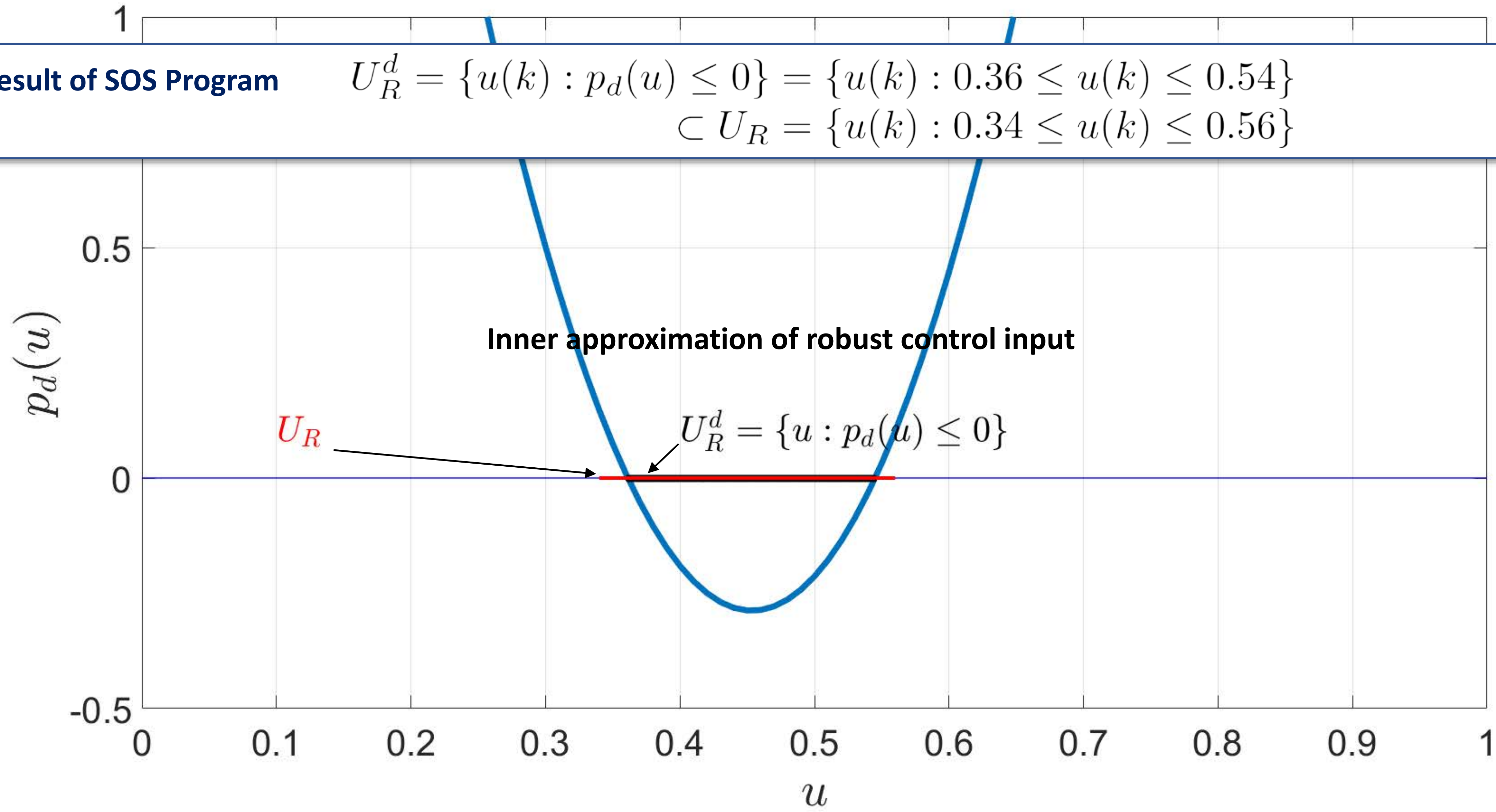
$$U_R = \left\{ u(k) : \ 1 - \left(\frac{\omega(k)x_2(k)}{0.2}\right)^2 - \left(\frac{x_1(k)x_3(k)}{0.2}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k)}{0.4}\right)^2 \geq 0, \ \forall x_k, \omega_k \in \Omega_x \right\}$$

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_Set/Example_2_RobustSet_DynamicalSys.m

Example: Uncertain Nonlinear System

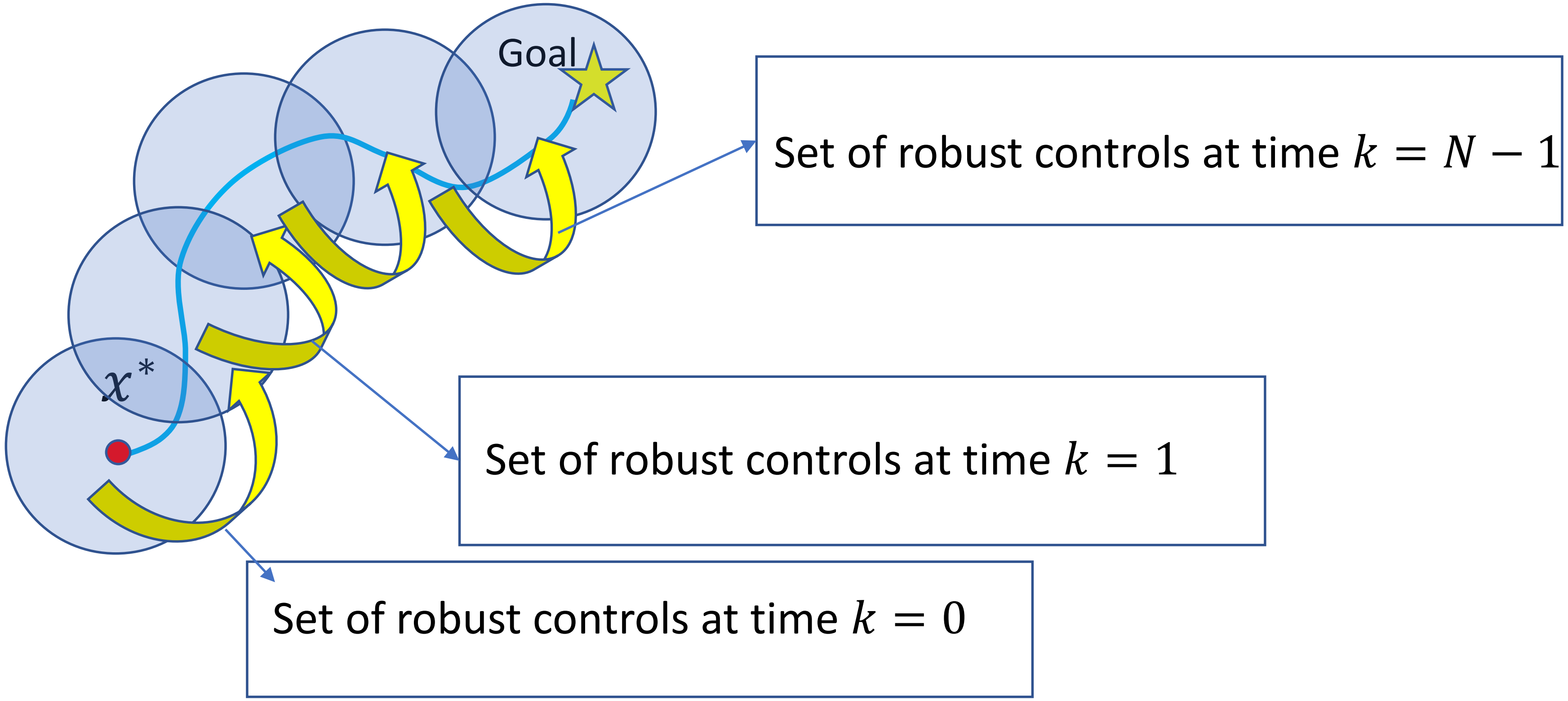
Result of SOS Program

$$U_R^d = \{u(k) : p_d(u) \leq 0\} = \{u(k) : 0.36 \leq u(k) \leq 0.54\}$$
$$\subset U_R = \{u(k) : 0.34 \leq u(k) \leq 0.56\}$$



- **Any control input that respects the obtained control bound, is robust in presence of uncertainties.**

- Hence, given a nominal trajectory for uncertain dynamical system, at each time k, we can find the set of control inputs that keeps the disturbed states in the flow-tube.



Robust Flow-tube

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Lyapunov based Stability and Control

- Lyapunov Stability
- SOS Lyapunov Conditions for stability of Systems
- SOS Lyapunov Conditions for stability of Uncertain systems

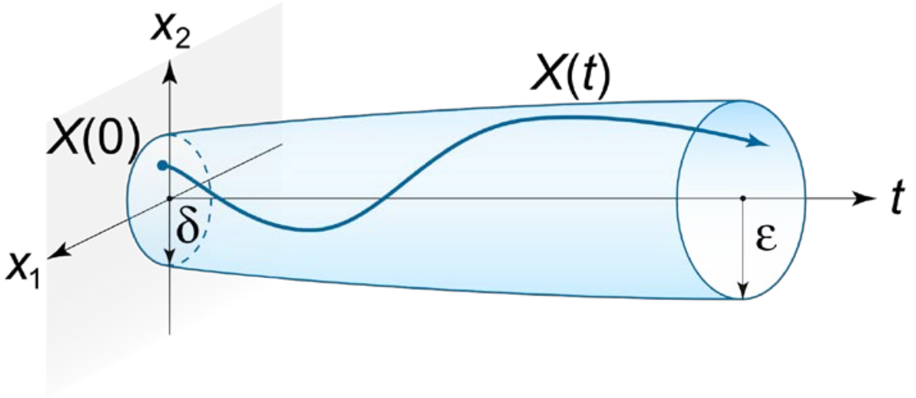
Lyapunov Stability

- Dynamical System: $\dot{x} = f(x)$ $\quad x \in \chi \quad f(0) = 0$

- Equilibrium point: x if $\dot{x} = 0$ (e.g., Origin x=0)

- **Lyapunov Stability:** The equilibrium point x_e is stable if for every $\epsilon > 0$ there exist a $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$ then $\|x(t) - x_e\| < \epsilon$

If the initial state is close to the equilibrium point, states remain close to the equilibrium point forever.



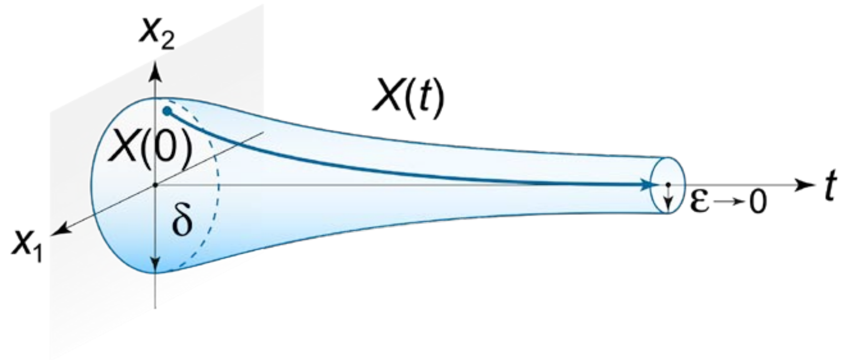
Lyapunov Stability

- ➢ Dynamical System: $\dot{x} = f(x)$ $x \in \chi$ $f(0) = 0$

- Equilibrium point: x if $\dot{x} = 0$ (e.g., Origin x=0)

➢ **Asymptotic Stability:** The equilibrium point x_e is asymptotically stable if it is **Lyapunov stable** and exist $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$ then $\mathrm{Lim}_{t\to\infty}\|x(t) - x_e\| = 0$

If the initial state is close to the equilibrium point, states not only remain close but also eventually converge to the equilibrium point .



Lyapunov Function

- Dynamical System: $\dot{x} = f(x)$ $x \in \chi$ $f(0) = 0$
  - Equilibrium point: x if $\dot{x} = 0$ (e.g., Origin x=0)

- The equilibrium point x=0 is **asymptotically stable** if there exist an energy function $V(x)$:

Lyapunov Function (Energy function):

$$V(x) = 0 \quad on \quad x = 0$$

$$V(x) > 0 \quad \forall x \neq 0$$

$$\dot{V}(x) < 0 \quad \forall x \neq 0$$ [1] ⟹

1: Stability Cond: $\dot{V}(x) \leq 0 \;\; \forall x \neq 0$

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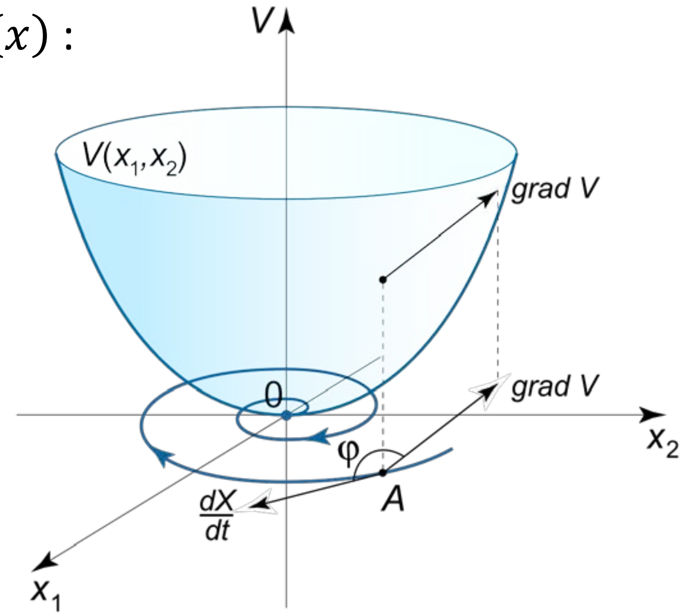

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Lyapunov Function

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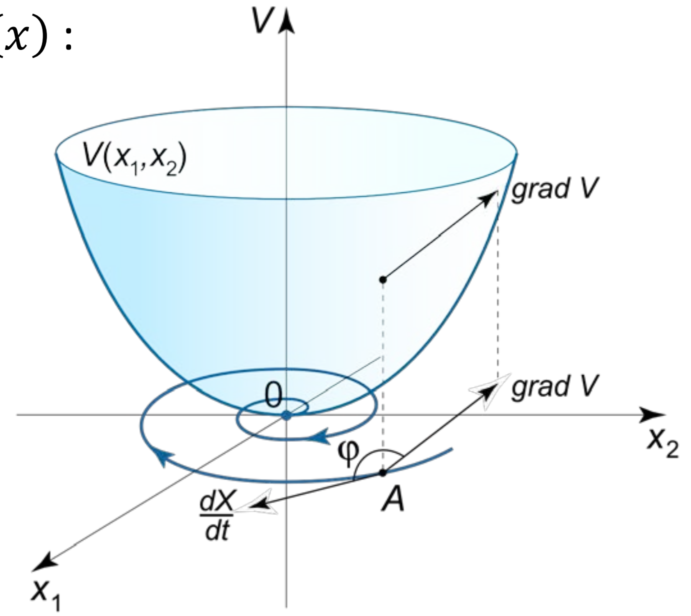
- Equilibrium point: x if $\dot{x} = 0$ (e.g., Origin x=0)

- ➢ The equilibrium point x=0 is **asymptotically stable** if there exist an energy function $V(x)$:



Lyapunov Function (Energy function):

$$V(x) = 0 \quad on \quad x = 0$$

$$V(x) > 0 \quad \forall x \neq 0$$

$$\dot{V}(x) < 0 \quad \forall x \neq 0 \quad \Longrightarrow$$

- $V(x)$ is decreasing along the trajectories of the system
- $\dot{V}(x) = \frac{\partial V(x)}{\partial x} \dot{x} = \frac{\partial V(x)}{\partial x} f(x) < 0 \Longrightarrow \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle < 0$
- The angle between the gradient vector $\frac{\partial V(x)}{\partial x}$ and the velocity vector $f(x)$ is greater than 90^o

Lyapunov Function

- Dynamical System: $\dot{x} = f(x)$ $x \in \chi$ $f(0) = 0$

- We look for polynomial $V(x) = \sum_{i=0}^{d} c_0 x^i$ using SOS program:

Lyapunov Function:

$V(x) = 0 \quad on \quad x = 0$ → $V(x)$: polynomial with no constant term, i.e., $c_0 = 0$

$V(x) > 0 \;\; \forall x \neq 0$ → $V(x) \geq \epsilon \|x\|_2^2$ $\epsilon > 0$ →

$\dot{V}(x) < 0 \;\; \forall x \neq 0$ → $-\dot{V}(x) \geq \epsilon \|x\|_2^2$ $\epsilon > 0$ →

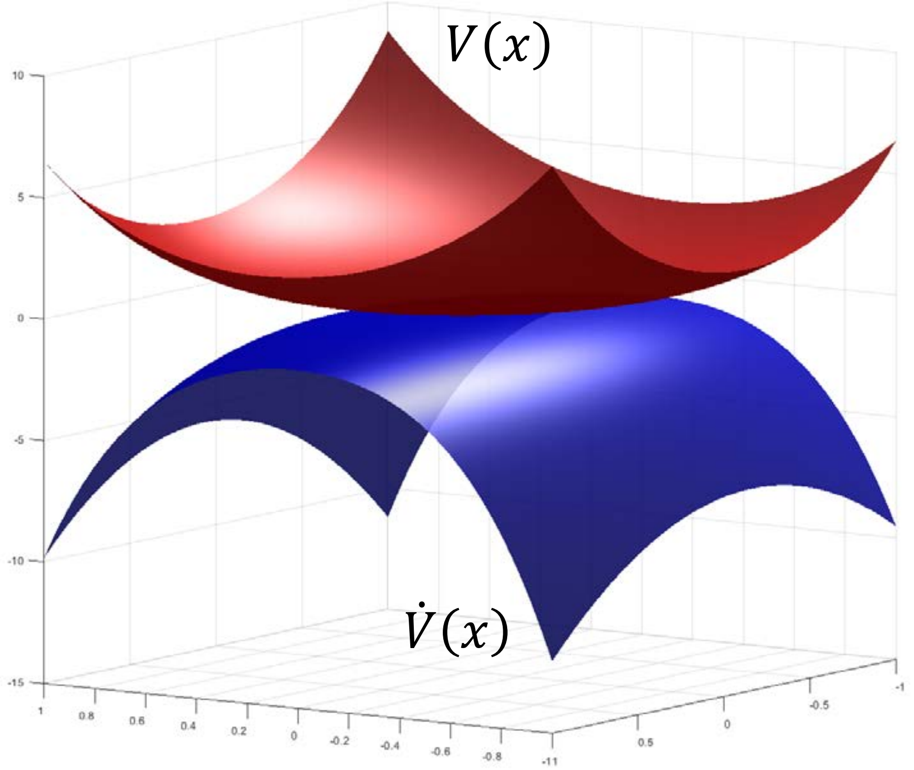
SOS Conditions:

$$V(x) = \sum_{i=0}^{d} c_0 x^i, c_0 = 0$$

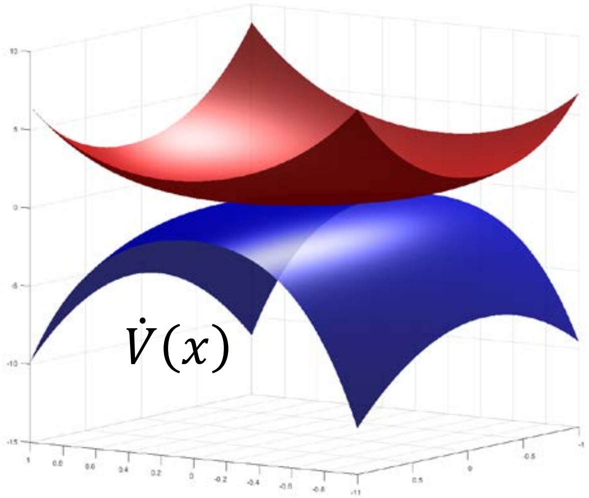
$$V(x) - \epsilon \|x\|_2^2 \in SOS$$

$$-\dot{V}(x) - \epsilon \|x\|_2^2 \in SOS$$

$$\dot{x}_1 = -x_1 + (1 + x_1)x_2$$
$$\dot{x}_2 = -(1 + x_1)x_1$$

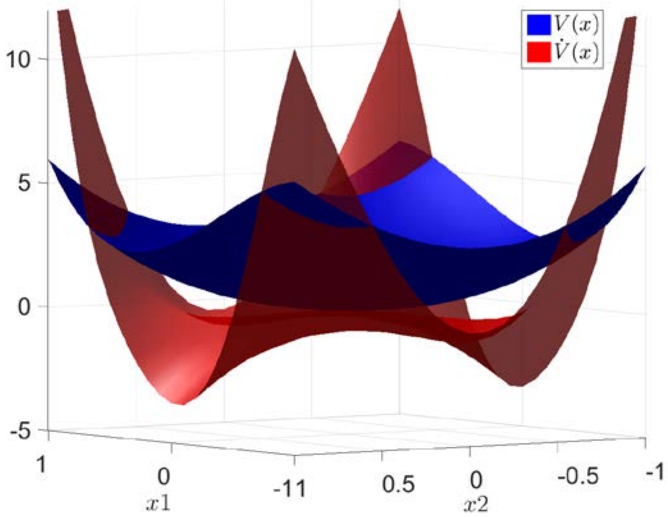


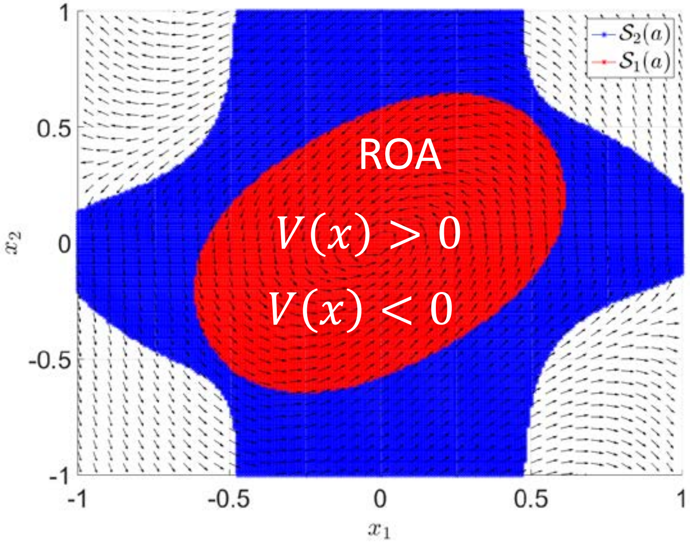
- In the provided formulation, we look for **global stability**, i.e., $\dot{V}(x) < 0 \;\; \forall x$



$V(x) > 0, \dot{V}(x) < 0$

- Instead, we can look for **local stability.** (i.e., Region of Attraction (ROA) set)





Lyapunov Function Under Uncertainty

- Uncertain Dynamical System: $\dot{x} = f(x,\omega)$ $x \in \chi$ $\omega \in \Omega = \{\omega: g_{\omega_i}(\omega) \geq 0, i = 1, \ldots, n_\omega\}$ $f(0,0) = 0$
- We look for polynomial $V(x) = \sum_{i=0}^{d} c_0 x^i$ using SOS program:

Uncertain system $\dot{x} = f(x,\omega)$ is stable for $\omega \in \Omega$, system if

Lyapunov Function:

$$V(x) = 0 \quad on \quad x = 0$$

$$V(x) > 0 \quad \forall x \neq 0$$

$$\dot{V}(x,\omega) < 0 \quad \forall x \neq 0 \quad \forall \omega \in \Omega$$

$$\left(\dot{V}(x,\omega) = \frac{\partial V(x)}{\partial x} \dot{x} = \frac{\partial V(x)}{\partial x} f(x,\omega) \right)$$

$V(x)$: polynomial with no constant term

$$V(x) \geq \epsilon \|x\|_2^2 \quad \epsilon > 0$$

$$-\dot{V}(x,\omega) \geq \epsilon \|x\|_2^2 \quad \epsilon > 0$$

SOS Conditions:

$$V(x) = \sum_{i=0}^{d} c_0 x^i, c_0 = 0$$

$$V(x) - \epsilon \|x\|_2^2 \in SOS$$

$$-\dot{V}(x,\omega) - \epsilon \|x\|_2^2 - \sum_{i=1}^{n_\omega} \sigma_i(x,\omega) g_{\omega_i}(\omega) \in SOS$$

$$\sigma_i(x,\omega) \in SOS$$

Example: Stability Analysis of Uncertain Nonlinear Systems

Uncertain Nonlinear Dynamical System:

$$\dot{x}_1 = -\frac{3}{2}x_1^3 - \frac{1}{2}x_1^2 - x_2$$

$$\dot{x}_2 = 6x_1 - \omega x_2$$

- Uncertain Parameter: $\omega \in [3 \quad 5]$

$$\Omega = \{\omega : (\omega - 3)(5 - \omega) \geq 0\}$$

➢ We look for polynomial Lyapunov function $V(x) = \sum_{\alpha=0}^{2d} c_\alpha x^\alpha$

Stability Conditions:

$V(x) = 0 \quad on \quad x = 0$ → $c_0 = 0$

$\epsilon > 0$

$V(x) > 0 \quad \forall x \neq 0$ → $V(x) \geq \epsilon \|x\|_2^2$ → $V(x) - \epsilon||x||_2^2 \in SOS$

$-\dot{V}(x,\omega) > 0 \quad \forall x \neq 0 \quad \forall \omega \in \Omega$ → $-\dot{V}(x,\omega) \geq \epsilon \|x\|_2^2$ → $-\dot{V}(x) - \epsilon||x||_2^2 - \sigma_1(x,\omega)(\omega - 3)(5 - \omega) \in SOS$

SOS Conditions:

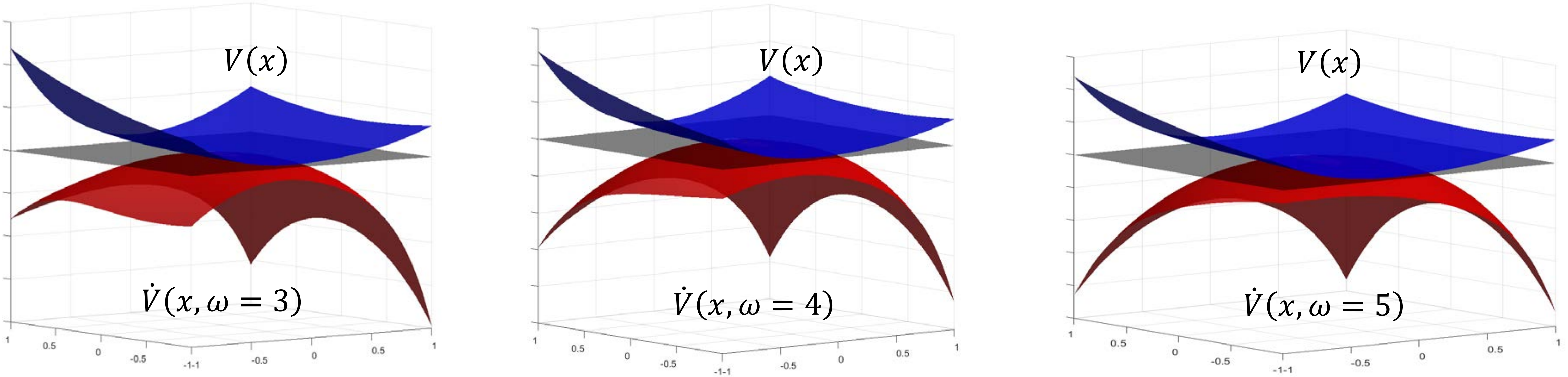
$c_0 = 0$

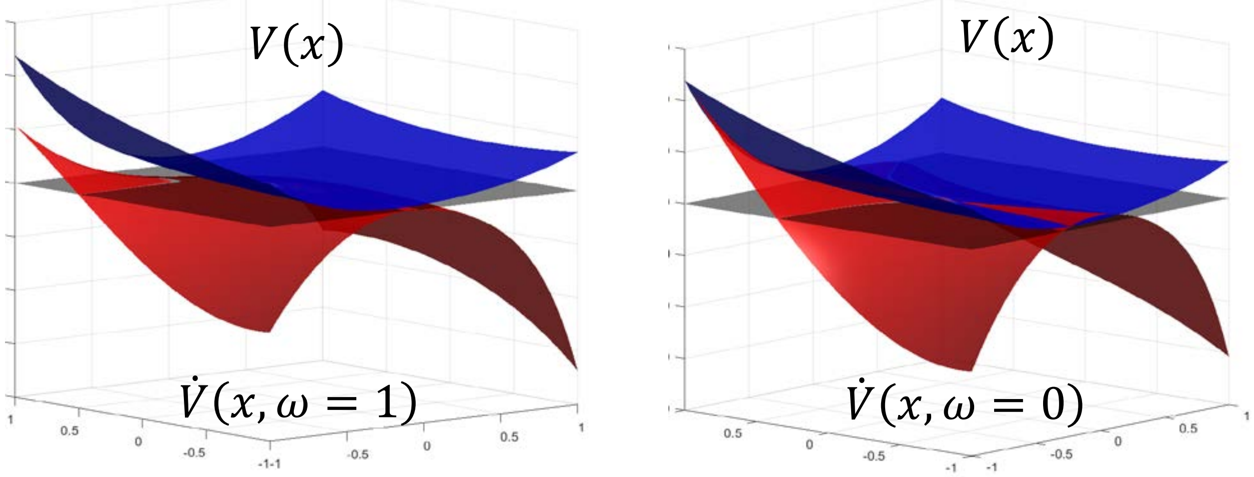
$V(x) - \epsilon||x||_2^2 \in SOS$

$-\dot{V}(x) - \epsilon||x||_2^2 - \sigma_1(x,\omega)(\omega - 3)(5 - \omega) \in SOS$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 = \frac{\partial V(x)}{\partial x_1}\left(-\frac{3}{2}x_1^3 - \frac{1}{2}x_1^2 - x_2\right) + \frac{\partial V(x)}{\partial x_2}(6x_1 - \omega x_2)$$

- Uncertain Nonlinear system is stable for Uncertain Parameter: $\omega \in [3 \quad 5]$





- For : $\omega \notin [3 \quad 5]$
  $V(x)$ does not satisfy the stability conditions.

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_SOS/Example_1_Lyapunov.m

Based on : J. Lofberg "Modeling and solving uncertain optimization problems in YALMIP" 17th World Congress, The International Federation of Automatic Control, Seoul, Korea, July 6-11, 2008

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_SOS/Example_2_Lyapunov.m

Existence of SOS Lyapunov function:

- A. A. Ahmadi, P. A. Parrilo,"Sum of Squares Certificates for Stability of Planar, Homogeneous, and Switched Systems", IEEE Transactions on Automatic Control, Volume: 62 , Issue: 10 , 2017.

Convex Lyapunov function:

- A. A. Ahmadi, Raphael M. Jungers,"SOS-Convex Lyapunov Functions and Stability of Difference Inclusions", 2018 https://arxiv.org/pdf/1803.02070.pdf

- A. A. Ahmadi, R. M. Jungers, "SOS-Convex Lyapunov Functions with Applications to Nonlinear Switched Systems", Conference on Decision and Control (CDC), 2013

Stability of large-scale nonlinear systems

- S. Shen, R. Tedrake" Compositional verification of large-scale nonlinear systems via sums-of-squares optimization", In Proceedings of the American Control Conference (ACC), USA, 2018

Application in Robotics

- M. Posa, M. Tobenkin, and R. Tedrake, "Lyapunov analysis of rigid body systems with impacts and friction via sums-of-squares", In Proceedings of the 16th International Conference on Hybrid Systems: Computation and Control (HSCC 2013), 2013

Topics:

- SOS optimization for Robust Control
- Lyapunov Stability and SOS optimization
- Barrier Function based Safety and SOS optimization
- Region of attraction Set Estimation and Design
- Invariant Set Estimation and Deign
- Funnel Based Robust Control
- Reachable Sets
- Constrained Volume Optimization

Barrier Function Based Safety

Example: Safety Verification

- Uncertain nonlinear dynamical system $\dot{x} = f(x, \omega) \quad x \in \chi$
- Bounded Uncertainty $\omega \in \Omega$
- Unsafe Set χ_{obs}
- Initial state $x(0) \in \chi_0$

Example: Safety Verification

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- Bounded Uncertainty $\omega \in \Omega$
- Unsafe Set χ_{obs}
- Initial state $x(0) \in \chi_0$

➢ Uncertain system is safe if function $B(x)$ satisfies: (Barrier function)

$B(x) \leq 0 \;\; \forall x \in \chi_0$ $\quad B(x) > 0 \;\; \forall x \in \chi_{obs}$[1] $\quad \dot{B}(x, \omega) = \frac{\partial B(x)}{\partial x} f(x, \omega) \leq 0 \;\; \forall x \in \chi, \quad \forall \omega \in \Omega$

1 $B(x) \geq 1 \;\; \forall x \in \chi_{obs}$

Example: Safety Verification

- Uncertain nonlinear dynamical system $\dot{x} = f(x, \omega) \quad x \in \chi$
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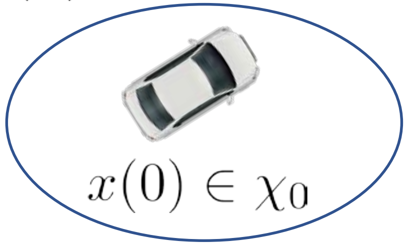
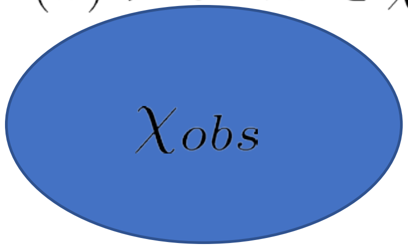
χ

$B(x) \leq 0 \quad \forall x \in \chi_0$



To reach to the obstacle
$B(x)$ should be increasing.

$B(x) > 0 \quad \forall x \in \chi_{obs}$



Example: Safety Verification

$x \in \chi = \{x : g_{x_i}(x) \geq 0, i = 1, ..., n_x\}$ $\quad \omega \in \Omega = \{\omega : g_{\omega_i}(\omega) \geq 0, i = 1, ..., n_\omega\}$

$\chi_{obs} = \{x : g_{obs_i}(x) \geq 0, i = 1, ..., n_{obs}\}$ $\quad \chi_0 = \{x : g_{0_i}(x) \geq 0, i = 1, ..., n_0\}$

- Polynomial barrier function: $B(x) = \sum_{i=0}^{d} c_0 x^i$

SOS Conditions:

$B(x) \leq 0 \quad \forall x \in \chi_0 \quad \Rightarrow \quad -B(x) - \sum_{i=1}^{n_0} \sigma_{0_i}(x) g_{0_i}(x) \in SOS \qquad \sigma_{0_i}(x) \in SOS \;\; i = 1, \ldots, n_0$

$B(x) > 0 \quad \forall x \in \chi_{obs} \quad \Rightarrow \quad B(x) - \sum_{i=1}^{n_{obs}} \sigma_{obs_i}(x) g_{obs_i}(x) \in SOS \qquad \sigma_{obs_i}(x) \in SOS \;\; i = 1, \ldots, n_{obs}$

$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \leq 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega \quad \Rightarrow \quad -\frac{\partial B(x)}{\partial x} f(x,\omega) - \sum_{i=1}^{n_x} \sigma_{x_i}(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \sigma_{\omega_i}(x,\omega) g_{\omega_i}(\omega) \in SOS$

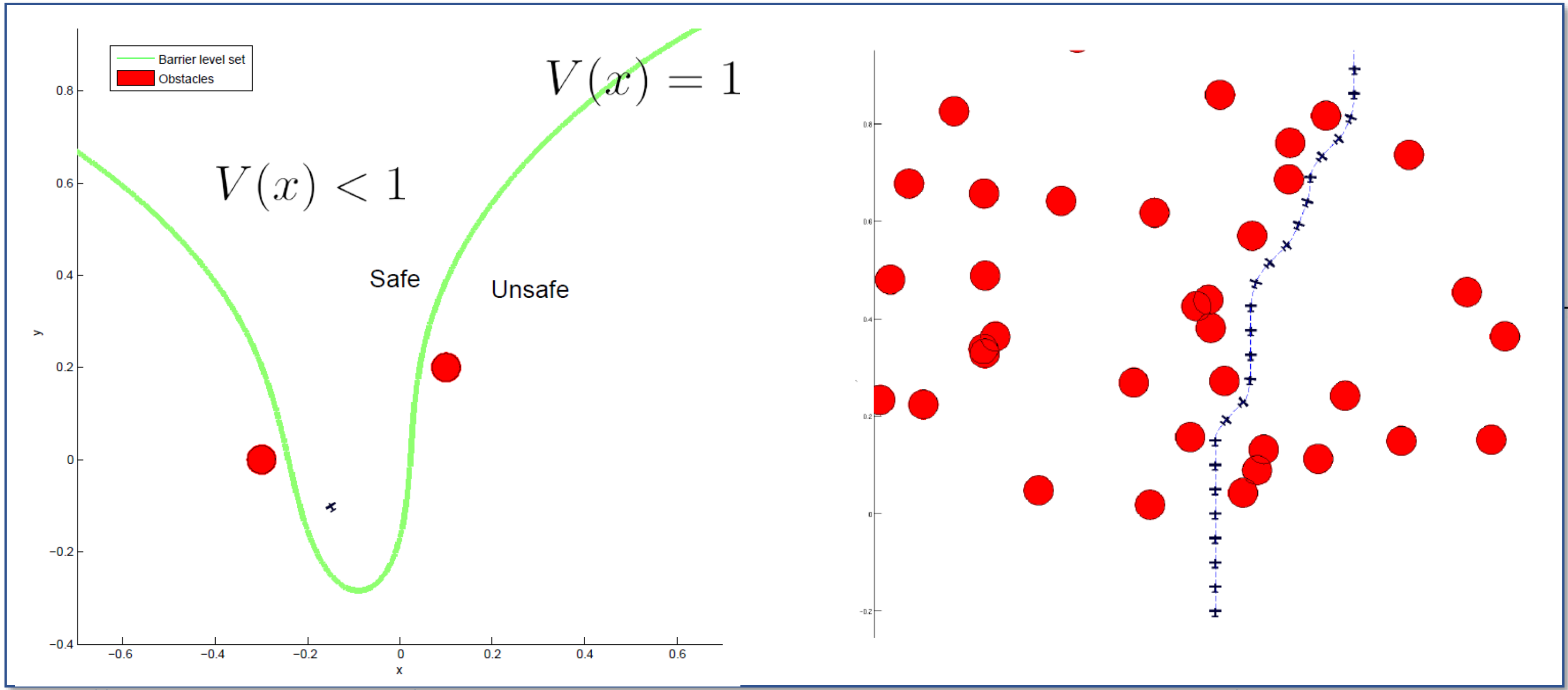
$\sigma_{x_i}(x,\omega) \in SOS \;\; i = 1, \ldots, n_x$

$\sigma_{\omega_i}(x,\omega) \in SOS \;\; i = 1, \ldots, n_\omega$

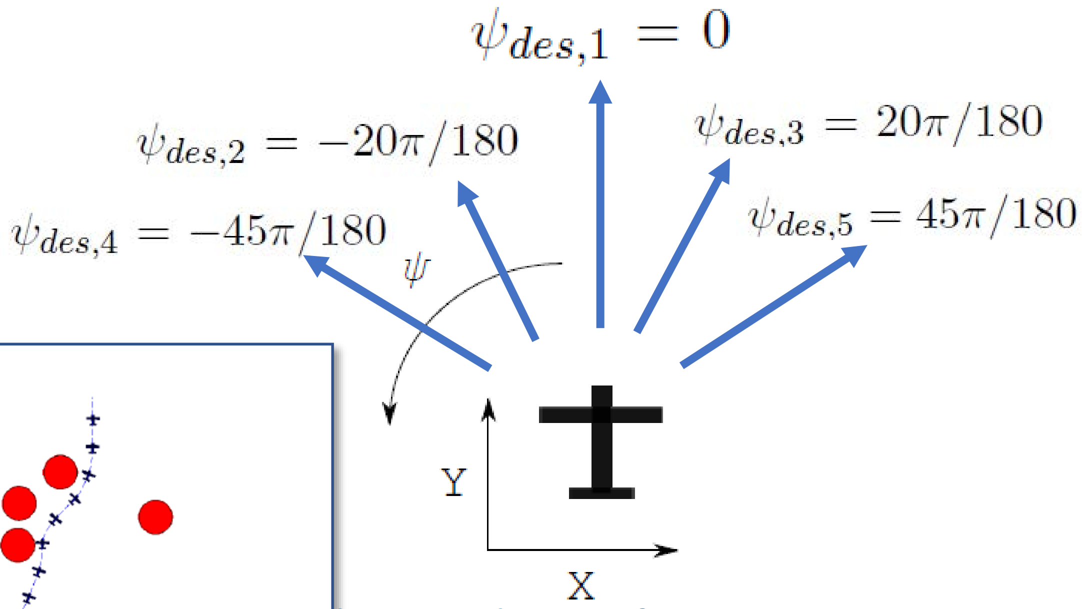
- Uncertain nonlinear dynamical system

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}, \qquad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -v \sin\psi + w \\ v \cos\psi \\ u \end{bmatrix}$$

- Wind Disturbance: $\omega \in [-0.05 \;\; 0.05]$



- Library of motion primitives



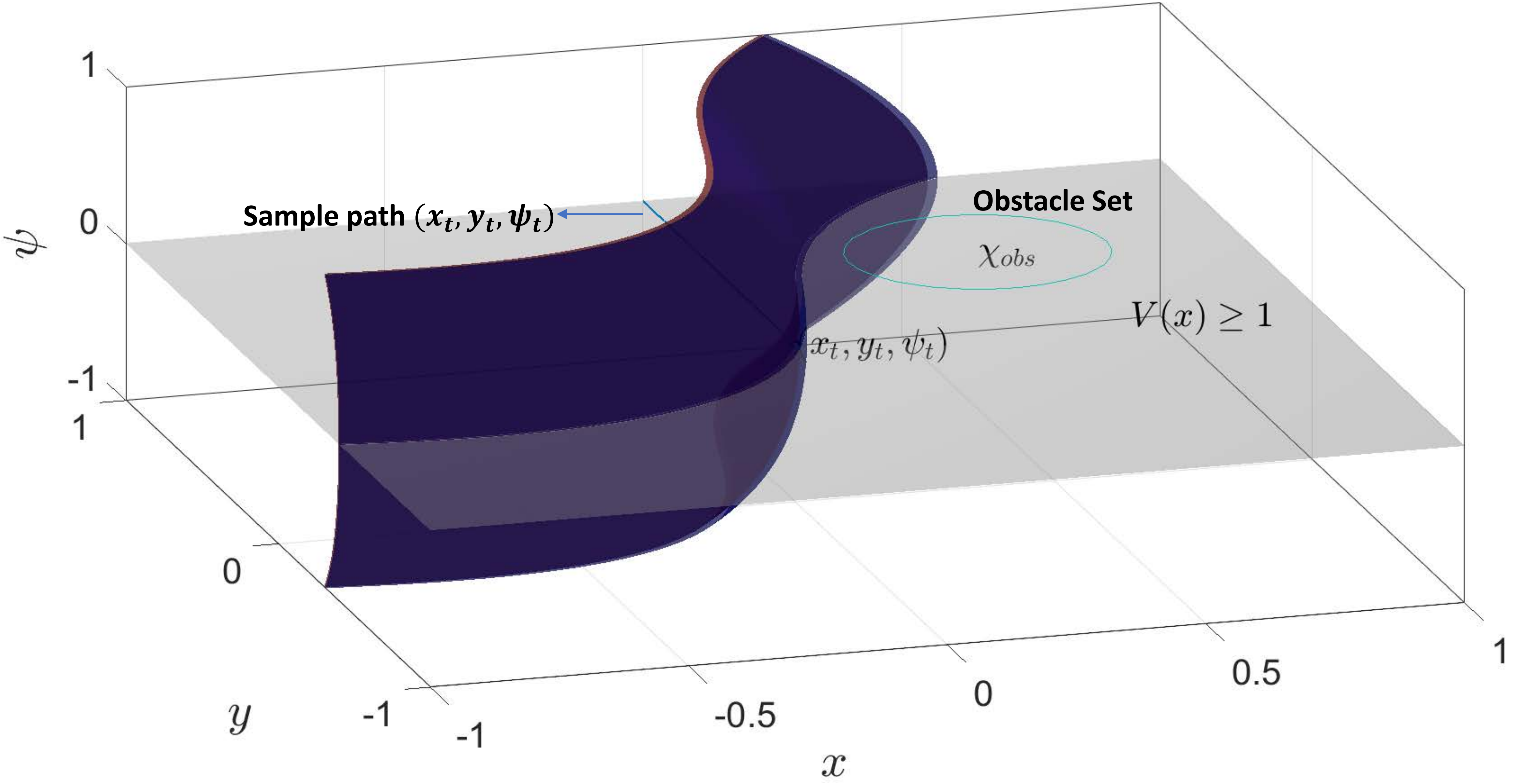
k control input for motion primitives:

$-K(\psi - \psi_{des,i}), \; i = 1, \ldots, 5.$

https://www.quantamagazine.org/a-classical-math-problem-gets-pulled-into-the-modern-world-20180523/

A. Ahmadi, A. Majumdary, “Some applications of polynomial optimization in operations research and real-time decision making”, Optimization Letters, Volume 10, Issue 4, pp 709–729, 2016.

Barrier Function



https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_SOS/Example_3_Barrier_SafetyVerification.m

- The conditions might be conservative as the derivative inequality needs to be satisfied on the whole state set χ

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \leq 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega$$

- The conditions might be conservative as the derivative inequality needs to be satisfied on the whole state set χ

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \le 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega$$

- For safety, $B(x)$ should be decreasing only on and near the set of $x \in \chi$ for which $B(x) = 0$

χ

$B(x) \le 0 \quad \forall x \in \chi_0$

$x(0) \in \chi_0$

- Condition to reach to the obstacle: $B(x) > 0$.
- Boundary of unsafe region: $B(x) = 0$
- At initial states $B(x) \le 0$

$B(x) > 0 \quad \forall x \in \chi_{obs}$

χ_{obs}

Safety Conditions:

$$B(x) \le 0 \quad \forall x \in \chi_0 \qquad B(x) > 0 \quad \forall x \in \chi_{obs}$$

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \le 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega \quad \text{s.t } B(x) = 0$$

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \leq 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega$$

SOS Constraints:

$$-\frac{\partial B(x)}{\partial x} f(x,\omega) - \sum_{i=1}^{n_x} \sigma_{x_i}(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \sigma_{\omega_i}(x,\omega) g_{\omega_i}(\omega) \in SOS$$

$$\sigma_{\omega_i}(x,\omega) \in SOS \;\; i = 1, \dots, n_\omega$$

$$\sigma_{x_i}(x,\omega) \in SOS \;\; i = 1, \dots, n_x$$

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \leq 0 \quad \forall x \in \chi, \quad \forall \omega \in \Omega \;\; \text{s.t} \;\; B(x) = 0$$

SOS Constraints:

$$-\frac{\partial B(x)}{\partial x} f(x,\omega) - \sigma_B(x,\omega) B(x) - \sum_{i=1}^{n_\omega} \sigma_{\omega_i}(x,\omega) g_{\omega_i}(\omega) \in SOS \qquad \sigma_{\omega_i}(x,\omega) \in SOS \;\; i = 1, \dots, n_\omega$$

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \le 0 \ \ \forall x \in \chi, \quad \forall \omega \in \Omega$$

SOS (Convex Constraint):

$$-\frac{\partial B(x)}{\partial x} f(x,\omega) - \sum_{i=1}^{n_x} \sigma_{x_i}(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \sigma_{\omega_i}(x,\omega) g_{\omega_i}(\omega) \in SOS$$

$$\sigma_{\omega_i}(x,\omega) \in SOS \ \ i = 1, \dots, n_\omega$$

$$\sigma_{x_i}(x,\omega) \in SOS \ \ i = 1, \dots, n_x$$

$$\dot{B}(x,\omega) = \frac{\partial B(x)}{\partial x} f(x,\omega) \le 0 \ \ \forall x \in \chi, \quad \forall \omega \in \Omega \ \ \text{s.t} \ B(x) = 0$$

SOS (bilinear):

$$-\frac{\partial B(x)}{\partial x} f(x,\omega) - \sigma_B(x,\omega) B(x) - \sum_{i=1}^{n_\omega} \sigma_{\omega_i}(x,\omega) g_{\omega_i}(\omega) \in SOS \qquad \sigma_{\omega_i}(x,\omega) \in SOS \ \ i = 1, \dots, n_\omega$$

Multiplication of 2 unknown polynomial

- We need Iterative algorithm. e.g.,
- Fix $\sigma_B(x,\omega)$, solve the SOS program with respect to $B(x)$ and $\sigma_{\omega_i}(x,\omega)$
- Fix $B(x)$, solve the SOS program with respect to $\sigma_B(x,\omega)$, and $\sigma_{\omega_i}(x,\omega)$

Safety of Uncertain Hybrid dynamical system using barrier function:

- S. Prajna and A. Jadbabaie, “Safety verification of hybrid systems using barrier certificates,” in *Hybrid Systems: Computation and Control*. Heidelberg: Springer-Verlag, 2004.

Barrier Function Based Control:

- A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, “Control Barrier Functions: Theory and Applications” 18th European Control Conference (ECC), 2019

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Region of Attraction Set

- Estimation
- Design

Region of Attraction Set Estimation

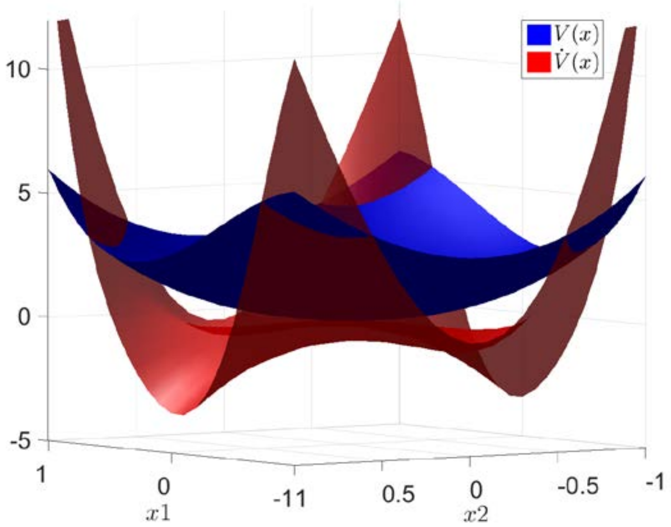
Dynamical System: $\dot{x} = f(x)$

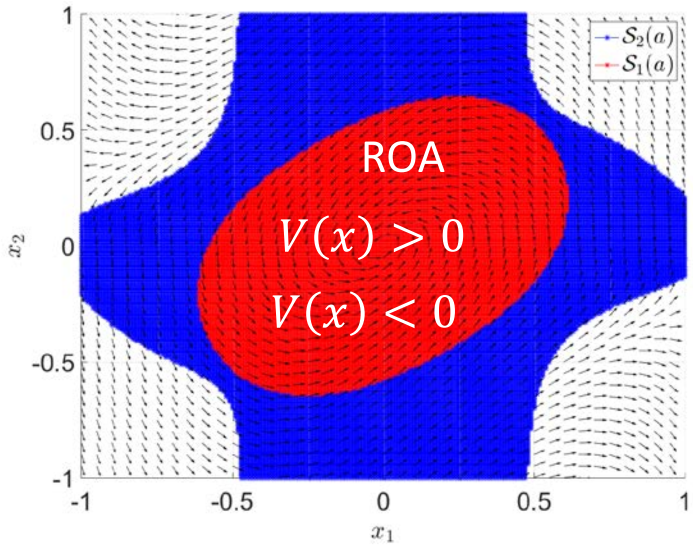
Region of attraction (ROA) Set: Largest set of all initial states whose trajectories converge to the origin

- Let $V(x) > 0, V(0) = 0,$

➢ The level set $\chi_{ROA} = \{x: V(x) \leq \rho\}$ is an **inner** approximation of ROA if

$$\dot{V}(x) < 0 \qquad \forall x \in \chi_{ROA}$$





H. K. Khalil, “Nonlinear systems, 3rd,” New Jewsey, Prentice Hall, vol. 9, 2002.

Dynamical System: $\dot{x} = f(x)$

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$$\dot{V}(x) < 0 \qquad \forall x \in \chi_{ROA}$$

Conditions:

$$V(0) = 0$$

$$V(x) > 0$$

$$\dot{V}(x) < 0 \quad \forall x \in \{x: V(x) \leq \rho\}$$

SOS Conditions:

$$V(x) = \sum_{i=0}^{d} c_0 x^i, c_0 = 0$$

$$V(x) \in SOS$$

$$-\frac{\partial V(x)}{\partial x} - L(x)(\rho - V(x)) \in SOS \qquad L(x) \in SOS$$

SOS Program:

$$\chi_{ROA} = \{x : V(x) \leq \rho\}$$

$$\max_{V(x), L(x), \rho} \rho$$

Subject to $V(x) = \sum_{i=0}^{d} c_0 x^i, c_0 = 0$

$V(x) \in SOS$

$-\dot{V}(x) + L(x)(V(x) - \rho) \in SOS$ $\qquad$ $L(x) \in SOS$

$V\left(\sum_j e_j\right) = 1$ $\qquad$ e_j: j-th standard basis vector for the state space

(Sum of coefficients of $V(x)$)

- We aim to find the largest ROA set by maximizing ρ subject to a normalization constraint on $V(x)$.
- If one does not normalize $V(x)$, ρ can be made arbitrarily large simply by scaling the coefficients of $V(x)$.

- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

SOS Program:

$$\max_{V(x), L(x), \rho} \rho$$

Subject to $V(x) = \sum_{i=0}^{d} c_0 x^i \, , c_0 = 0$

$V(x) \in SOS$

$-\dot{V}(x) + L(x)(V(x) - \rho) \in SOS \qquad L(x) \in SOS$

$V\left(\sum_j e_j\right) = 1 \qquad e_j$: j-th standard basis vector for the state space

(Sum of coefficients of $V(x)$)

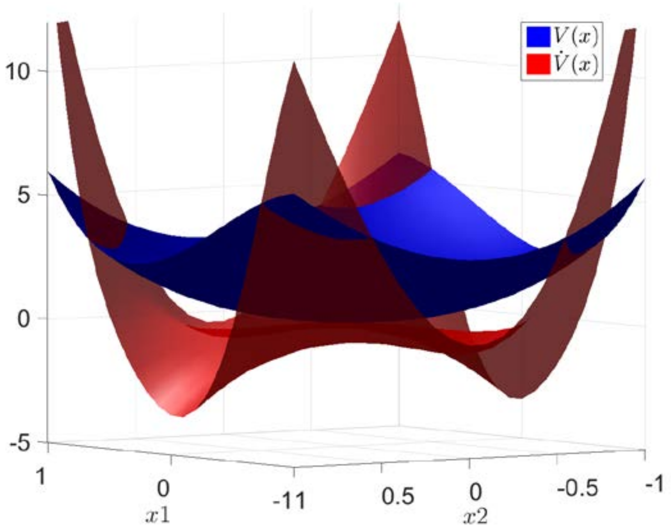
- $-\dot{V}(x) + L(x)(V(x) - \rho) \in SOS$ is bilinear constraints.
- It contains multiplication of unknowns $L(x)\, V(x)$ and $L(x)\, \rho$.

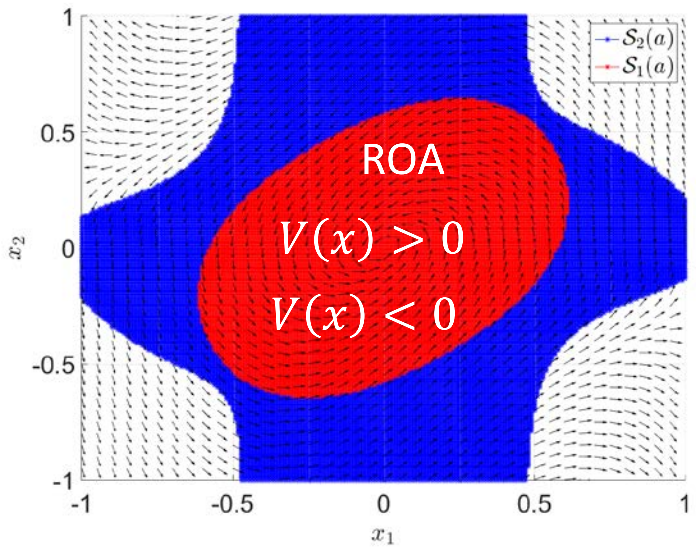
➢ We need Iterative algorithm. e.g.,

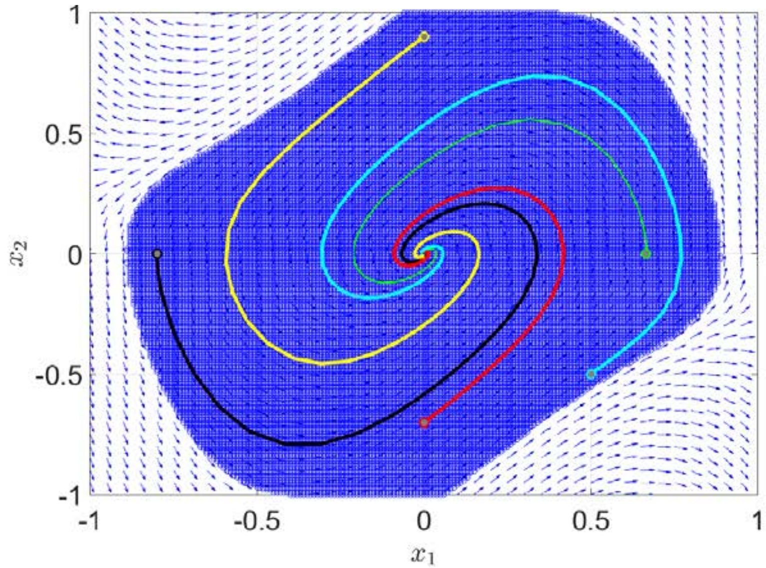
- Fix $L(x)$, solve the SOS program with respect to $V(x)$ and ρ
- Fix $V(x)$, solve the SOS program with respect to $L(x)$ and Binary search over ρ in order to maximize it.

- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (4x_1^2 - 1)x_2$$







True ROA set

Region of Attraction Set Design

Control affine system $\dot{x} = f(x) + g(x)u$

- Find a controller $u(x)$ that stabilizes the system to a fixed point (e.g., $x = 0$) and that produce the “largest” region of attraction.

- $u(x)$: polynomials in the state variables.

- ROA Conditions:

Inner approximation $\chi_{ROA} = \{x: V(x) \leq \rho\}$

$V(0) = 0$

$V(x) > 0$

$\dot{V}(x,u) < 0 \quad \forall x \in \{x: V(x) \leq \rho\}$

where $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}\big(f(x) + g(x)u(x)\big) = \frac{\partial V}{\partial x} f_{cl}(x, u(x))$

SOS program:

$$\begin{aligned} \underset{\rho, L(x), V(x), u(x)}{\text{maximize}} \quad & \rho \\ \text{subject to} \quad & V(x) \text{ SOS} \\ & -\dot{V}(x) + L(x)(V(x) - \rho) \text{ SOS} \\ & L(x) \text{ SOS} \\ & V(\sum_j e_j) = 1 \end{aligned}$$

- $-\overbrace{\frac{\partial V}{\partial x}\big(f(x) + g(x)u(x)\big)}^{\dot{V}(x)} + L(x)(V(x) - \rho) \in SOS$ is nonconvex constraints.
- It contains multiplication of unknowns $\frac{\partial V(x)}{\partial x} u(x), L(x)\, V(x)$ and $L(x)\, \rho$.

➢ We need Iterative algorithm. e.g.,

- Fix $V(x)$ and solve the SOS program with respect to $u(x)$ and $L(x)$ and Binary search over ρ in order to maximize it.
- Fix $u(x)$ and $L(x)$ and solve the SOS program with respect to $V(x)$ and ρ.

➢ Initial guess:

- $V_{guess}(x) = x^T S x$ where S is the solution of Riccati equation $Q + SA + A^T S = 0$,
  A is linearized system about the origin, Q is PSD cost-matrices.

- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

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Invariant Set

- Estimation
- Design

Invariant Set Estimation

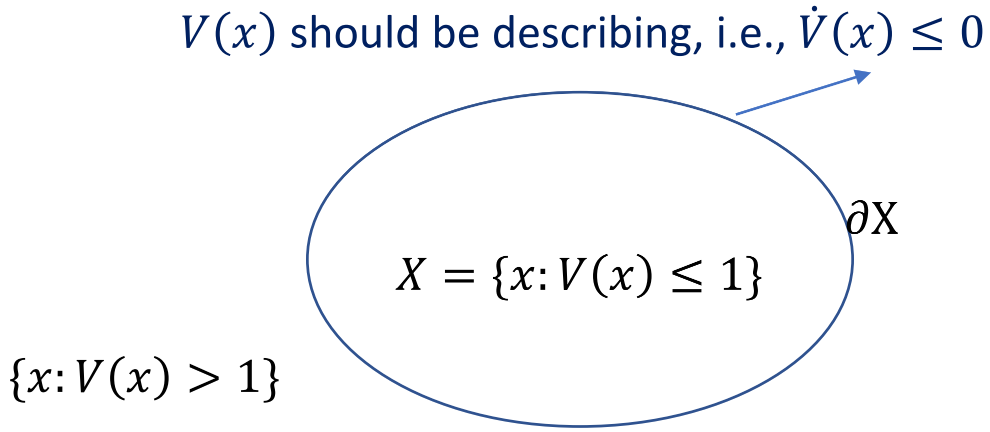
Dynamical System: $\dot{x} = f(x)$

Invariant Set: Set χ_{inv} is invariant if $x(0) \in \chi_{inv}$ then $x(t) \in \chi_{inv} \quad \forall t$

- Let $V(x) > 0, V(0) = 0,$

➢ The level set $\chi_{INV} = \{x: V(x) \leq \rho\}$ is an **inner** approximation of invariant set if

$$\dot{V}(x) < 0 \qquad \forall x \in \partial \chi_{INV}$$



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Invariant Set: Set χ_{inv} is invariant if $x(0) \in \chi_{inv}$ then $x(t) \in \chi_{inv} \quad \forall t$

- Let $V(x) > 0, V(0) = 0,$

➢ The level set $\chi_{INV} = \{x: V(x) \leq \rho\}$ is an **inner** approximation of invariant set if

$$\dot{V}(x) < 0 \qquad \forall x \in \partial \chi_{INV}$$

Conditions:

$$V(0) = 0$$

$$V(x) > 0$$

$$\dot{V}(x) < 0 \quad \forall x \in \{x: V(x) = \rho\}$$

SOS Conditions:

$$V(x) = \sum_{i=0}^{d} c_0 x^i, c_0 = 0$$

$$V(x) \in SOS$$

$$-\frac{\partial V(x)}{\partial x} - L(x)(\rho - V(x)) \in SOS$$

SOS Program:

$$\max_{V(x),L(x),\,\rho} \rho$$

$$\chi_{INV} = \{x: V(x) \le \rho\}$$

Subject to $V(x) = \sum_{i=0}^{d} c_0 x^i \text{ , } c_0 = 0$

$$V(x) \in SOS$$

$$-\dot{V}(x) + L(x)(V(x) - \rho) \in SOS$$

$$V\left(\sum_j e_j\right) = 1$$

e_j: j-th standard basis vector for the state space

(Sum of coefficients of $V(x)$)

- $-\dot{V}(x) + L(x)(V(x) - \rho) \in SOS$ is bilinear constraints.
- It contains multiplication of unknowns $L(x)\ V(x)$ and $L(x)\ \rho$.

➢ We need Iterative algorithm. e.g.,

- Fix $L(x)$, solve the SOS program with respect to $V(x)$ and ρ
- Fix $V(x)$, solve the SOS program with respect to $L(x)$ and Binary search over ρ in order to maximize it.

- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

Invariant Set Design

- Uncertain Dynamical System $\dot{x} = f(x) + g_\omega(x)\omega$ $\omega \in \Omega \subset \mathbb{R}^{n_\omega}$

Invariant Set: Set χ_{inv} is invariant if $x(0) \in \chi_{inv}$ then $x(t) \in \chi_{inv}$ $\forall t, \forall \omega \in \Omega$

- Let the peak of a signal ω be bounded by $\|\omega\|_\infty = \sup_t |\omega(t)| \leq \sqrt{\gamma}$ $\Omega = \{\omega \in \mathbb{R}^{n_\omega}: \omega^T\omega \leq \gamma\}$
- Given Set $\{x: V(x) \leq 1\}$

➢ Find the **maximum peak** disturbance value γ such that a given set remains **invariant** under these bounded disturbances.

- Jarvis-Wloszek Z., Feeley R., Tan W., Sun K., Packard A. Control Applications of Sum of Squares Programming. In: Henrion D., Garulli A. (eds) Positive Polynomials in Control. Lecture Notes in Control and Information Science, vol 312. Springer, Berlin, Heidelberg

- Uncertain Dynamical System $\dot{x} = f(x) + g_\omega(x)\omega$ $\omega \in \Omega \subset \mathbb{R}^{n_\omega}$
- Let the peak of a signal ω be bounded by $\|\omega\|_\infty = \sup_t |\omega(t)| \leq \sqrt{\gamma}$ $\Omega = \{\omega \in \mathbb{R}^{n_\omega} : \omega^T\omega \leq \gamma\}$
- Given Set $X = \{x : V(x) \leq 1\}$

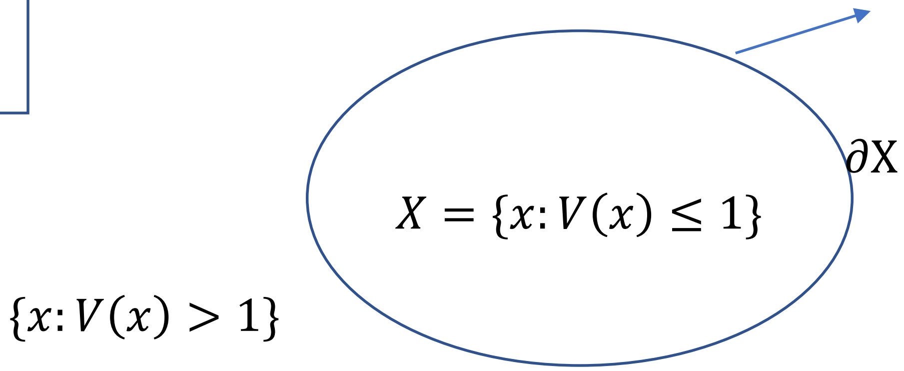
➢ Find the **maximum peak** disturbance value γ such that a given set remains **invariant** under these bounded disturbances.

➢ $V(x)$ should satisfy $\dot{V}(x, \omega) \leq 0 \quad \forall x \in \partial X \text{ and } \forall \omega \in \Omega$

Boundary of set X

Where, $\dot{V}(x, \omega) = \frac{\partial V(x)}{\partial x}(f(x) + g_\omega(x)\omega)$

$V(x)$ should be describing, i.e., $\dot{V}(x) \leq 0$



- $V(x)$ should satisfy $\dot{V}(x,\omega) \leq 0 \quad \forall x \in \{V(x) = 1\}$ and $\forall \omega \in \{\omega^T \omega \leq \gamma\}$

- SOS Conditions:

$$-\frac{\partial V(x)}{\partial x}(f(x) + g_\omega(x)\omega) - \sigma_V(x,\omega)(V(x) - 1) - \sigma_\omega(x,\omega)(\gamma - \omega^T\omega) \in SOS$$

$$\sigma_\omega(x,\omega) \in SOS$$

- **SOS Program for maximum peak** disturbance**:**

$$\max_{\gamma,\sigma_V(x,\omega),\sigma_\omega(x,\omega)} \gamma$$

Subject to

$$-\frac{\partial V(x)}{\partial x}(f(x) + g_\omega(x)\omega) - \sigma_V(x,\omega)(V(x) - 1) - \sigma_\omega(x,\omega)(\gamma - \omega^T\omega) \in SOS$$

$$\sigma_\omega(x,\omega) \in SOS$$

- Bilinear constraint (multiplication of $\gamma, \sigma_\omega(x,\omega)$)
- We need Iterative algorithm.

Invariant Set Design Using Controller

- Uncertain Dynamical System $\dot{x} = f(x,\omega) + g_\omega(x)\omega + g_u(x)u$ $\omega \in \Omega = \{\omega: \omega^T\omega \leq \gamma\}$ (γ: given)
- Given Set $X = \{x: V(x) \leq 1\}$

➢ Find the **control input** u such that a given set remains **invariant** under these bounded disturbances, $\omega \in \Omega$.

➢ $V(x)$ should satisfy $\dot{V}(x,\omega,u) \leq 0 \quad \forall x \in \partial X$ and $\forall \omega \in \Omega$

(∂X: Boundary of set X)

➢ SOS Conditions:

$$-\frac{\partial V(x)}{\partial x}\left(f(x,\omega) + g_\omega(x)\omega + g_u(x)u\right) - \sigma_V(x,\omega)(V(x)-1) - \sigma_\omega(x,\omega)(\gamma - \omega^T\omega) \in SOS$$

$$\sigma_\omega(x,\omega) \in SOS$$

$Unknow: u(x), \sigma_V(x,\omega), \sigma_\omega(x,\omega)$

Invariant Set Design Using Controller

Discrete-time formulation

- Uncertain Dynamical System $x(k+1) = f(x(k), u(k), \omega(k))$ $\omega \in \Omega \subset \mathbb{R}^{n_\omega}$

- Given Set $\mathrm{X} = \{x : V(x) \geq 0\}$

- $u(x)$: Polynomial control input in x

➢ Find $u(x)$ such that that a given set remains **invariant** under bounded disturbances

- Set X is **invariant** if $x(k) \in X$ then $x(k+1) \in X \quad \forall \omega \in \Omega$

$$f(x, u(x), \omega) \in X \quad \forall x \in X, \forall \omega \in \Omega$$

$$V\big(f(x, u(x), \omega)\big) \geq 0 \quad \forall x \in X, \forall \omega \in \Omega$$

- Given Set $X = \{x: V(x) \geq 0\}$
- Let $\Omega = \{\omega: g(\omega) \geq 0\}$

Invariance condition: $V\big(f(x, u(x), \omega)\big) \geq 0 \quad \forall x \in X, \forall \omega \in \Omega$

SOS condition: $V\big(f(x, u(x), \omega)\big) - \sigma_x(x, \omega)V(x) - \sigma_\omega(x, \omega)g(\omega) \in SOS$

- This is nonlinear constraint. It is polynomial function of $u(x)$, i.e. V($f(u(x), ., .)$)

- Let $u(x) = \sum_i^d c_i x^i$ $\quad c_i \in C$ $\quad f_{cl}(x, c, \omega) = f\left(x, \sum_i^d c_i x^i, \omega\right)$

  ↓ Control parameters

- Invariance condition $V\big(f_{cl}(x, c, \omega)\big) \geq 0 \quad \forall x \in X, \forall \omega \in \Omega$ ⇒ Nonlinear Condition

- Invariance condition $V\big(f_{cl}(x,c,\omega)\big) \geq 0 \quad \forall x \in X, \forall \omega \in \Omega$ ⇒ $\min V\big(f_{cl}(x,c,\omega)\big) \geq 0 \quad x \in X, \forall \omega \in \Omega$

- Let $J(c)$ be a lower bound polynomial $J(c) \leq V\big(f_{cl}(x,c,\omega)\big) \quad \forall x \in X \quad \forall \omega \in \Omega \quad \forall c \in C$

The set $\{c\colon J(c) \geq 0\}$ is the set of all control parameters c that makes the given set X invainant.

$$V\big(f_{cl}(x,c,\omega)\big) - J(c) \geq 0 \quad \forall x \in X \quad \omega \in \Omega \quad c \in C$$

↓ Polynomial in x, c, ω

↓ Polynomial in c

We look for coefficient of polynomial J. This is convex SOS condition:

$$V\big(f_{cl}(x,c,\omega)\big) - J(c) - \sigma_x(x,\omega,c)V(x) - \sigma_\omega(x,\omega,c)g_\omega(\omega) - \sigma_c(x,\omega,c)g_c(c) \in SOS$$

$$V\big(f_{cl}(x,c,\omega)\big) - J(c) \geq 0 \quad \forall x \in X \quad \omega \in \Omega \quad c \in C$$

↓ Polynomial in x, c, ω ↓ Polynomial in c

We look for coefficient of polynomial J. This is convex SOS condition:

$$V\big(f_{cl}(x,c,\omega)\big) - J(c) - \sigma_x(x,\omega,c)V(x) - \sigma_\omega(x,\omega,c)g_\omega(\omega) - \sigma_c(x,\omega,c)g_c(c) \in SOS$$

- To find the best lower bound polynomial, we solve

$$\min_{J(c)} \int J(c)dc$$

Subject to $V\big(f_{cl}(x,c,\omega)\big) - J(c) - \sigma_x(x,\omega,c)V(x) - \sigma_\omega(x,\omega,c)g_\omega(\omega) - \sigma_c(x,\omega,c)g_c(c) \in SOS$

- The set $\{c\text{: } J(c) \geq 0\}$ is the inner approximation of the set of all control parameters c that makes the given set X invainant. and converges as the order of the polynomial increase.

- Ashkan Jasour, C. Lagoa, "Convex Relaxations of a Probabilistically Robust Control Design Problem", 52st IEEE Conference on Decision and Control, Florence, Italy, 2013

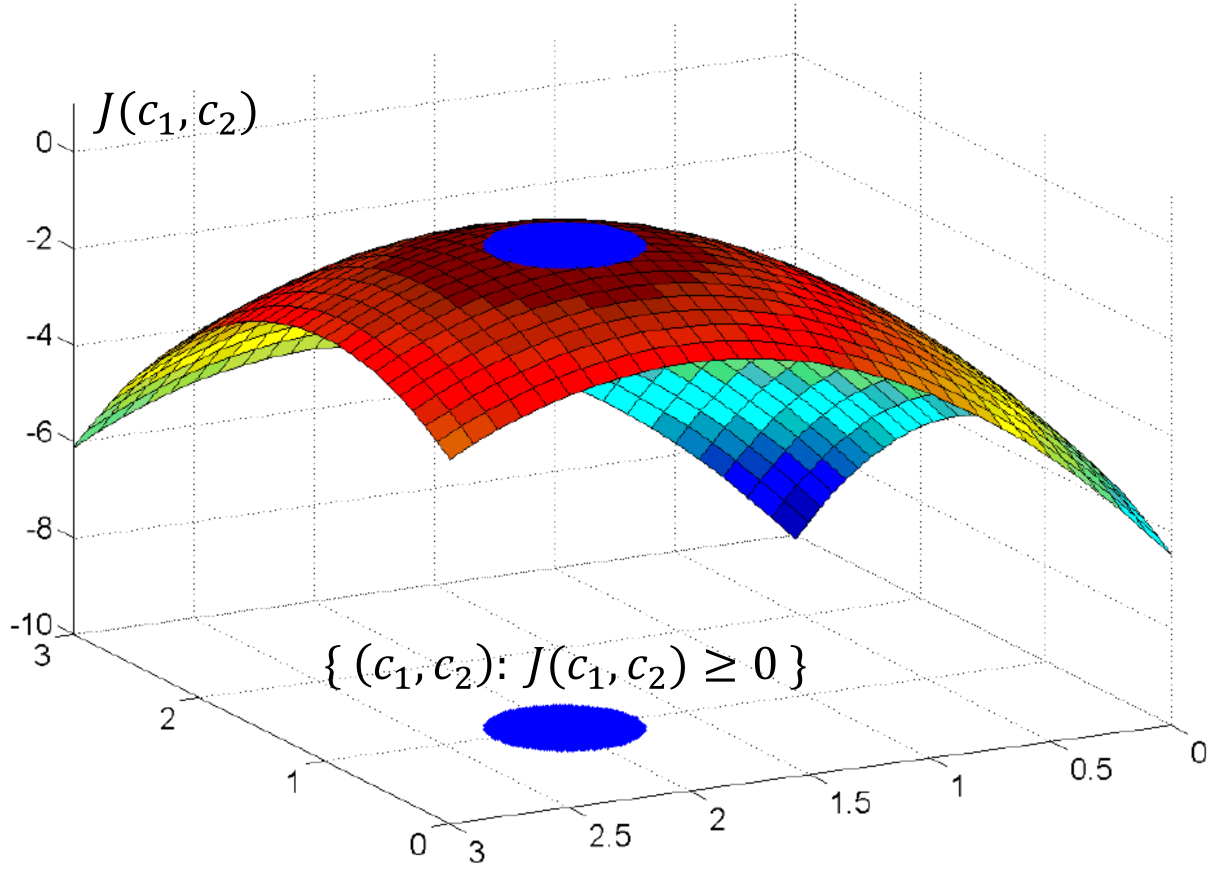
Example:

$$x_1(k+1) = \delta x_2(k)$$

$$x_2(k+1) = x_1(k) + 2x_2(k) + u(k) + \omega(k)$$

$$\delta \in [-0.5, 0.5] \quad , \omega(k) \in [-0.4, 0.4]$$

- Given Set $X = \{x: 1 - x_1^2 - x_2^2 \geq 0\}$
- Control Input $u(x) = -c_1 x_1(k) - c_2 x_2(k)$



- Ashkan Jasour, C. Lagoa, "Convex Relaxations of a Probabilistically Robust Control Design Problem", 52st IEEE Conference on Decision and Control, Florence, Italy, 2013

Topics:

- SOS optimization for Robust Control
- Lyapunov Stability and SOS optimization
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- Funnel Based Robust Control
- Reachable Sets
- Constrained Volume Optimization

Funnel Design Using Controller

- Control affine system $\dot{x} = f(x) + g(x)u$

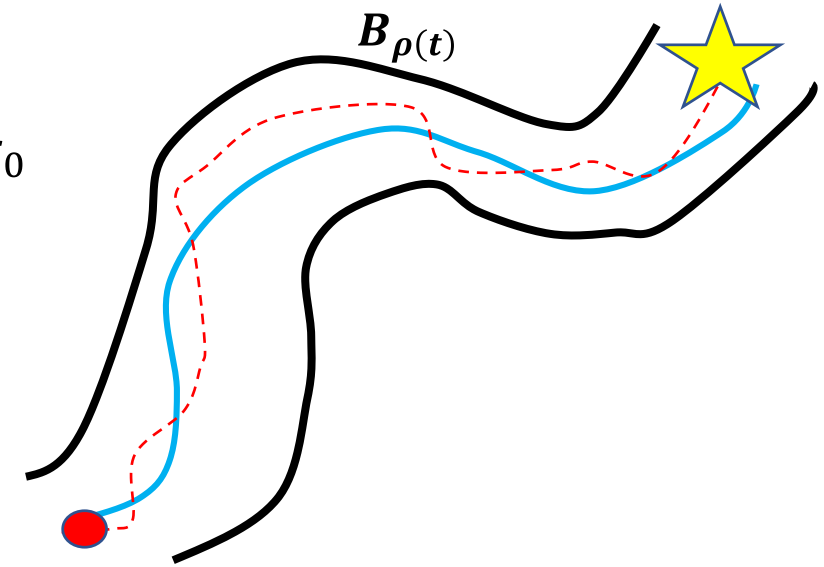
- Nominal trajectory and open loop control input: $\left(x_0(t), u_0(t)\right) \quad t = [0, T]$

➢ We look for time-varying controllers that maximize the size of the **funnel.**

➢ Define Error dynamics as follows:

$$\bar{x} = x - x_0(t) \qquad \bar{u} = u - u_0(t)$$

- Error dynamics: $\dot{\bar{x}} = \dot{x} - \dot{x}_0 = f(\bar{x} + x_0) + g(\bar{x} + x_0)(\bar{u} + u_0) - \dot{x}_0$



- Error dynamics: $\dot{\bar{x}} = f(\bar{x} + x_0) + g(\bar{x} + x_0)(\bar{u} + u_0) - \dot{x}_0$

  Given open loop trajectory (x_0, x_0, u_0, $\dot{x}_0$)

➢ We model the funnel as level set of time varying function $B_{\rho(t)} = \{\bar{x}: V(\bar{x}, t) \leq \rho(t)\}$

  V: Polynomial in t and x

Invariant Funnel: $\bar{x}(t_0) \in B_{\rho(t)}$ $\quad\quad$ $\bar{x} \in B_{\rho(t)} \forall t \in [t_0, T]$

- Our task will be to design time-varying controllers that maximize the size of this funnel.

➢ **Invariance condition:**

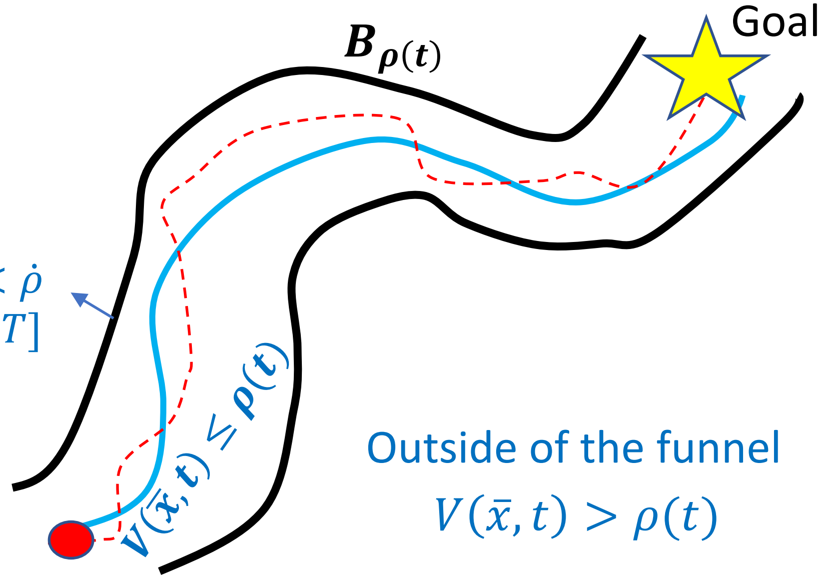
To stay inside the funnel:

$$V(\bar{x}, t) = \rho(t) \Rightarrow \dot{V}(\bar{x}, t) < \dot{\rho} \quad \forall t \in [0, T]$$

$V(\bar{x}, t) = \rho(t)$: On the boundary of the funnel

$\dot{V}(\bar{x}, t) < \dot{\rho}$: $V(\bar{x}, t) - \rho(t)$ should be decreasing

Where, $\dot{V}(\bar{x}, t) = \frac{\partial V(\bar{x},t)}{\partial \bar{x}} \dot{\bar{x}} + \frac{\partial V(\bar{x},t)}{\partial t}$



$V(\bar{x},t) = \rho(t)$ ⇨ $\dot{V}(\bar{x},t) < \dot{\rho} \quad \forall t \in [0,T]$

SOS Condition: $\dot{\rho} - \dot{V}(\bar{x},t) - L(x,t)(V(\bar{x},t) - \rho(t)) \in SOS$

SOS Program:

$$\max_{\rho, V(x,t), u(x), L(x)} \rho$$

Subject to: $V(\bar{x},t) \in SOS$

$$\dot{\rho} - \dot{V}(\bar{x},t) - L(x,t)(V(\bar{x},t) - \rho(t)) \in SOS$$

$$V\left(\sum_j e_j\right) = 1$$

- Continuous-time formulation is computationally expensive
- We check at sample points in time $t_i \in \ [0,T]\ i \ = \ 1, \dots, N$
- Instead of looking for $V(\bar{x},t), u(t,\bar{x}), \rho(t)$ we look for $V_i(\bar{x}), u_i(\bar{x}), \rho(t_i)\ \ i \ = \ 1, \dots, N$

- A.Majumdar and R. Tedrake. Funnel libraries for real-time robust feedback motion planning. International Journal of Robotics Research, 36(8):947 - 982, 2017
- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

SOS Program:

$$\max_{\rho, V(x,t), u(x), L(x)} \int \rho(t) dt$$

Subject to: $V(\bar{x}, t) \in SOS$

$$\dot{\rho}(t) - \dot{V}(\bar{x}, t) - L(x, t)(V(\bar{x}, t) - \rho(t)) \in SOS$$

$$V\left(\sum_j e_j\right) = 1$$

- $V(\bar{x}, t), u(t, \bar{x}), \rho(t)$ → $V_i(\bar{x}), u_i(\bar{x}), \rho(t_i) \quad i = 1, \dots, N$

SOS Program:

$$\underset{\rho(t_i), L_i(\bar{x}), V_i(\bar{x}), \bar{u}_i(\bar{x})}{\text{maximize}} \quad \sum_{i=1}^{N} \rho(t_i)$$

subject to $V_i(\bar{x})$ SOS

$$-\dot{V}_i(\bar{x}) + \dot{\rho}(t_i) + L_i(\bar{x})(V_i(\bar{x}) - \rho(t_i)) \text{ SOS}$$

$$V_i\left(\sum_j e_j\right) = V_{guess}\left(\sum_j e_j, t_i\right)$$

where:

$$\dot{\rho}(t_i) = \frac{\rho(t_{i+1}) - \rho(t_i)}{t_{i+1} - t_i} \qquad \dot{V}(\bar{x}, t) = \frac{\partial V(\bar{x}, t)}{\partial \bar{x}} \dot{\bar{x}} + \frac{\partial V(\bar{x}, t)}{\partial t} \qquad \frac{\partial V(\bar{x}, t)}{\partial t} \approx \frac{V_{i+1}(\bar{x}) - V_i(\bar{x})}{t_{i+1} - t_i}$$

SOS Program:

$$\underset{\rho(t_i), L_i(\bar{x}), V_i(\bar{x}), \bar{u}_i(\bar{x})}{\text{maximize}} \quad \sum_{i=1}^{N} \rho(t_i)$$

$$\text{subject to} \quad V_i(\bar{x}) \ \text{SOS}$$

$$-\dot{V}_i(\bar{x}) + \dot{\rho}(t_i) + L_i(\bar{x})(V_i(\bar{x}) - \rho(t_i)) \ \text{SOS}$$

$$V_i(\sum_j e_j) = V_{guess}(\sum_j e_j, t_i)$$

where:

$$\dot{\rho}(t_i) = \frac{\rho(t_{i+1}) - \rho(t_i)}{t_{i+1} - t_i} \qquad \dot{V}(\bar{x}, t) = \frac{\partial V(\bar{x}, t)}{\partial \bar{x}} \dot{\bar{x}} + \frac{\partial V(\bar{x}, t)}{\partial t} \qquad \frac{\partial V(\bar{x}, t)}{\partial t} \approx \frac{V_{i+1}(\bar{x}) - V_i(\bar{x})}{t_{i+1} - t_i}$$

- The above optimization involves constraints that are bilinear in the decision variables.
- Constrainnts are linear in $L_i(\bar{x})$ and $u_i(\bar{x})$ for a fixed $V_i(\bar{x})$ and $\rho(t_i)$.
- We need Iterative algorithm.

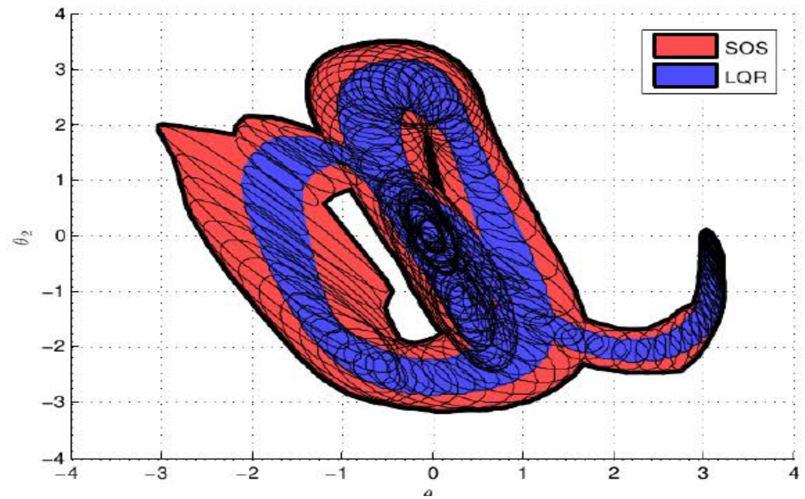
- A.Majumdar and R. Tedrake. Funnel libraries for real-time robust feedback motion planning. International Journal of Robotics Research, 36(8):947 - 982, 2017
- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

Example: Acrobat

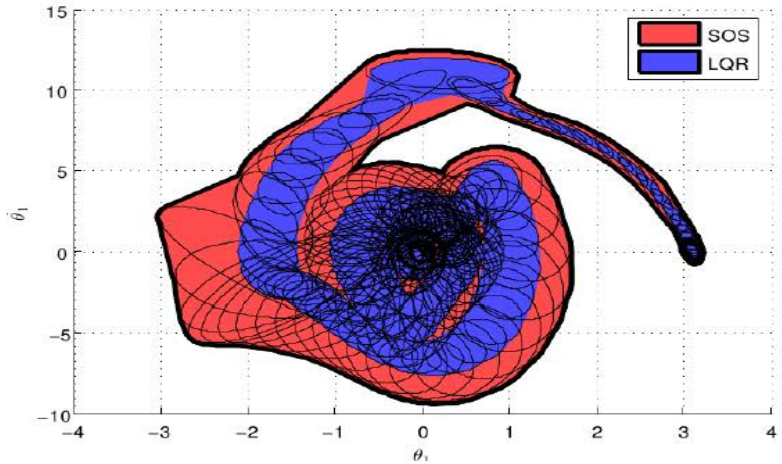
- States $x = [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]$
- Open-loop trajectory for the swing-up from initial state $[0,0,0,0]$ and final states $[\pi, 0,0,0]$.
- For a given open-loop trajectory, we design an invariant funnel.

- A.Majumdar and R. Tedrake. Funnel libraries for real-time robust feedback motion planning. International Journal of Robotics Research, 36(8):947 - 982, 2017
- A. Majumdar, A. A. Ahmadi, and R. Tedrake. Control design along trajectories with sums of squares programming. In Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA), pages 4054-4061, 2013.

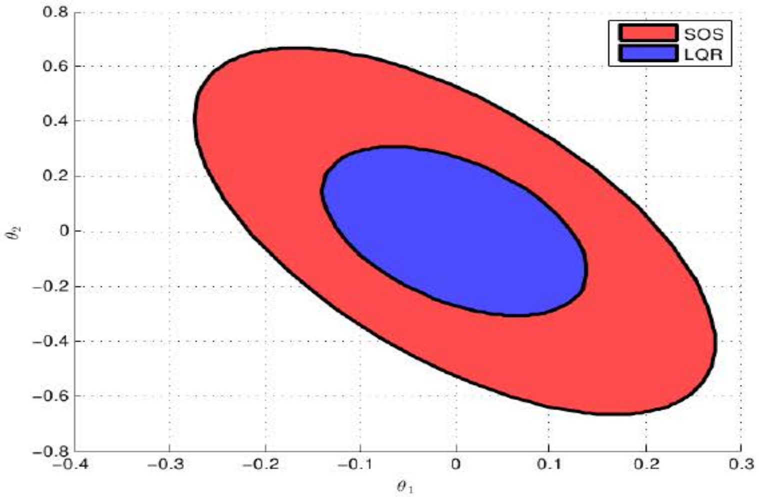
(θ_1, θ_2) Projection of the designed funnel



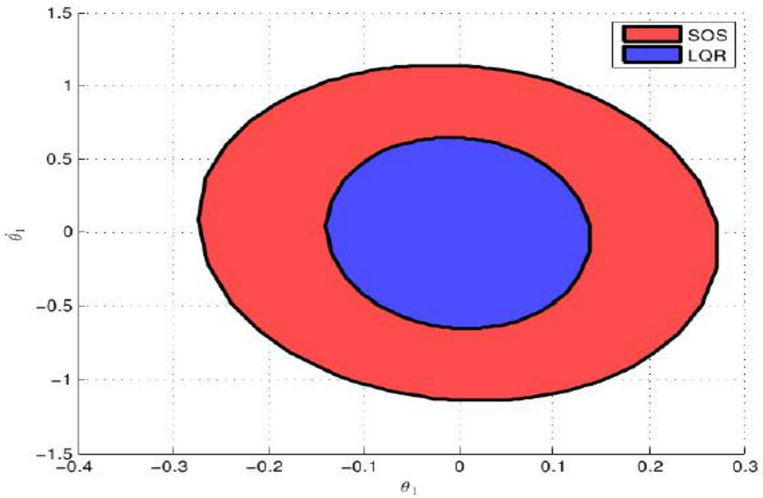
$(\theta_1, \dot{\theta}_1)$ Projection of the designed funnel



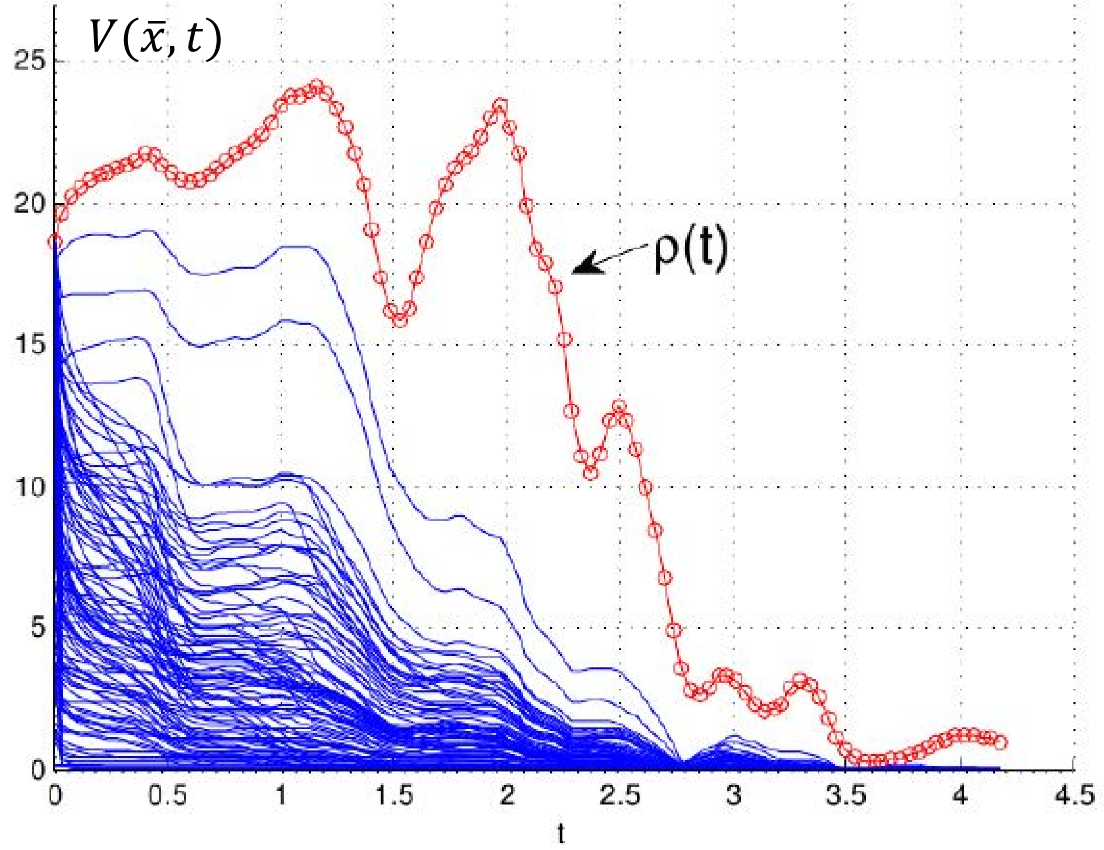
(θ_1, θ_2) Projection of the designed funnel at $t = 0$



$(\theta_1, \dot{\theta}_1)$ Projection of the designed funnel at $t = 0$



- any initial state inside the funnel will reach to the goal point under the designed states controller.



The robot is started off from random initial conditions

- A.Majumdar and R. Tedrake. Funnel libraries for real-time robust feedback motion planning. International Journal of Robotics Research, 36(8):947 - 982, 2017
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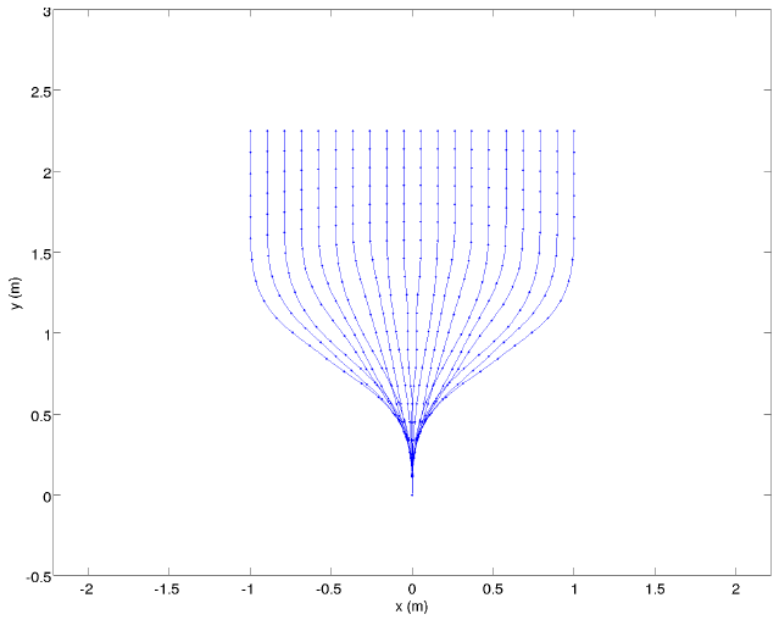

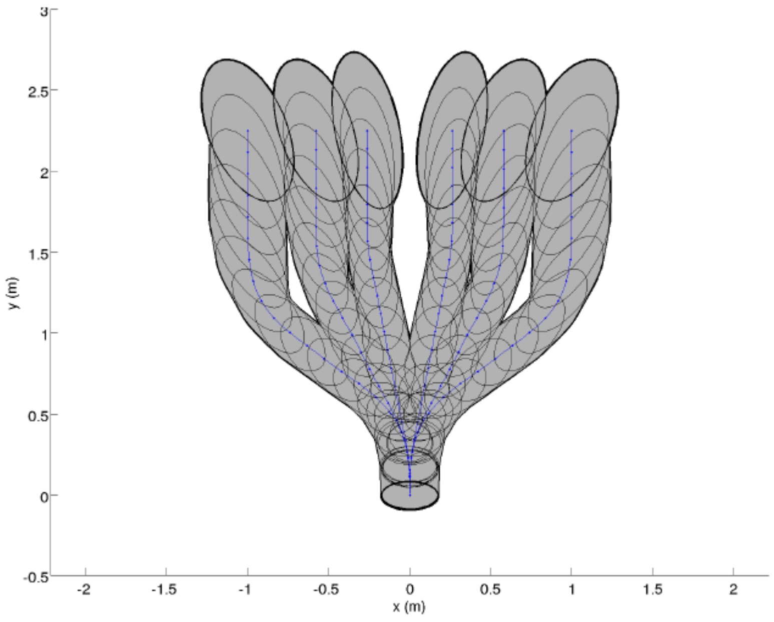

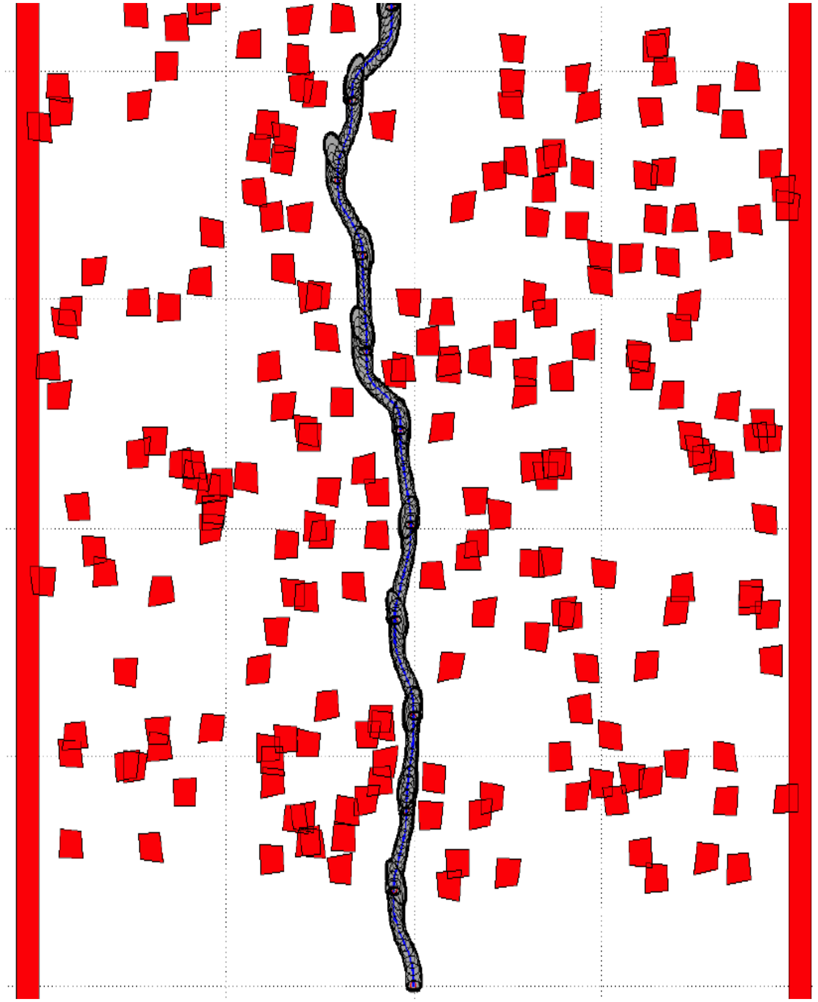

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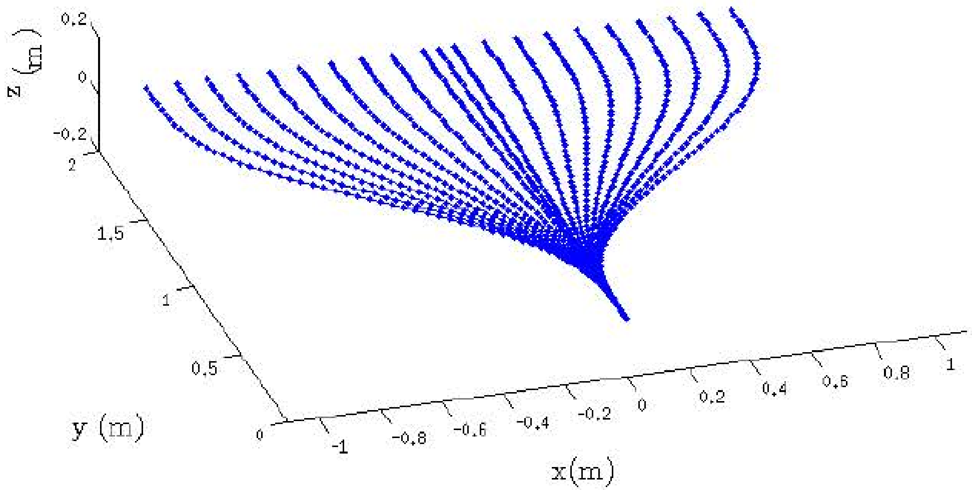

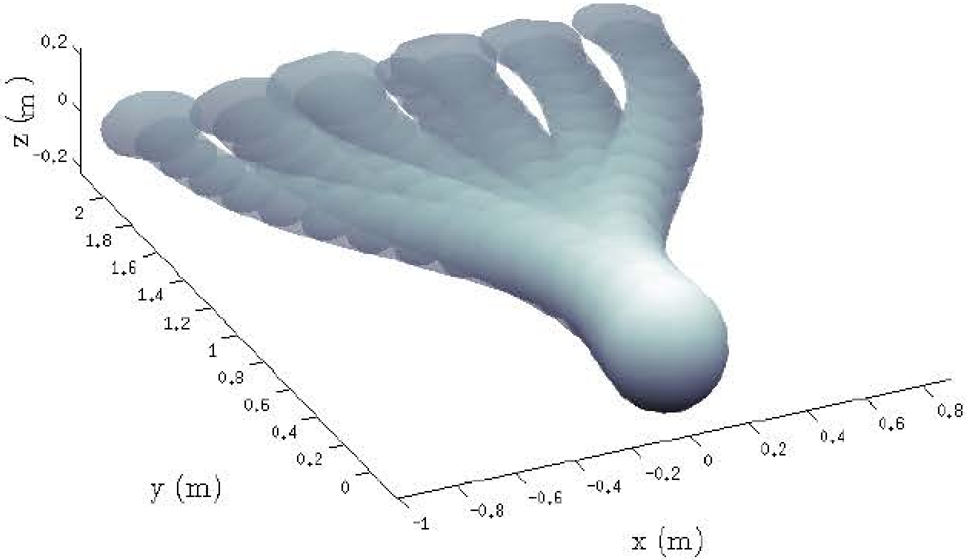

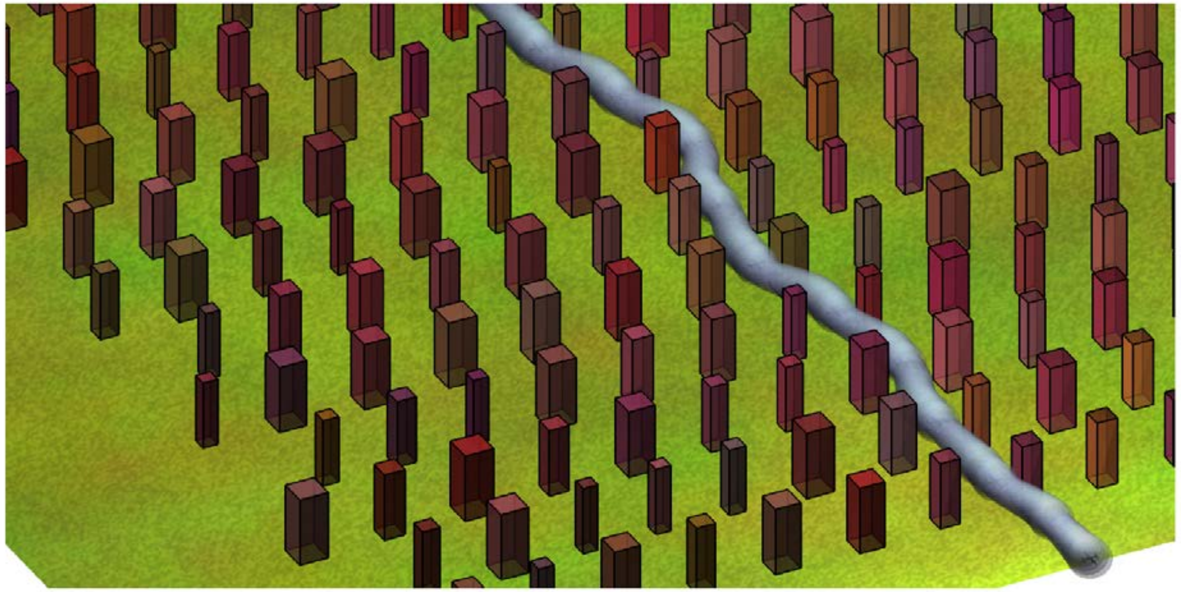

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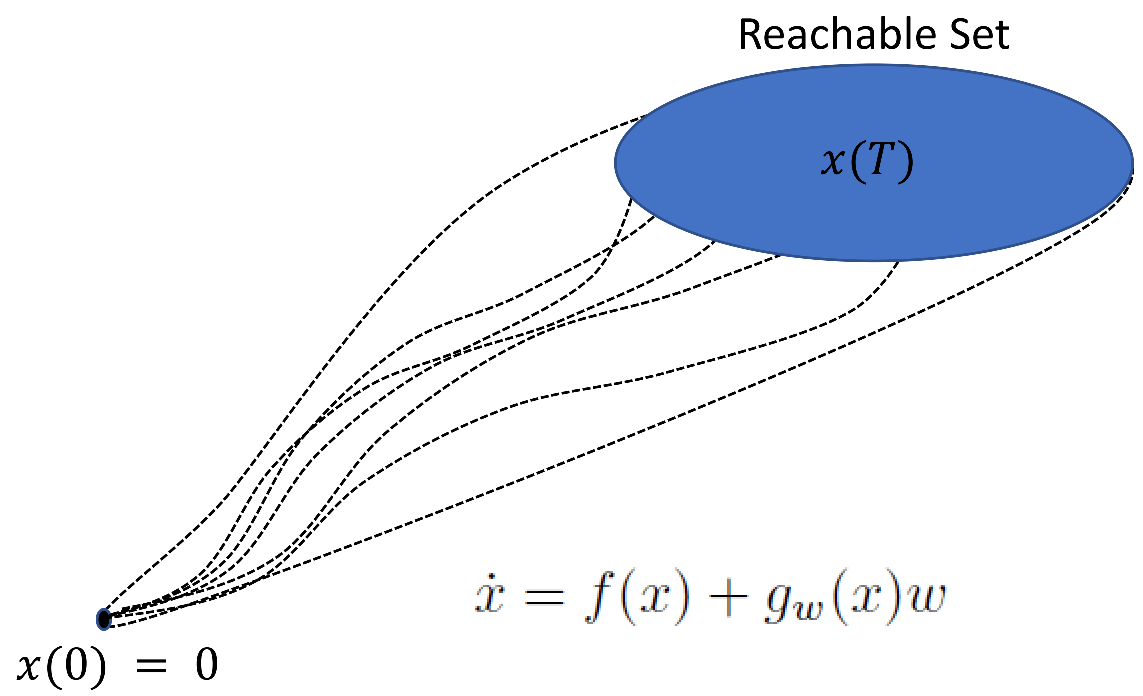
Reachable sets

- Uncertain dynamical System $\dot{x} = f(x) + g_w(x)w \qquad f(0) = 0,$

- Bounded disturbance: $\Omega = \int_0^T \left(\omega_1^2(t) + \cdots + \omega_{n_\omega}^2(t)\right) dt \leq R$

➢ We want to compute a set that is reachable at time T from $x(0) = 0$ in the presence of bound disturbance.



- Uncertain dynamical System $\dot{x} = f(x) + g_w(x)w$ $f(0) = 0,$

- Bounded disturbance: $\Omega = \int_0^T \left(\omega_1^2(t) + \cdots + \omega_{n_\omega}^2(t)\right) dt \leq R$

➢ We want to compute a set that is reachable at time T from $x(0) = 0$ in the presence of bound disturbance.

➢ We model reachable set as $\chi_R = \{x \colon V(x) \leq R\}$ $x(T) \in \chi_R = \{x \colon V(x) \leq R\}$

- Uncertain dynamical System $\dot{x} = f(x) + g_w(x)w \qquad f(0) = 0,$

- Bounded disturbance: $\Omega = \int_0^T \left(\omega_1^2(t) + \cdots + \omega_{n_\omega}{}^2(t)\right) dt \leq R$

➢ We want to compute a set that is reachable at time T from $x(0) = 0$ in the presence of bound disturbance.

➢ We model reachable set as $\chi_R = \{x\colon V(x) \leq R\}$ $\qquad x(T) \in \chi_R = \{x\colon V(x) \leq R\}$

- $V(x)$ satisfies:

$$V(0) = 0$$

$$V(x) > 0 \;\; \forall x, x \neq 0$$

$$\dot{V}(x) \leq \omega^T \omega \quad \forall x, \omega \longrightarrow \frac{\partial V}{\partial x}(f(x) + g_\omega(x)\omega) \leq \omega_1^2 + \cdots + \omega_{n_\omega}{}^2 \quad \forall x, \quad \forall \omega \in \Omega$$

- We model reachable set as $\chi_R = \{x\colon V(x) \leq R\}$ $\quad x(T) \in \chi_R = \{x\colon V(x) \leq R\}$

- $V(x)$ satisfies:

$$V(0) = 0$$

$$V(x) > 0 \ \ \forall x, x \neq 0$$

$$\dot{V}(x) \leq \omega^T \omega \ \ \forall x, \omega \longrightarrow \frac{\partial V}{\partial x}(f(x) + g_\omega(x)\omega) \leq \omega_1^2 + \cdots + \omega_{n_\omega}^2 \ \ \forall x, \quad \forall \omega \in \Omega$$

Then:

$$\frac{\partial V(x)}{\partial t} \leq \omega^T \omega \ \ \forall x, \omega \xrightarrow{\text{Integration } \int_0^T} V(x(T)) - V(x(0)) \leq \int_0^T \omega^T \omega \, dt \leq R \longrightarrow V(x(T)) \leq R$$

Hence: $\quad x(T) \in \{x\colon V(x) \leq R\}$

- Jarvis-Wloszek Z., Feeley R., Tan W., Sun K., Packard A. Control Applications of Sum of Squares Programming. In: Henrion D., Garulli A. (eds) Positive Polynomials in Control. Lecture Notes in Control and Information Science, vol 312. Springer, Berlin, Heidelberg

- We look for tightest set $\chi_R = \{x\colon V(x) \leq R\}$ that contains $x(T)$.
- Previously, we used level R and normalization constraints to min/max the size of the set.
- Since R is given, we use level sets of another polynomial to minimize the size of the χ_R.

Let $\{p(x) \leq \beta\}$ be a level set of given polynomial with unknow level β,
e.g., $\{p(x) \leq \beta\}$ is an ellipsoid with unknown radius β.

$$\min_{V(x),\beta} \beta$$

Such that

$$\frac{\partial V}{\partial x}(f(x) + g_\omega(x)\omega) \leq \omega^T\omega \quad \forall x \in \chi_R, \forall \omega \in R^n$$

$$V > 0 \qquad V(0) = 0$$

$$\{x\colon V \leq R\} \subset \{x\colon p(x) \leq \beta\} \longrightarrow \{p(x) \leq \beta\} \quad \forall x \in \{x\colon V \leq R\}$$

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$$\min_{V(x),\beta} \beta$$

Such that

$$\frac{\partial V}{\partial x}(f(x) + g_\omega(x)\omega) \leq \omega^T\omega \quad \forall x \in \chi_R, \forall \omega \in R^n$$

$$V > 0 \qquad V(0) = 0$$

$$\{x: V \leq R\} \subset \{x: p(x) \leq \beta\} \longrightarrow \{p(x) \leq \beta\} \quad \forall x \in \{x: V \leq R\}$$

SOS Program:

$$\min_{V(x),\beta} \beta$$

Such that

$$-\frac{\partial V}{\partial x}(f(x) + g_\omega(x)\omega) - \sigma_x(x,\omega)(V(x) - R) - \sigma_\omega(x,\omega)(R - \omega^T\omega) \in SOS$$

$$\sigma_x(x,\omega), \sigma_\omega(x,\omega) \in SOS$$

$$V(x) \in SOS \qquad V(0) = 0$$

$$(\beta - p(x)) - \sigma_V(x)(R - V(x)) \in SOS \qquad \sigma_V(x) \in SOS$$

- The above optimization involves constraints that are bilinear in the decision variables.
- We need Iterative algorithm.

Reachable set of linear uncertain systems:

- Linear uncertain system: $\dot{x} = Ax + B_\omega \omega$
- Bounded disturbance: $\Omega = \int_0^T \left(\omega_1^2(t) + \cdots + \omega_{n_\omega}^2(t)\right) dt \leq 1$

➢ We model reachable set as $\chi_R = \{x\colon V(x) \leq 1\}$ where $V(x) = x^T P x$

By applying same methodology, P should satisfy

$$P \succ 0 \qquad \begin{bmatrix} A^T P + PA & PB_\omega \\ B_\omega^T P & -I \end{bmatrix} \preccurlyeq 0$$

- S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan (1994). Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia.

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- Reachable Sets
- Constrained Volume Optimization

Constrained Volume Optimization

Let:

- Set 1 : $S_1(a) = \left\{x\colon p_{1_j}(x,a) \geq 0, j = 1, \dots, o_1\right\}$
- Set 2 : $S_2(a) = \left\{x\colon p_{2_j}(x,a) \geq 0, j = 1, \dots, o_2\right\}$
- $a \in A$: unknown parameters

Constrained volume optimization:

$$\max_{a \in A} vol(S_1(a)) = \int_{S_1(a)} dx$$

Subject to $S_1(a) \subseteq S_2(a)$

➢ We can reformulate different problems as a particular case of constrained volume optimization.

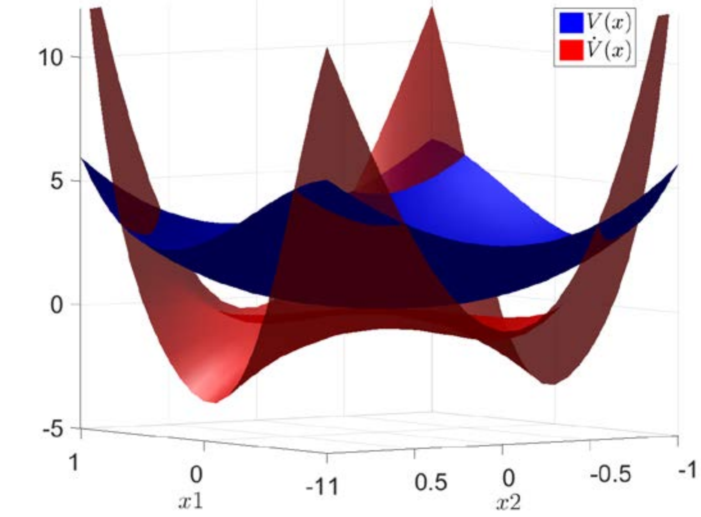
- Ashkan Jasour, C. Lagoa, "Convex Constrained Semialgebraic Volume Optimization: Application in Systems and Control", (submitted) IEEE Transaction on Automatic Control, 2017, (arXiv:1701.08910)
- "Convex Approximation of Chance Constrained Problems: Application in Systems and Control", School of Electrical Engineering and Computer Science, The Pennsylvania State University, 2016.

Region of attraction set $\chi_{ROA} = \{x: V(x) \leq \rho\}$

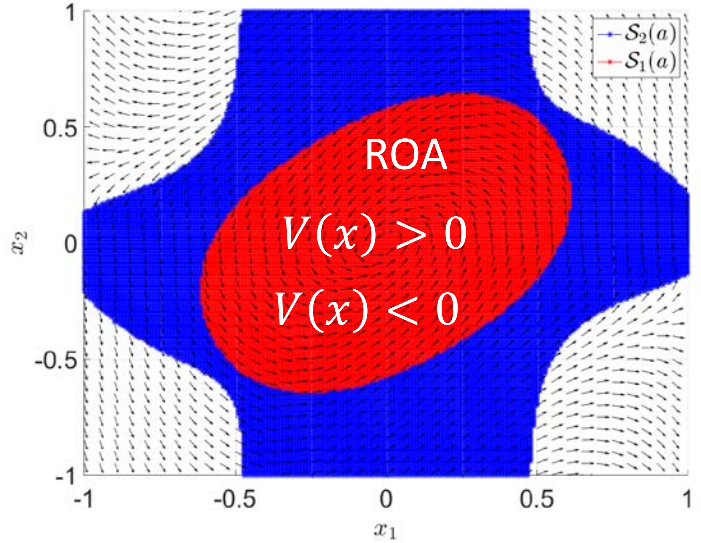
$V(0) = 0 \quad V(x) > 0 \qquad \dot{V}(x) < 0 \quad \forall x \in \{x: V(x) \leq \rho\}$

$$\{x: V(x) \leq \rho\} \subset \{x: \dot{V}(x) < 0\}$$

Inside the ROA set, $V(x)$ should be decreasing



- $S_1(a) = \{x: 0 \leq V(x) \leq \rho\}$
- $S_2(a) = \{x: \dot{V}(x) \leq -\epsilon\|x\|_2^2\}$
- a: Coefficient of polynomial $V(x)$
- $vol(S_1) = \int_{S_1} dx$



- With same reasoning, we can extend the problem in to discreet-time systems $x_{k+1} = f(x_k)$ by replacing the derivative of Lyapunov function $\dot{V}(x)$ with difference Lyapunov function $\Delta V(x) = V(x_{k+1}) - V(x_k)$

Invariant set- Continuous-time

$\dot{x} = f(x, \omega)$

$\omega \in \Omega = \{\omega: g(\omega) \geq 0\}$

$\chi_{inv} = \{x: V(x) \leq \rho\}$

$x(0) \in \chi_{inv} \rightarrow x(t) \in \chi_{inv}$

Condition: $\dot{V}(x, \omega) \leq 0 \quad \forall x \in \partial X \;\text{ and }\; \forall \omega \in \Omega$

On the boundary

$$\{(x, \omega): V(x) = \rho, g(\omega) \geq 0\} \subseteq \{(x, \omega): \dot{V}(x, \omega) \leq 0\}$$

- $S_1(a) = \{(x, \omega): V(x) = \rho, g(\omega) \geq 0\}$
- $S_2(a) = \{(x, \omega): \dot{V}(x, \omega) \leq 0\}$
- a: Coefficient of polynomial $V(x)$ and ρ
- $vol(S_1) = \int_{S_1} dx$

Invariant set-discrete time

$x_{k+1} = f(x_k, \omega_k)$

$\omega_k \in \Omega = \{\omega : g(\omega) \geq 0\}$

$\chi_{inv} = \{x : V(x) \leq \rho\}$

$x \in \chi_{inv} \rightarrow f(x, \omega) \in \chi_{inv}$

Condition: $V\big(f(x,\omega)\big) \leq \rho \quad \forall x \in \{x : V(x) \leq \rho\} \quad \forall \omega \in \{\omega : g(\omega) \geq 0\}$

$$\{(x,\omega) : V(x) \leq \rho, g(\omega) \geq 0\} \subseteq \{(x,\omega) : V(f(x,\omega)) \leq \rho\}$$

- $S_1(a) = \{(x,\omega) : V(x) \leq \rho, g(\omega) \geq 0\}$
- $S_2(a) = \{(x,\omega) : V(f(x,\omega)) \leq \rho\}$
- a: Coefficient of polynomial $V(x)$ and ρ
- $vol(S_1) = \int_{S_1} dx$

Sum-of-Squares

Look for polynomial $p(x,a)$ with unknow parameters a that satisfies

$$p(x,a) \geq 0 \quad \forall\, x \in \{x\colon\, p_{1_j}(x) \geq 0, j = 1, \dots, o_2\}$$

Nonnegative polynomial — Given Set

Generalized Sum-of-Squares

Look for nonnegative polynomial $p(x,a)$ with unknow parameters a

$$p(x,a) \geq 0 \quad \forall\, x \in \{x\colon\, p_{1_j}(x,a) \geq 0, j = 1, \dots, o_2\}$$

Nonnegative polynomial — Set with unknow parameters

- $S_1(a) = \left\{x\colon\, p_{1_j}(x,a) \geq 0, j = 1, \dots, o_2\right\}$
- $S_2(a) = \{p(x,a) \geq 0\}$

$$\max_{a \in A} vol(S_1(a)) = \int_{S_1(a)} dx$$

Subject to $\quad S_1(a) \subseteq S_2(a)$

Let:
- Set 1 : $S_1(a) = \left\{x\colon p_{1_j}(x,a) \geq 0, j = 1, \dots, o_1\right\}$
- Set 2 : $S_2(a) = \left\{x\colon p_{2_j}(x,a) \geq 0, j = 1, \dots, o_2\right\}$
- $a \in A$: unknown parameters

Constrained volume optimization:

$$\max_{a \in A} vol(S_1(a)) = \int_{S_1(a)} dx$$

Subject to $S_1(a) \subseteq S_2(a)$

To solve this problem:

1) We find the inner approximation of set of all parameters a for which $S_1(a) \subseteq S_2(a)$, i.e., $\{a\colon P(a) < 1\}$
2) Solve volume maximization problem

$$\max_{a} vol(S_1(a))$$

Subject to $\{a\colon P(a) < 1\}$

1) We find the inner approximation of set of all parameters a for which $S_1(a) \subseteq S_2(a)$

- Set 1 : $S_1(a) = \left\{x\colon p_{1_j}(x,a) \geq 0, j = 1, \dots, o_1\right\}$
- Set 2 : $S_2(a) = \left\{x\colon p_{2_j}(x,a) \geq 0, j = 1, \dots, o_2\right\}$

- $K_1 = \left\{(x,a)\colon p_{1_j}(x,a) \geq 0, j = 1, \dots, o_1\right\}$
- $K_2 = \left\{(x,a)\colon p_{2_j}(x,a) \geq 0, j = 1, \dots, o_2\right\}$

SOS Program:

$$\min_{P(a)} \int P_a(a)da$$

Complement set

Subject to $P_a(a) \geq 1 \quad on \quad K_1 \cap \overline{K_2}$ (SOS)

$P_a(a) \geq 0$ (SOS)

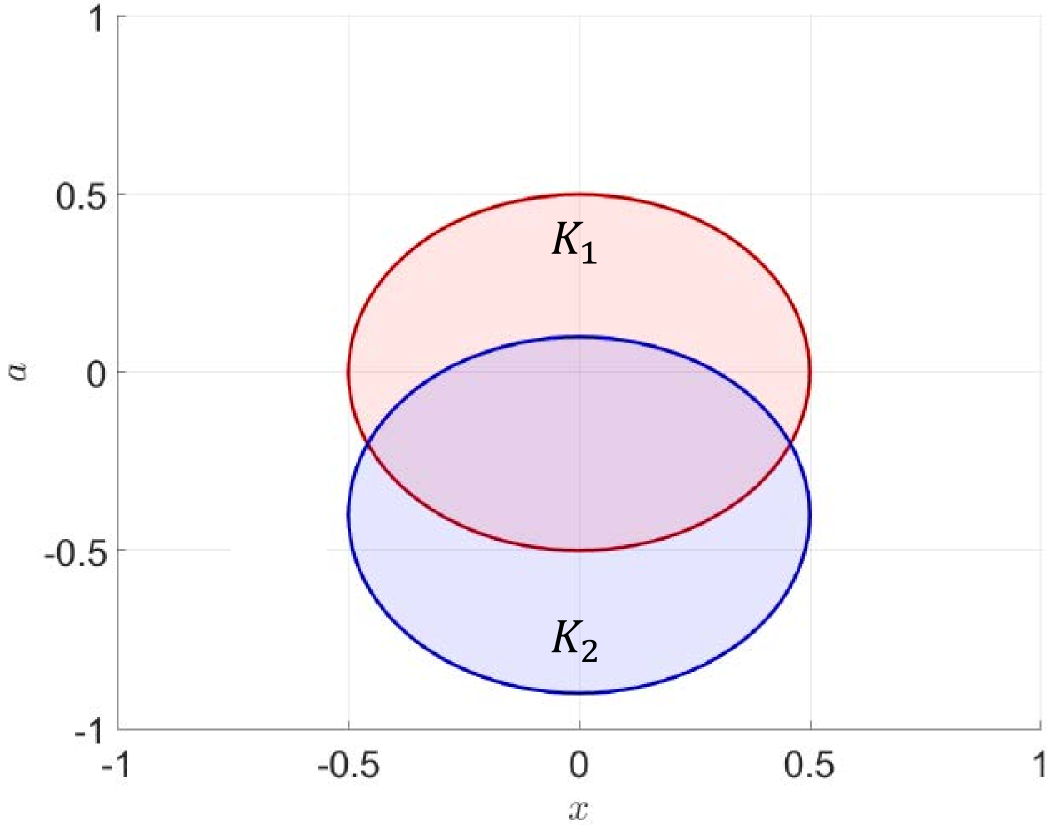
- $P_a(a)$ is upper bound approximation of the indicator function of the set $\{a\colon \; S_1(a) \nsubseteq S_2(a)\}$ $i.e., \{a\colon S_1(a) \; and \; \bar{S}_2(a)\}$

$\{a\colon P_a(a) \geq 1\}$: outer approximation of the set of all a for which $S_1(a) \nsubseteq S_2(a)$

$\{a\colon P_a(a) < 1\}$: inner approximation of the set of all a for which $S_1(a) \subseteq S_2(a)$

Example:

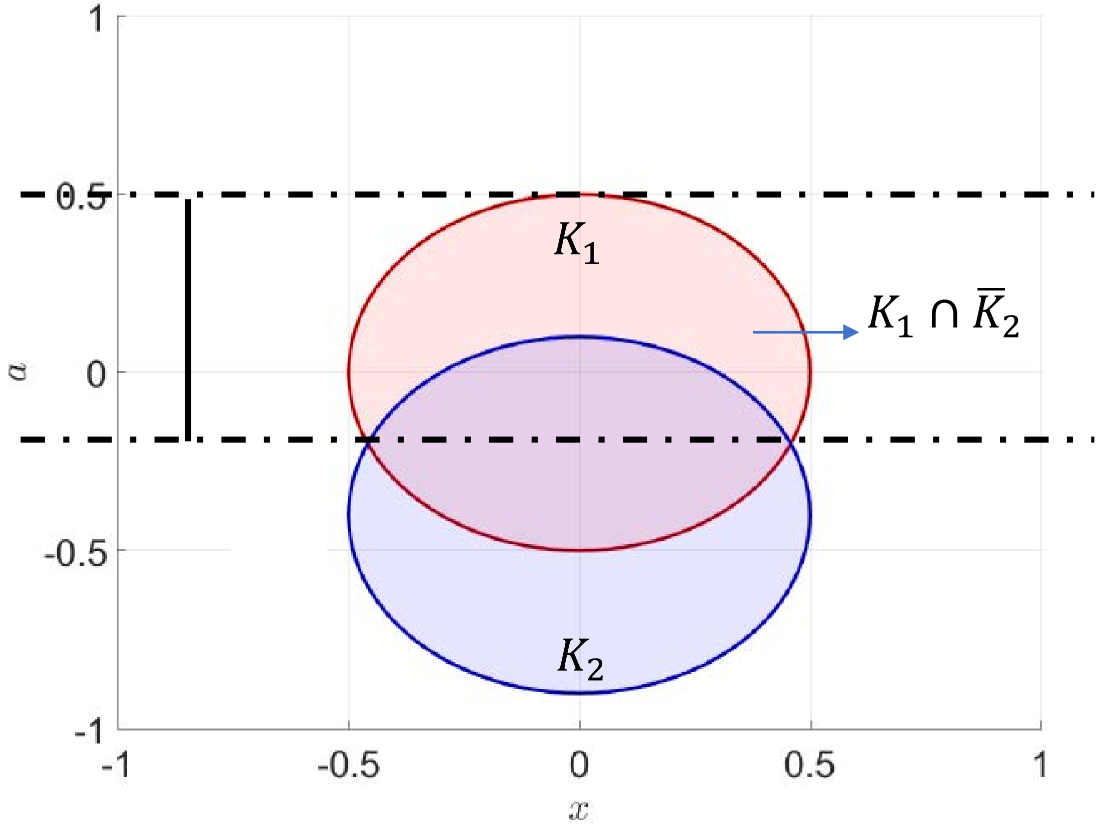
- Set 1 : $S_1(a) = \{x: 0.25 - a^2 - x^2 \geq 0\}$
- Set 2 : $S_2(a) = \{x: 0.09 - a^2 - 0.8a - x^2 \geq 0\}$



Example:

- Set 1 : $S_1(a) = \{x: 0.25 - a^2 - x^2 \geq 0\}$
- Set 2 : $S_2(a) = \{x: 0.09 - a^2 - 0.8a - x^2 \geq 0\}$

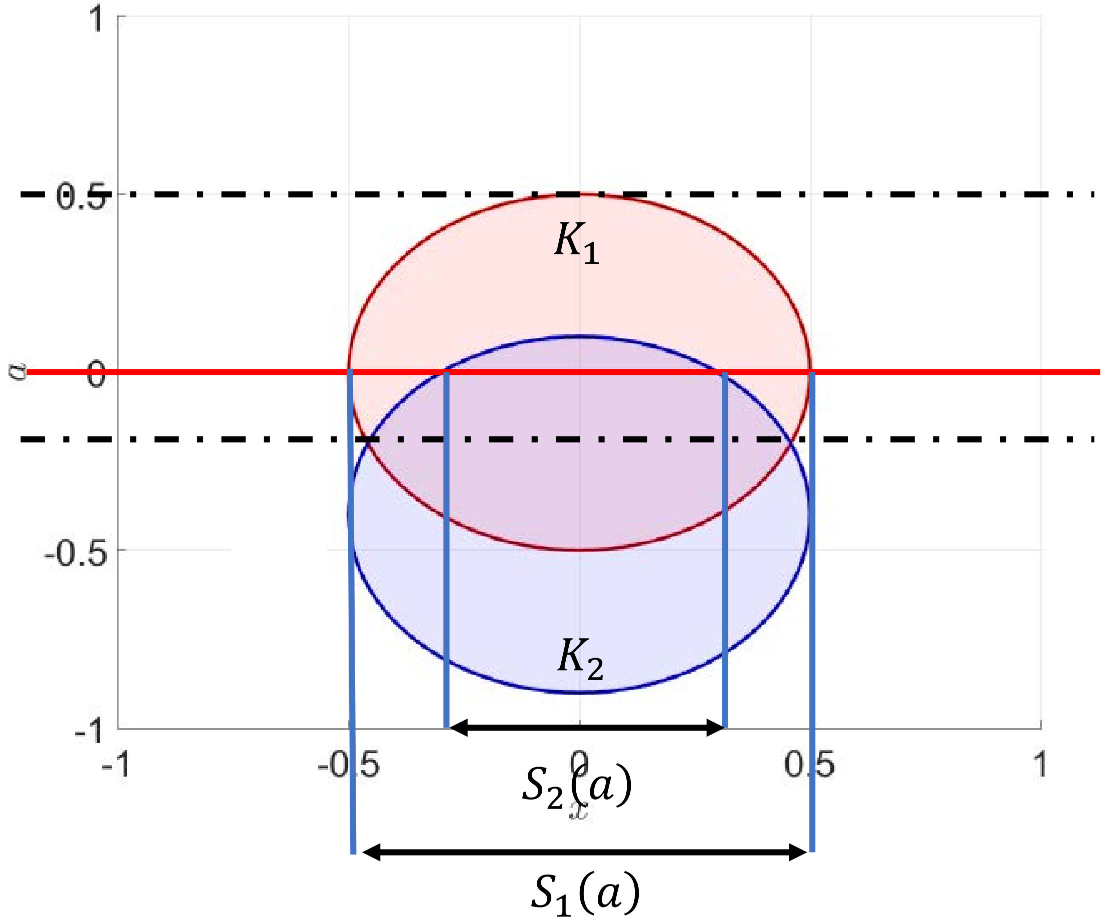
$\{a: \ S_1(a) \nsubseteq S_2(a)\}$

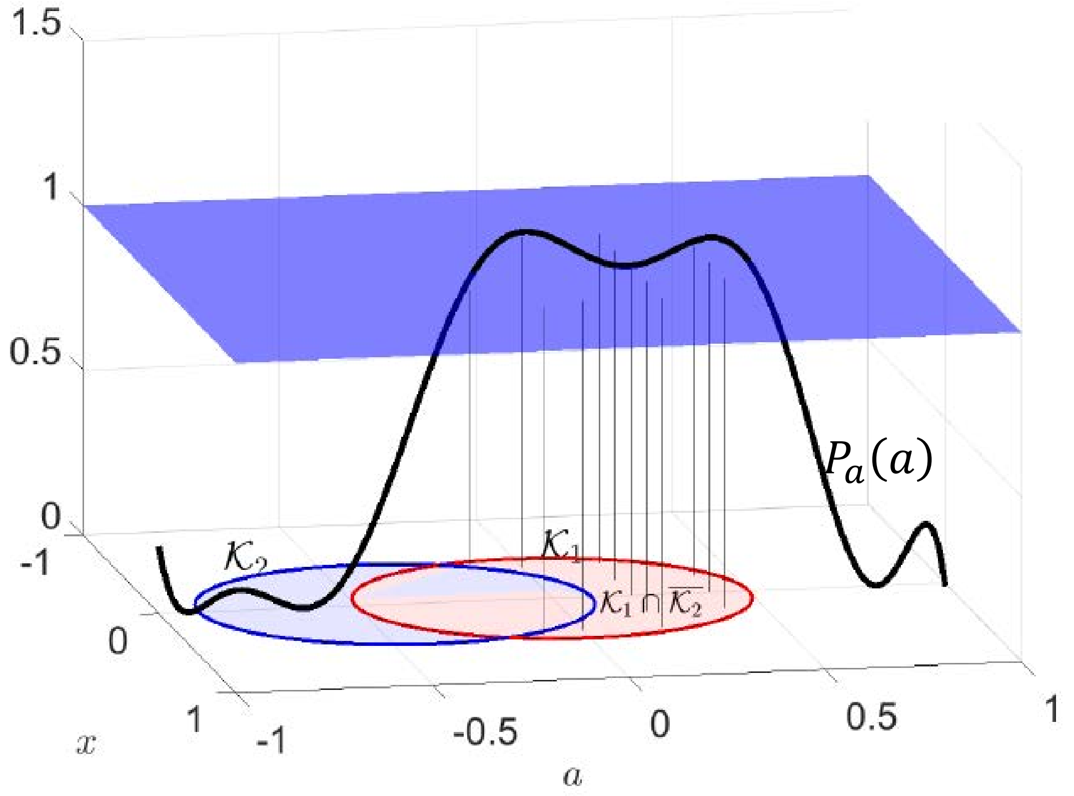


Example:

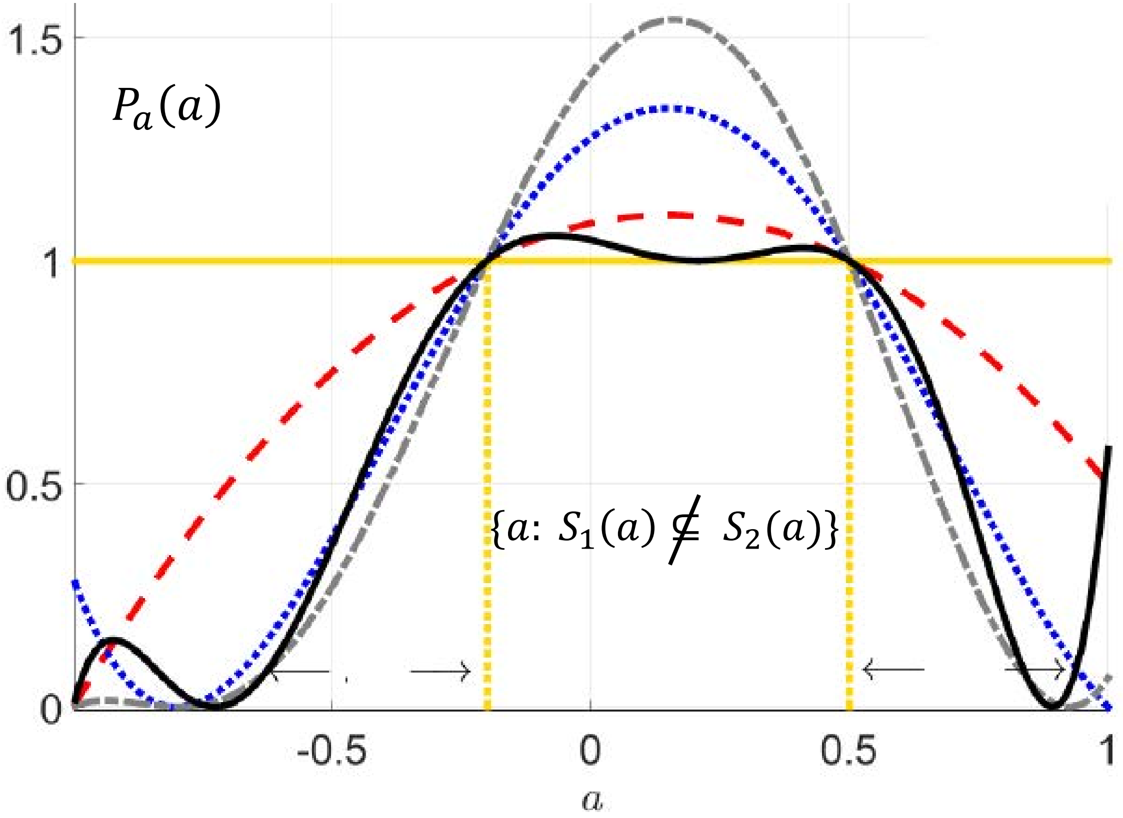
- Set 1 : $S_1(a) = \{x: 0.25 - a^2 - x^2 \geq 0\}$
- Set 2 : $S_2(a) = \{x: 0.09 - a^2 - 0.8a - x^2 \geq 0\}$

$\{a: \; S_1(a) \nsubseteq S_2(a)\}$





1.5
1
0.5
0
-1
0
1
x
-1
-0.5
0
0.5
1
a
$\mathcal{K}_2$
$\mathcal{K}_1$
$\mathcal{K}_1 \cap \overline{\mathcal{K}_2}$
$P_a(a)$



1.5
1
0.5
0
-0.5
0
0.5
1
a
$P_a(a)$
$\{a\colon S_1(a) \not\subseteq S_2(a)\}$

2) Solve volume maximization problem $\max_{a} vol(S_1(a))$

Subject to $\{a: P(a) < 1\}$

This is particular case of chance optimization, i.e., volume : probability with respect to Lebesgue measure

$$vol(S_1(a)) = \int_{S1} dx$$

LP in measure:

$$\max_{\mu_a, \mu} \int d\mu$$

Subject to $\mu \preccurlyeq \mu_a \times \mu_x$ ← Lebesgue measure

μ_a : probability measure

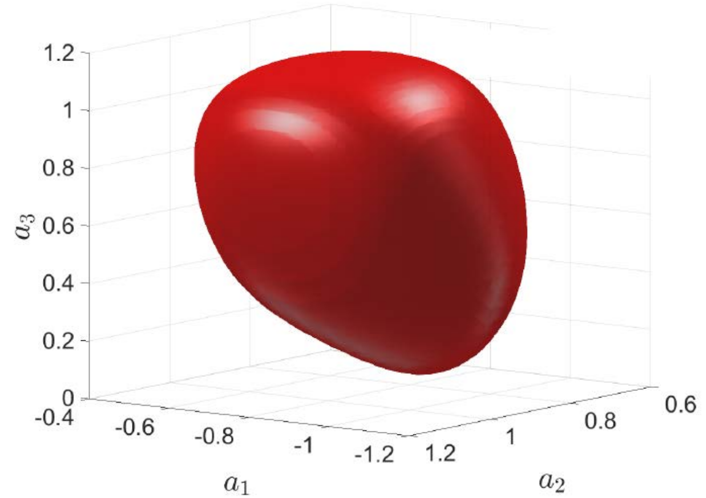
$supp(\mu_a) \subset \{a: P(a) < 1\}$

$supp(\mu) \subset K_1$

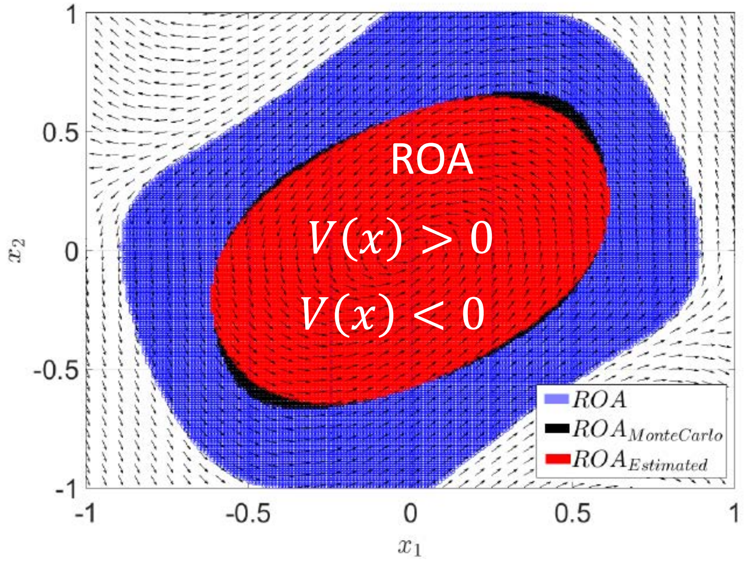
Moment SDP: moment representation of the measures.

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (4x_1^2 - 1)x_2$$

$$V(x) = 3\|x\|_2^2 + 3a_1x_1x_2 + 3a_2x_1^3x_2 + 3a_3x_1x_2^3$$



Step 1: Set of all (a_1, a_2, a_3) that makes function $V(x)$ a Lyapunov function.



Step 2: volume maximization problem

$$(a_1^*, a_2^*, a_3^*) = -0.999362, 0.853458, 0.132566$$

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Topics:

- SOS optimization for Robust Control
- Lyapunov Stability and SOS optimization
- Barrier Function based Safety and SOS optimization
- Region of attraction Set Estimation and Design
- Invariant Set Estimation and Deign
- Funnel Based Robust Control
- Reachable Sets
- Constrained Volume Optimization