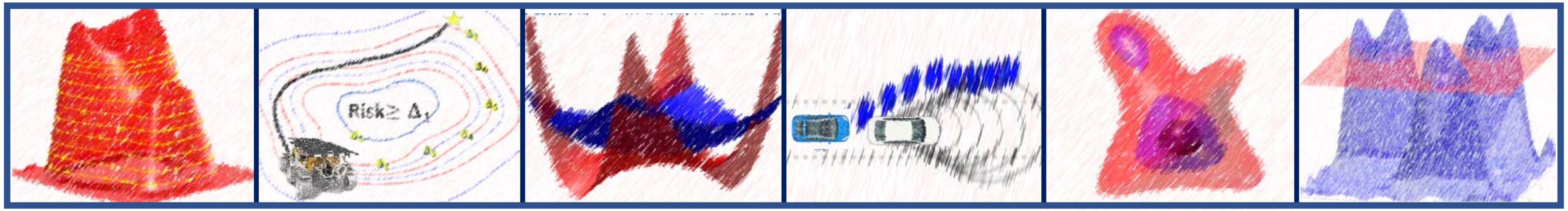
Lecture 1

Risk Aware and Robust Nonlinear Planning

Introduction and Course Overview

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

Ashkan Jasour

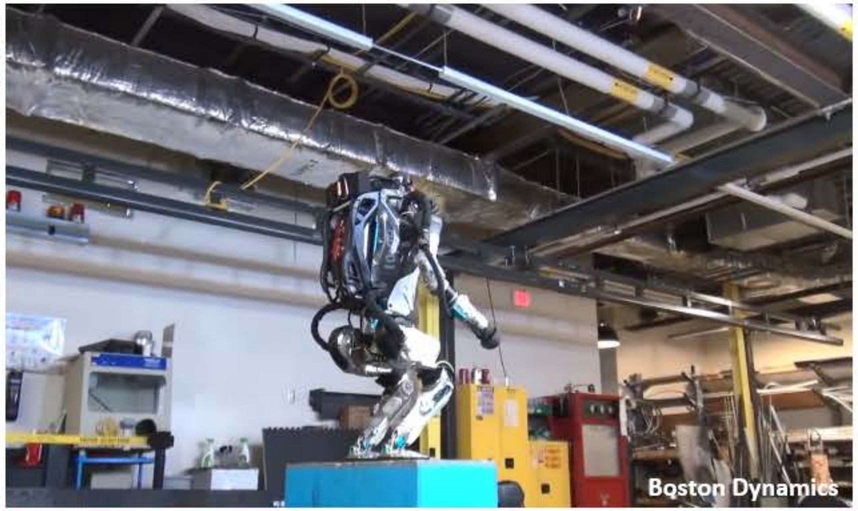


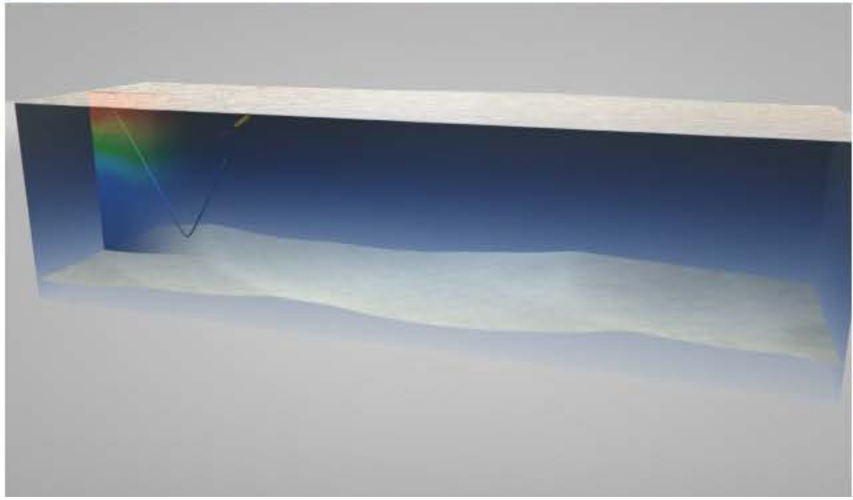
Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications

Introduction to Planning Under Uncertainty

Planning For Autonomous Systems

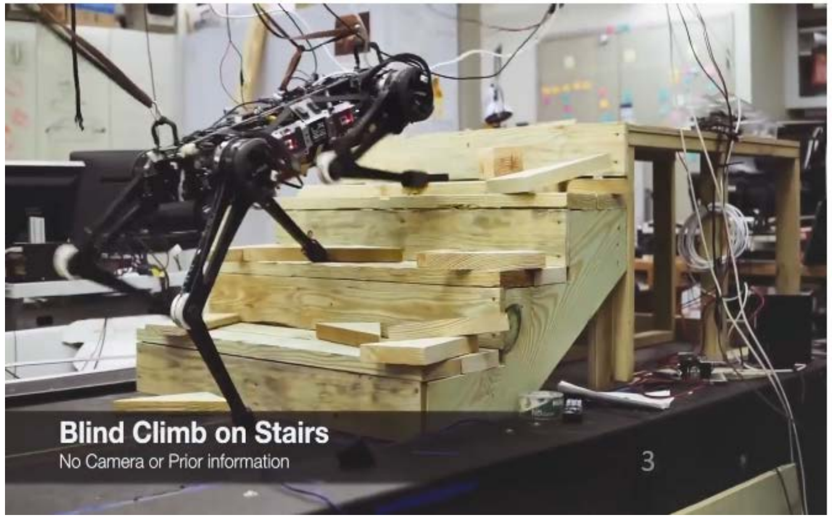




Source: public domain/NASA

Challenge: Uncertainty

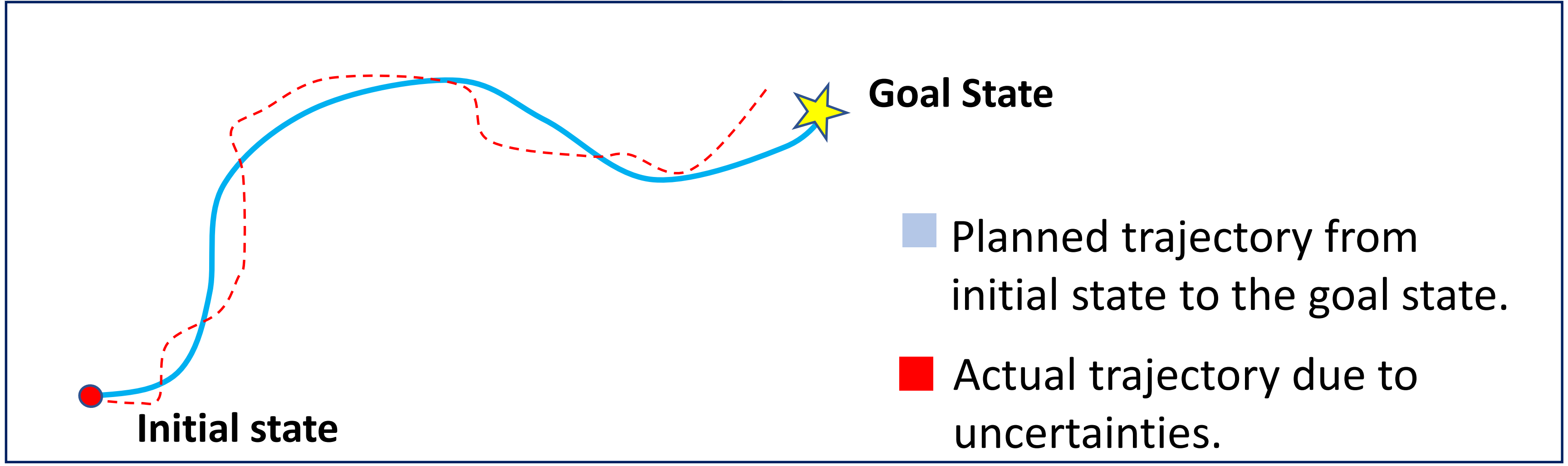
- **Planning under Uncertainty**: Planning in presence of imperfect or unknown information.



- **Due to uncertainty,** it is impossible to exactly describe the *"current situation"* or *"future behavior"* of the systems/environment.

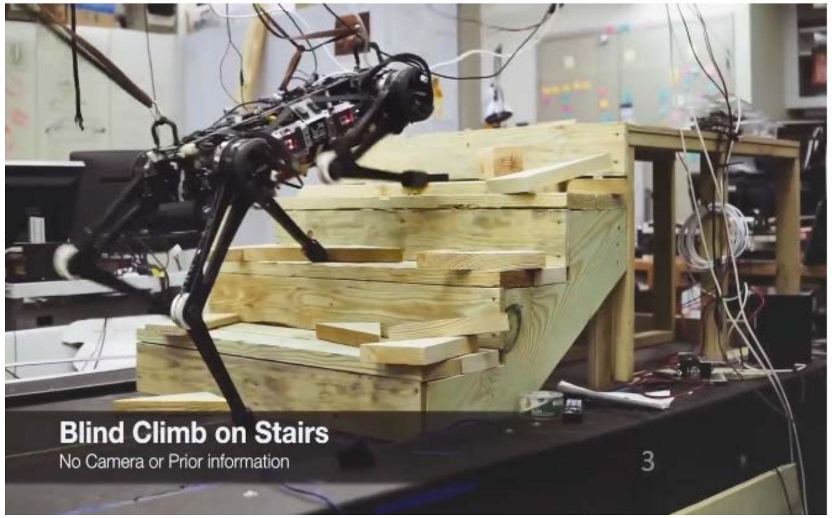
Challenge: Uncertainty

- **Planning under Uncertainty**: Planning in presence of imperfect or unknown information.



🟦 Planned trajectory from initial state to the goal state.

🟥 Actual trajectory due to uncertainties.



Source of Uncertainty

1. Environment:

i) **Sensor Noise**

e.g., localizing obstacles or the robot

ii) **Control Disturbance**

e.g., wind disturbances

iii) **Unmodeled Environment**

e.g., rough train

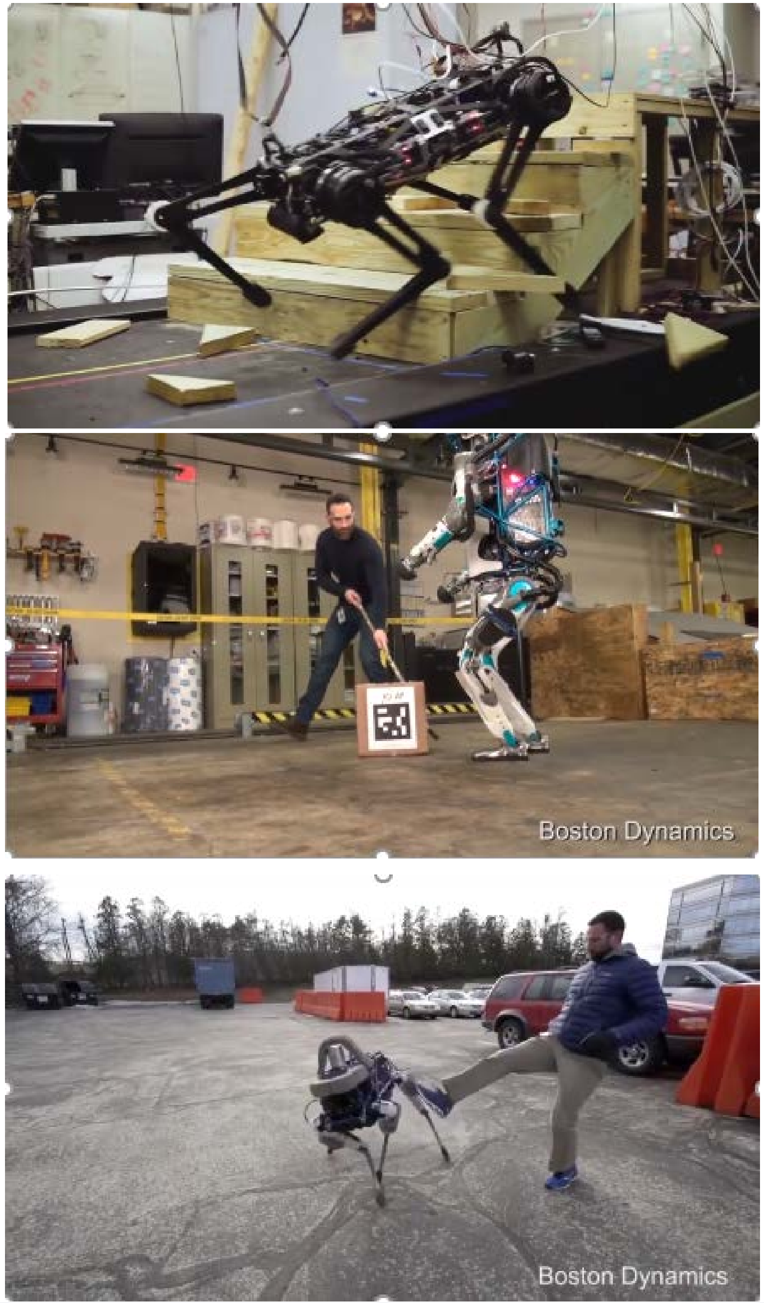
iii) **Intention**

e.g., future behavior of other agents (dynamic environment)

2. System:

i) **Imperfect system model**

e.g., unknow parameters of system model
unmodeled dynamics (linear model for nonlinear systems)



How to Deal with Uncertainty ?

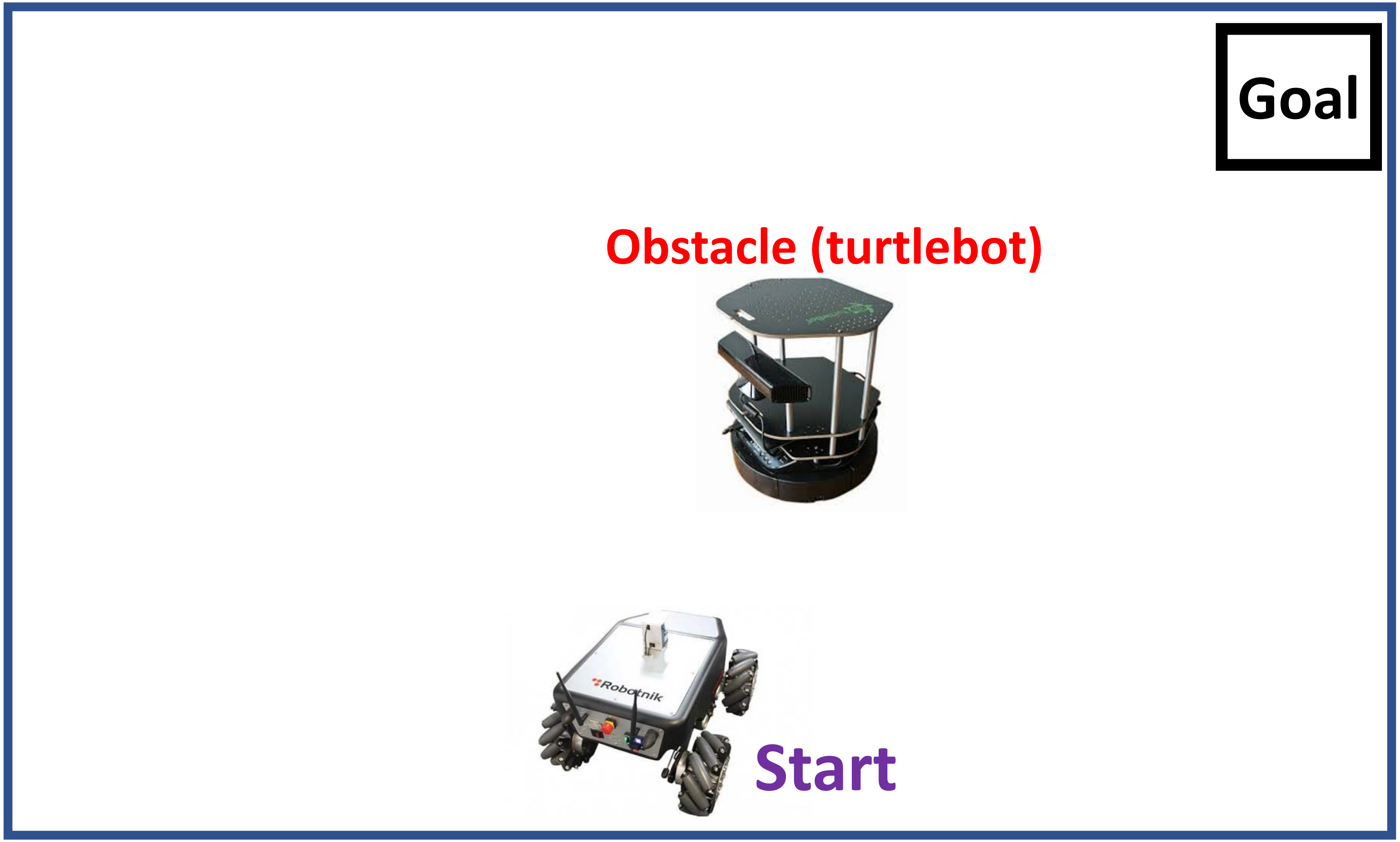
How to Deal with Uncertainty ?

- **Robust Approaches**
- **Risk Bounded Approaches**

Robust Approaches:

- Plan should be valid for all possible realization of uncertainty
- Look at the Uncertainty Set (Range of Uncertainty)

Motion Planning

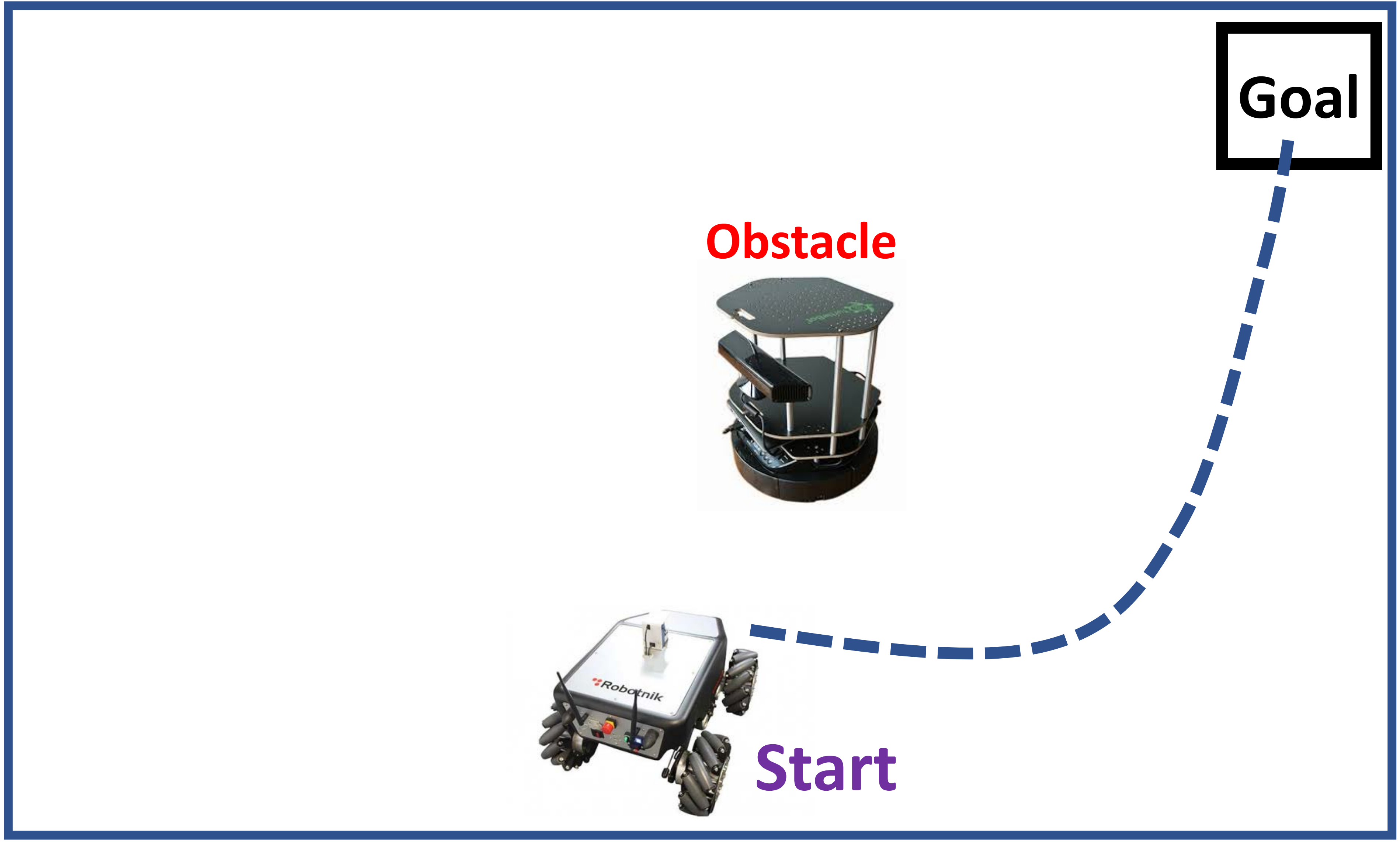


Goal

Obstacle (turtlebot)

Start

Motion Planning

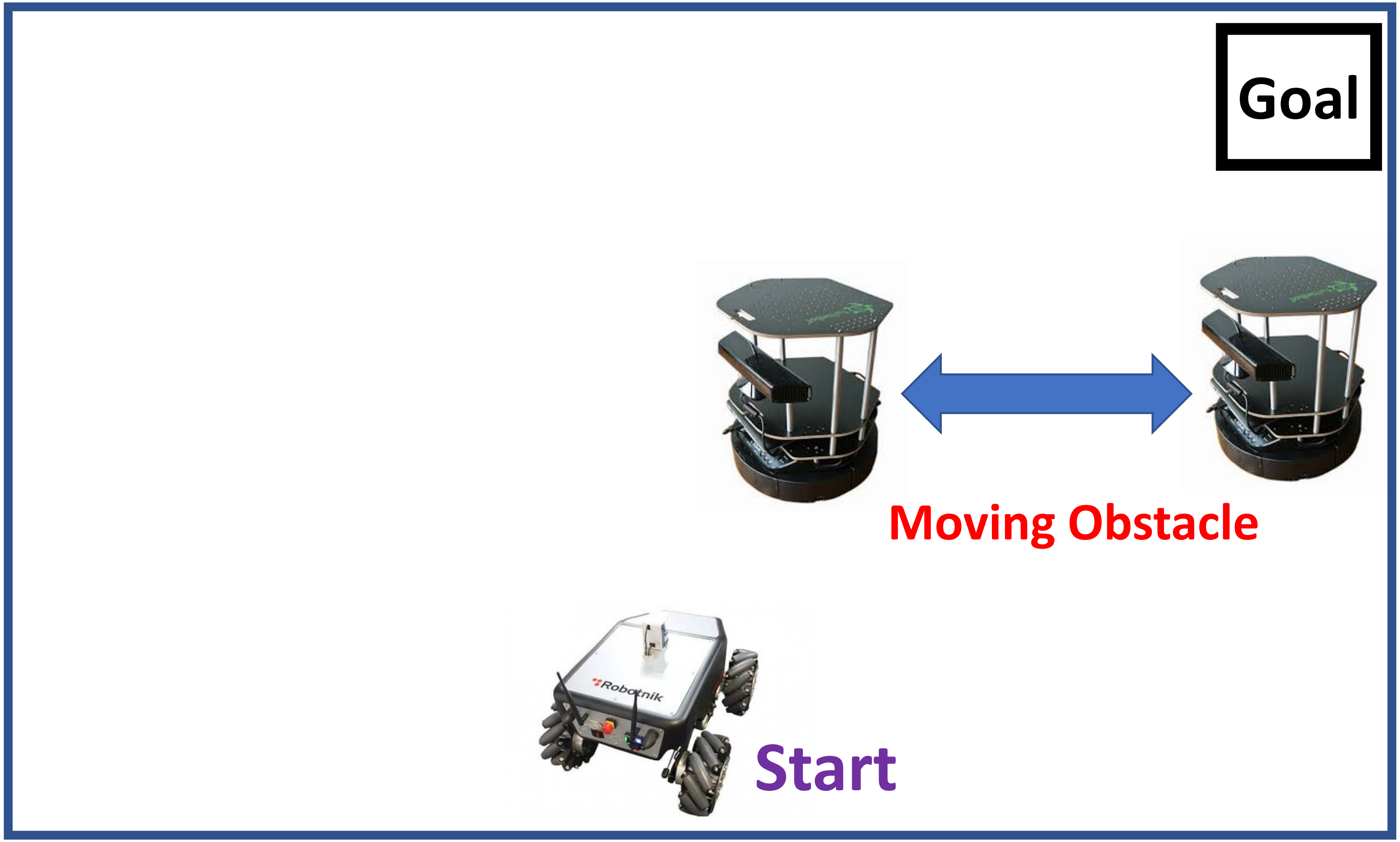


Goal

Obstacle

Start

Motion Planning Under Uncertainty:

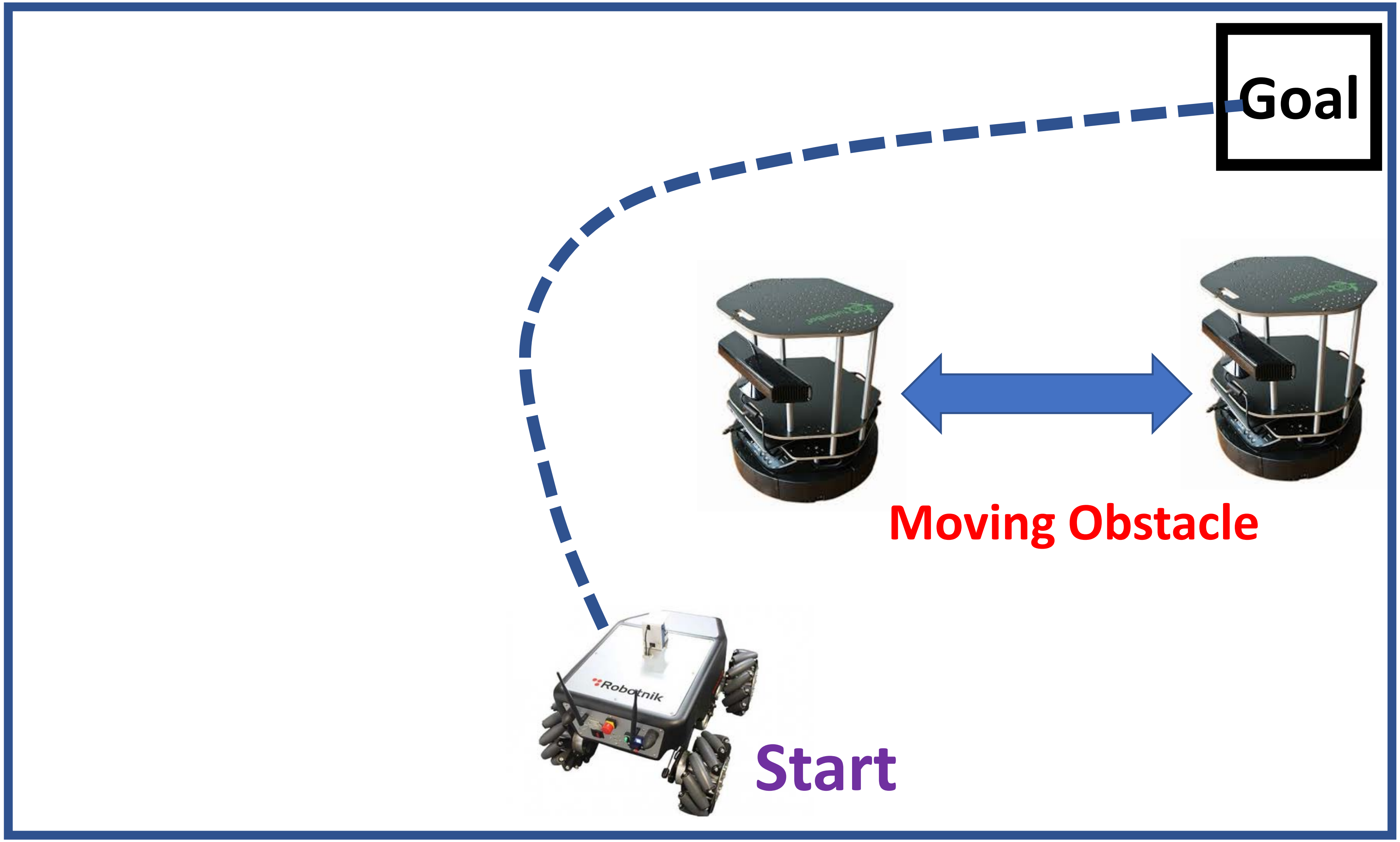


Goal

Moving Obstacle

Start

Motion Planning Under Uncertainty:



Motion Planning Under Uncertainty:

Goal

Moving Obstacle

Start

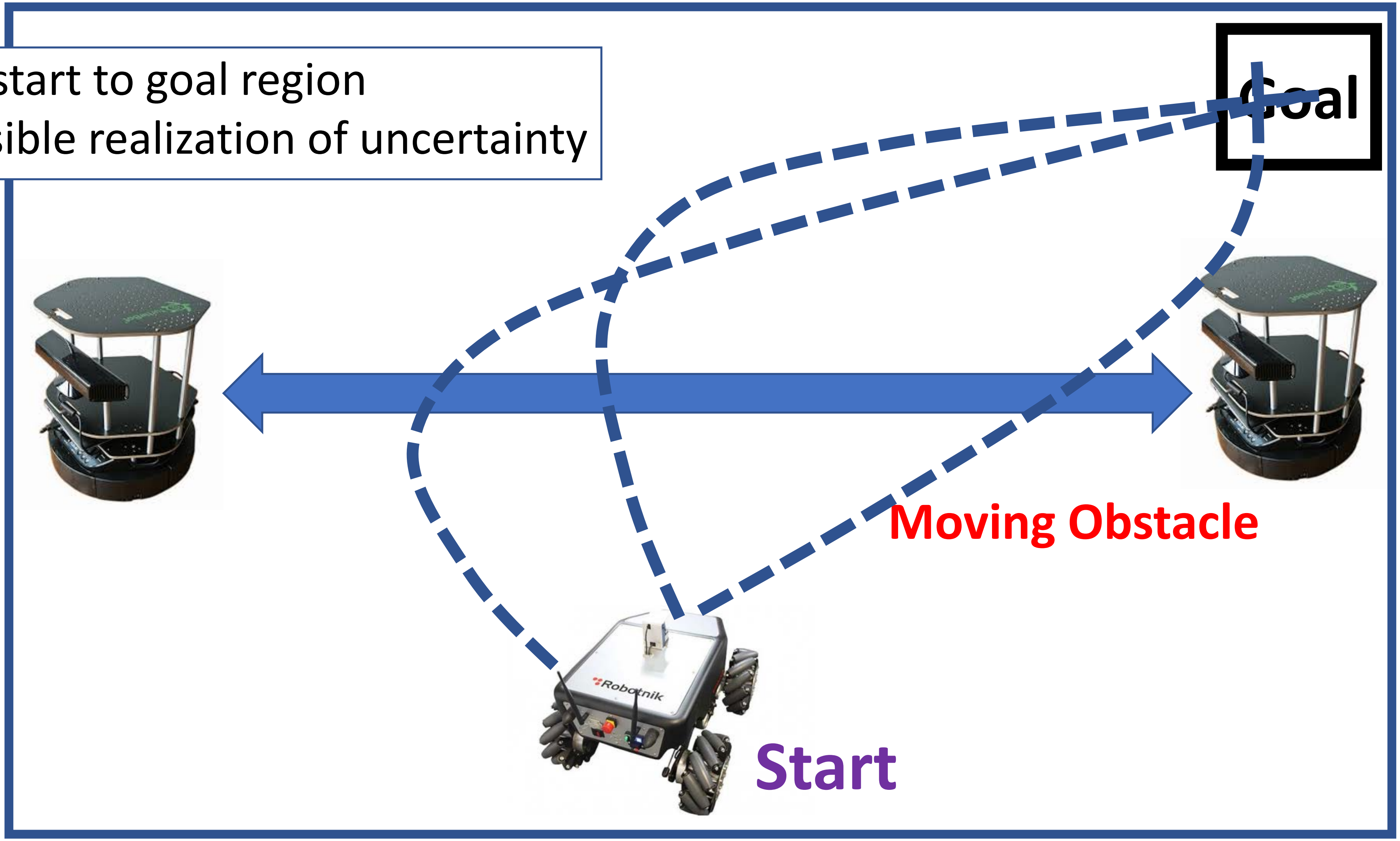
Motion Planning Under Uncertainty:

There is no path from start to goal region
that is valid for all possible realization of uncertainty

Robust approach

↓

Conservative solution



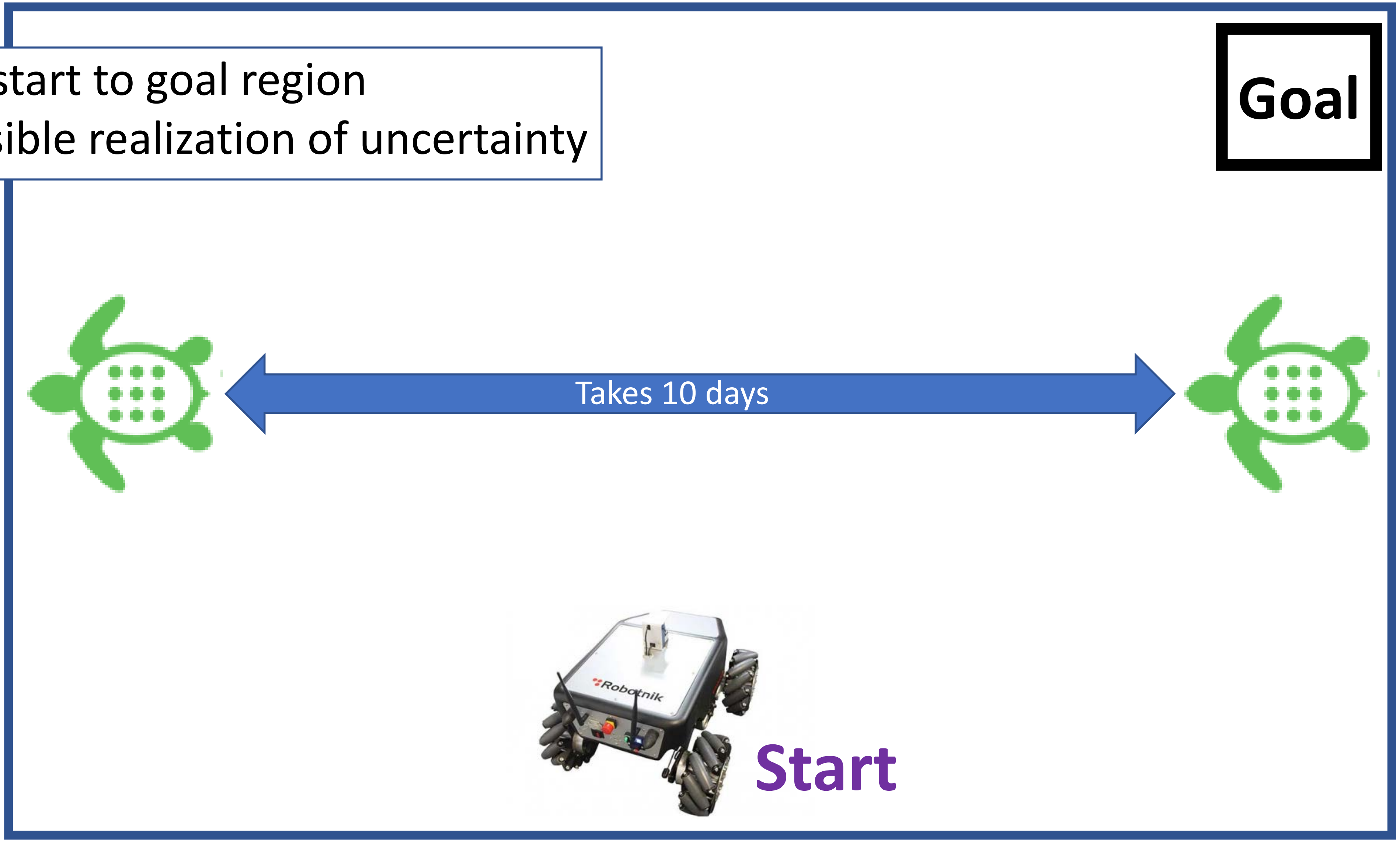
Motion Planning Under Uncertainty:

There is no path from start to goal region
that is valid for all possible realization of uncertainty

Robust approach

↓

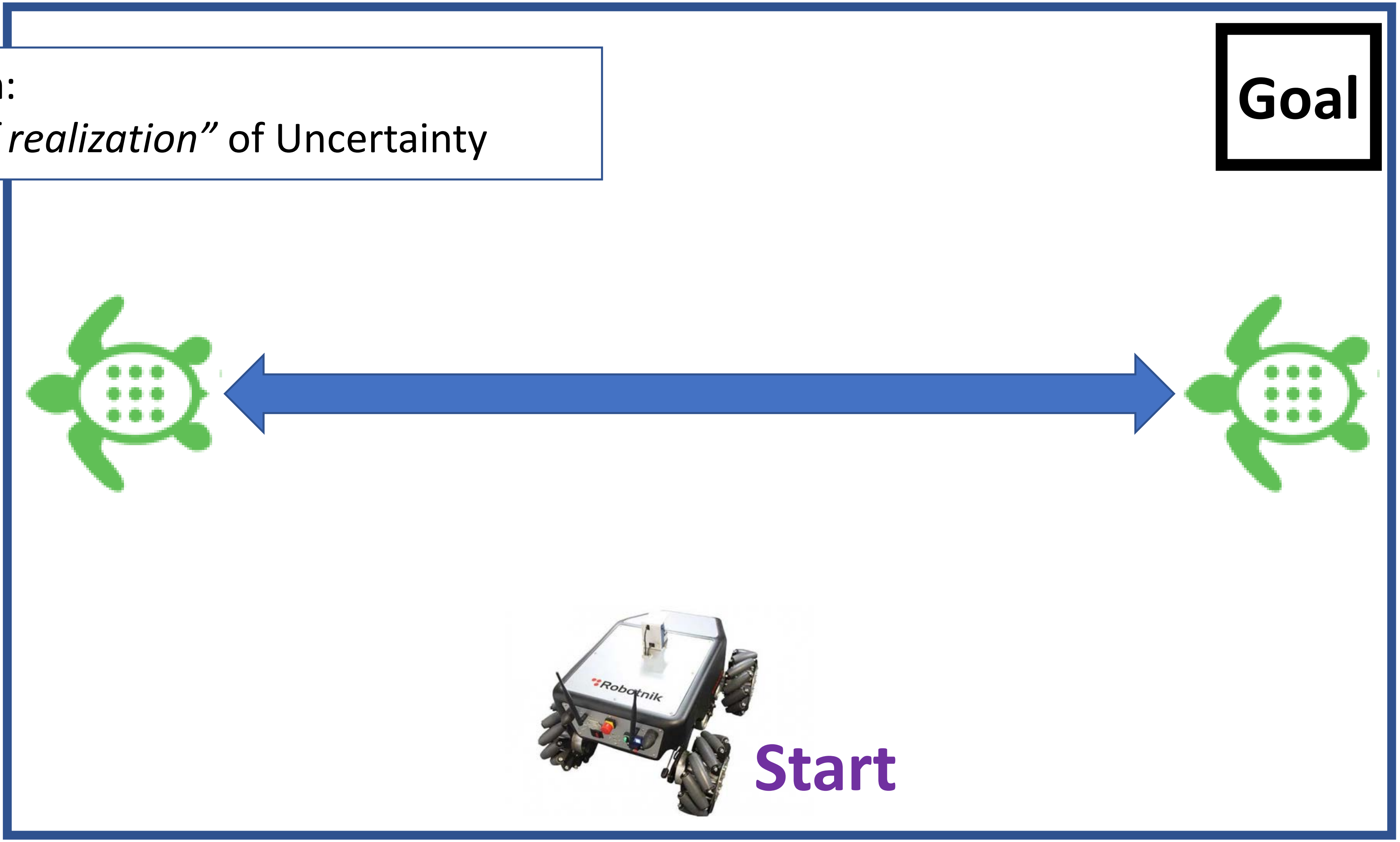
Conservative solution



Motion Planning Under Uncertainty:

Risk Bounded Approach:

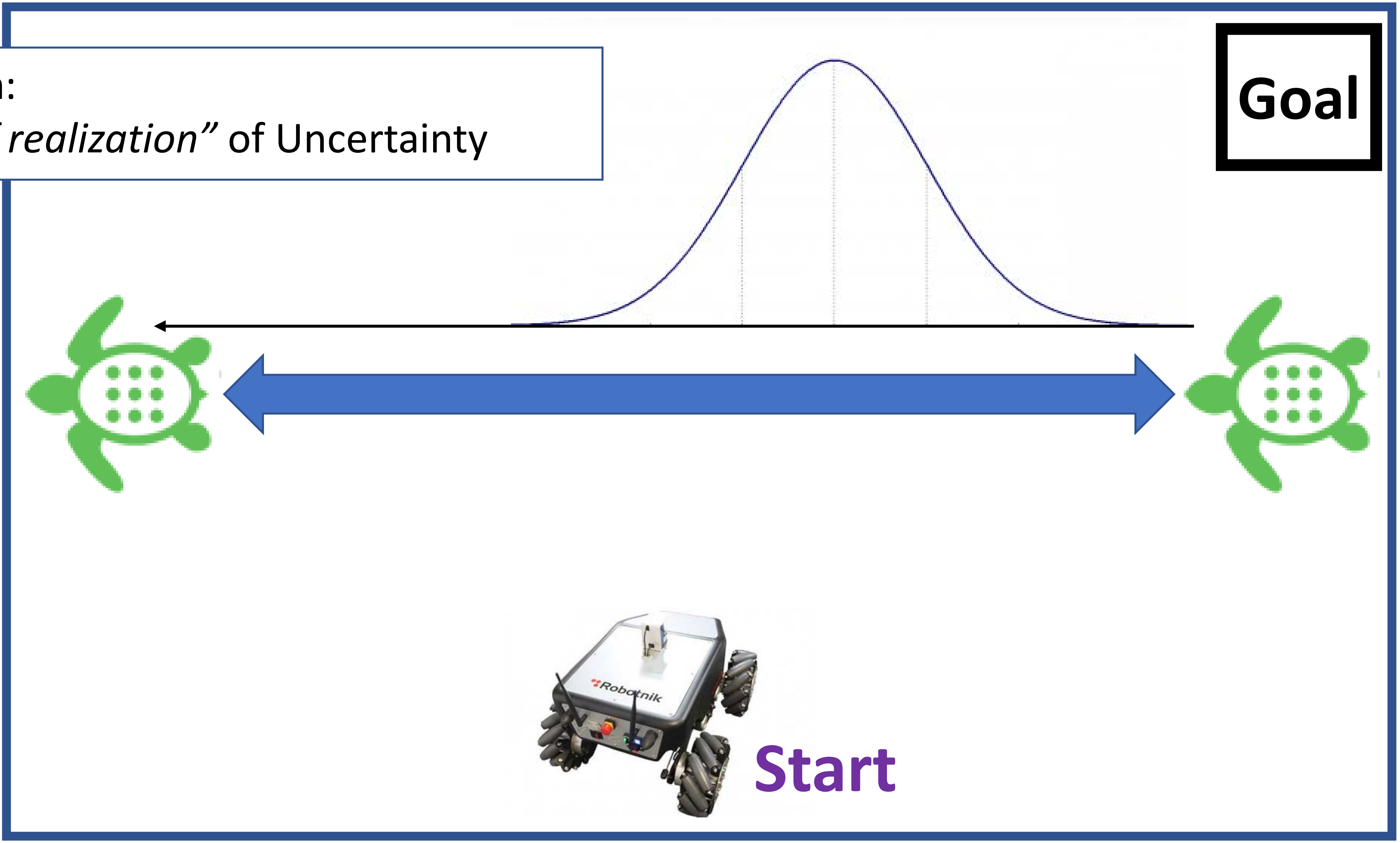
- Look at “*frequency of realization*” of Uncertainty



Motion Planning Under Uncertainty:

Risk Bounded Approach:

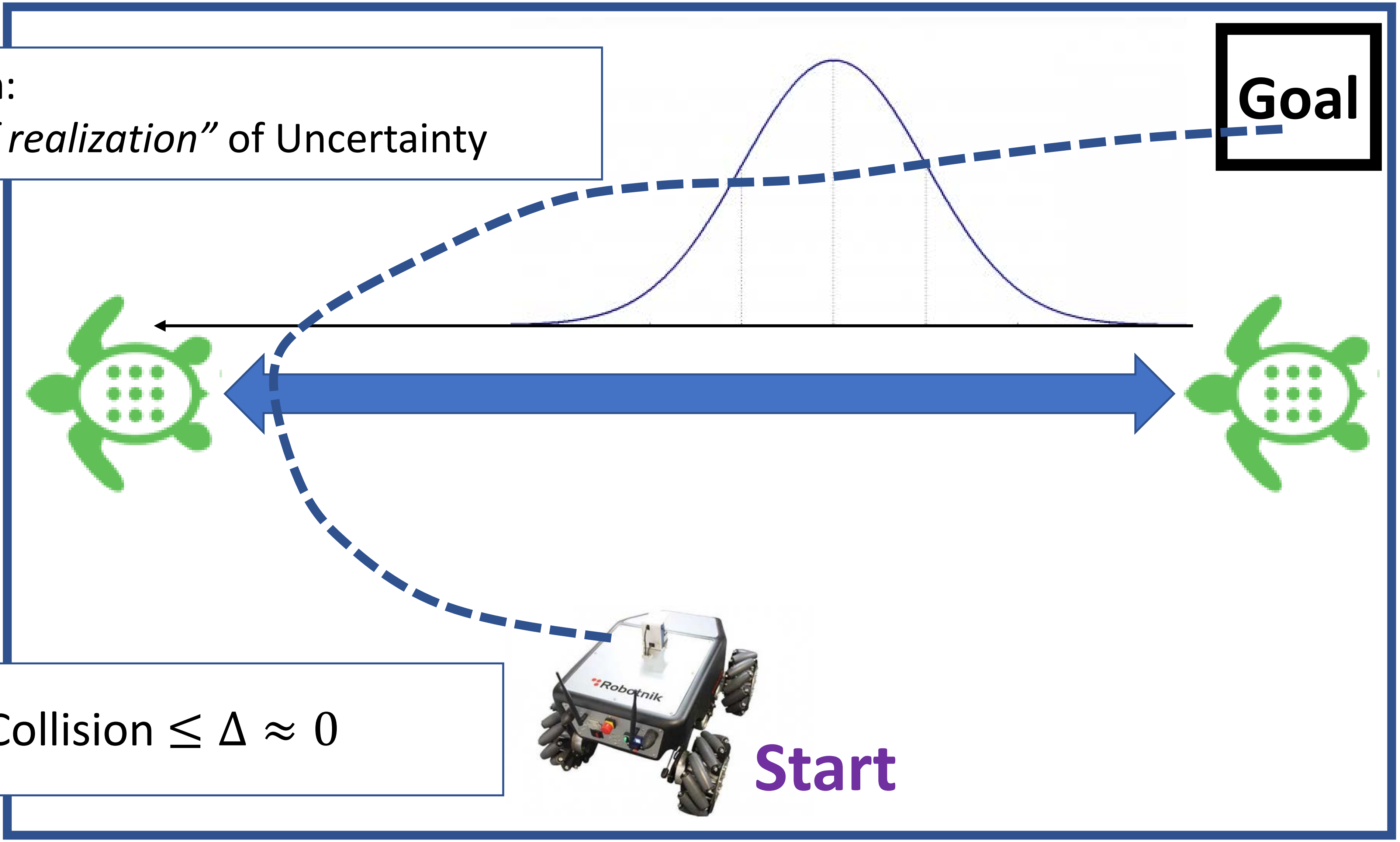
- Look at *"frequency of realization"* of Uncertainty



Motion Planning Under Uncertainty:

Risk Bounded Approach:

- Look at *"frequency of realization"* of Uncertainty



Probability of Collision $\leq \Delta \approx 0$

Robust Approaches

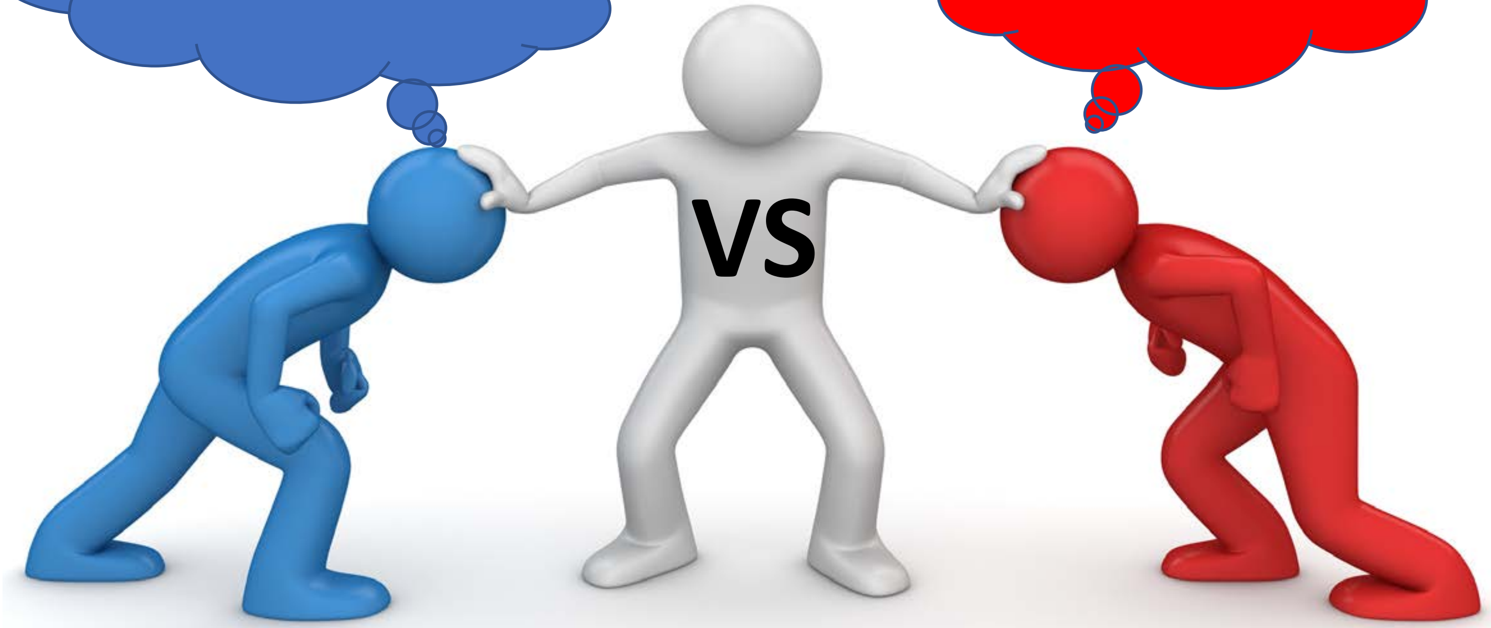
- Plan should be valid for all possible realization of uncertainty
- Look at the Uncertainty Set (Range of Uncertainty)

Risk Bounded Approaches

- Plan should be valid with high probability.
- Look at “*frequency of realization*” of Uncertainty (Probability Distribution)

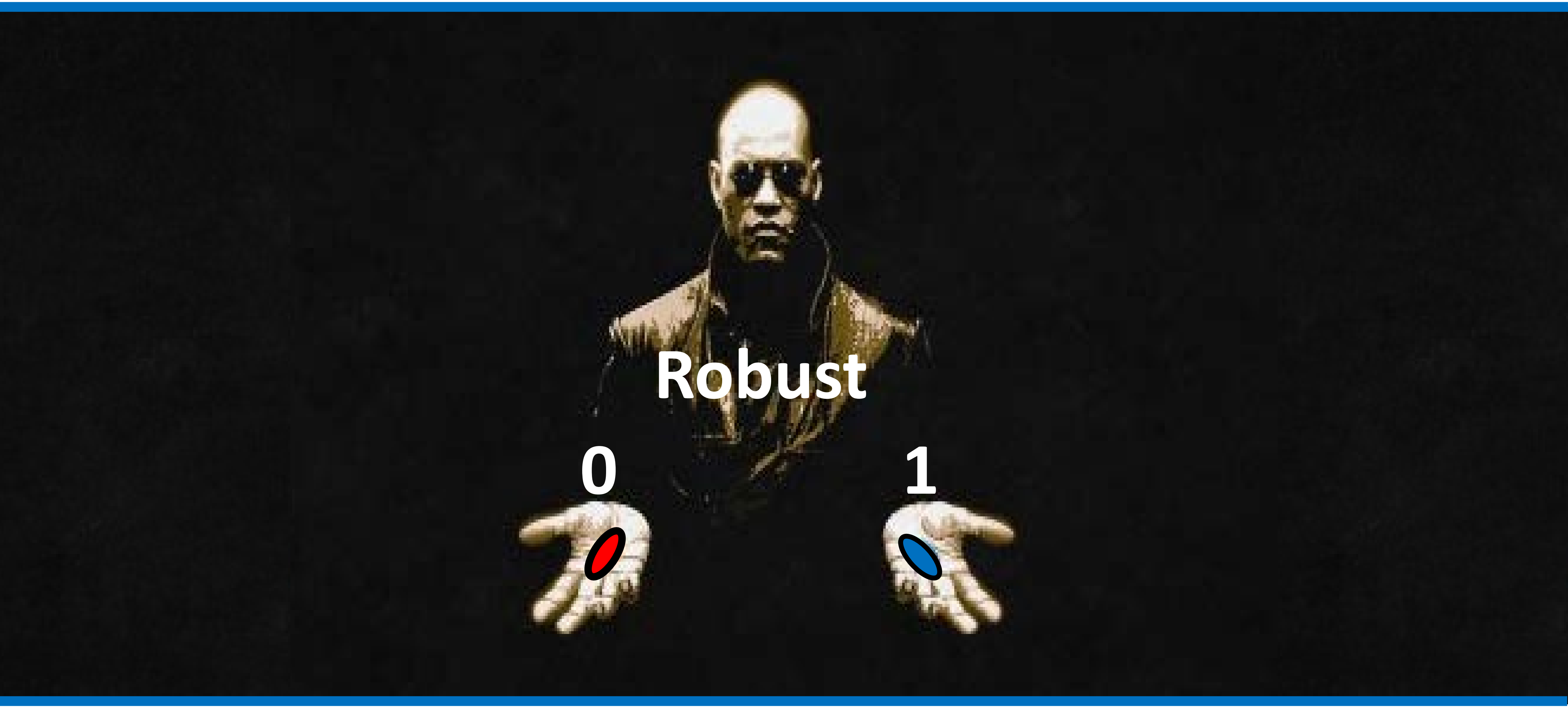
Risk Bounded

Robust



Probability

Possibility


Robust
0
1

Risk Bounded

Topics:

- Introduction to Planning Under Uncertainty
- **Approaches and Challenges**
- Technical Idea and Mathematical Tools
- Applications

Optimization Based Planning

Optimization Based Planning

$$\begin{array}{ll} \underset{\text{design parameters}}{\text{minimize}} & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} & \text{Constraints}(\text{design parameters}) \end{array}$$

Objective function: cost of execution

Constraints: safety constraints, resource constraints, dynamical constraints, temporal constraints

Example: Trajectory Planning

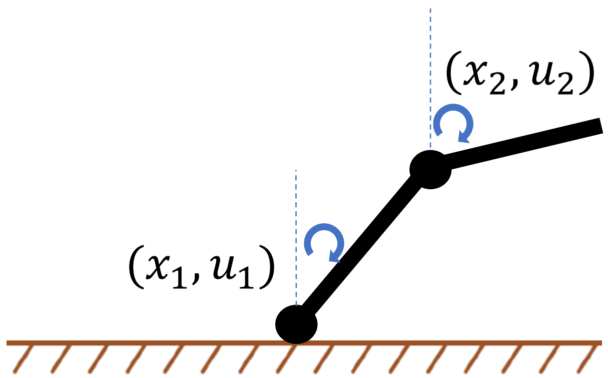
Dynamical Systems

Continuous State space model

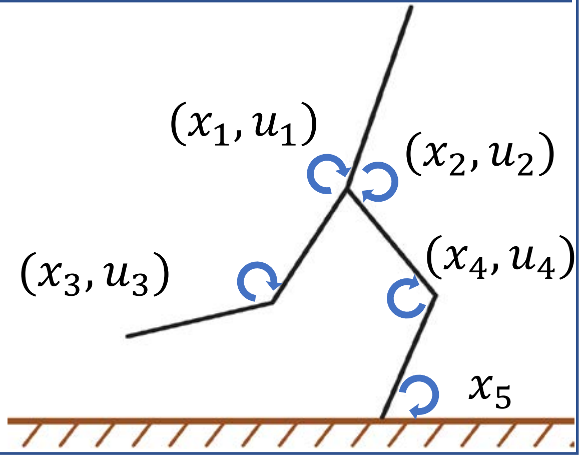
$$x_{k+1} = f(x_k, u_k)$$

states inputs

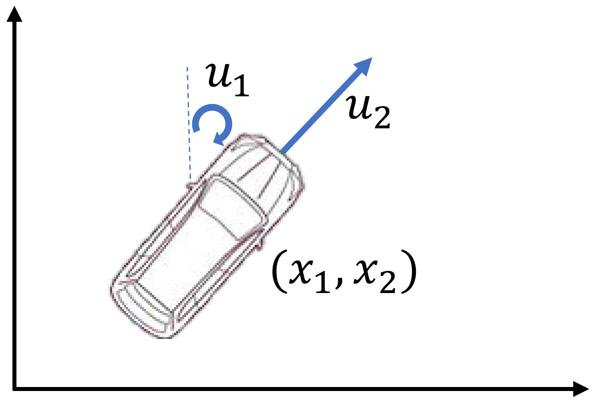
x: joint angles and angular velocities
u: Torque motor



x: joint angles and angular velocities
u: Torque motor



x: position and velocity
u: Steering angle, Torque



Example: Trajectory Planning

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$ to derive the robot to the goal point.

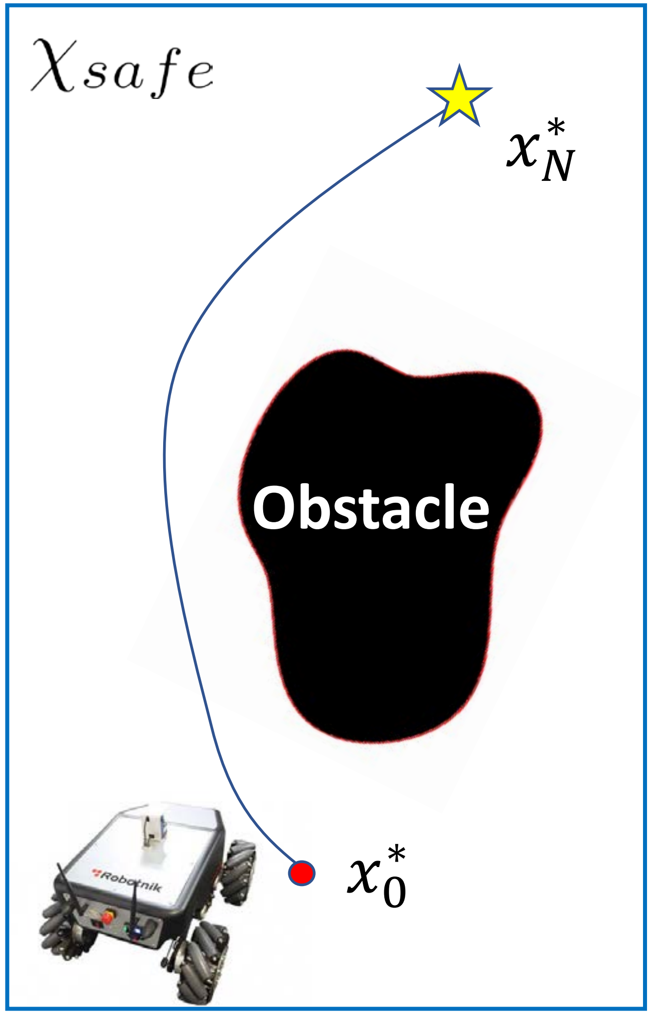
$$\underset{u_k|_{k=0}^{N-1}}{\text{minimize}} \quad \sum_{k=0}^{N-1} u^2(k) \longrightarrow \text{Control cost}$$

$$\text{subject to} \quad x_0 = x_0^*, x_N = x_N^* \longrightarrow \text{Boundary conditions}$$

$$x_{k+1} = f(x_k, u_k) \longrightarrow \text{Dynamical constraints}$$

$$x_k \in \mathcal{X}_{safe} \longrightarrow \text{Safety constraints}$$

$$u_k \in \mathcal{U} \longrightarrow \text{Resource constraints}$$



Optimization Based Planning

$$\underset{\text{design parameters}}{\text{minimize}} \quad \text{Objective Function}(\text{design parameters})$$
$$\text{subject to} \quad \text{Constraints}(\text{design parameters})$$

Mathematical Formulation

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad g_i(x) \geq 0, \;\; i = 1, \ldots, n_g$$

$$p(x) : \mathbb{R}^n \rightarrow \mathbb{R}, \;\; g_i(x) : \mathbb{R}^n \rightarrow \mathbb{R} \;\; i = 1, \ldots, n_g$$

Optimization Based Planning Under Uncertainty

- Robust Optimization
- Risk Aware Optimization, i.e., Chance optimization and Chance Constrained Optimization
- Distributionally Robust Optimization

Robust Optimization Based Planning

$$\underset{\text{design parameters}}{\text{minimize}} \quad \text{Objective Function(design parameters)}$$

$$\text{subject to} \quad \text{satisfy constraints for all possible valuse of uncetainties}$$

Mathematical Formulation

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x, \underset{\underset{\text{Uncertainty}}{\downarrow}}{\omega}) \geq 0, \;\; i = 1, \ldots, n_g, \quad \forall \omega \in \underset{\underset{\text{Uncertainty Set}}{\downarrow}}{\Omega} \end{aligned}$$

Example: Robust Trajectory Optimization

Uncertain Dynamical Systems and Uncertain Safety Constraints

Continuous State space model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states

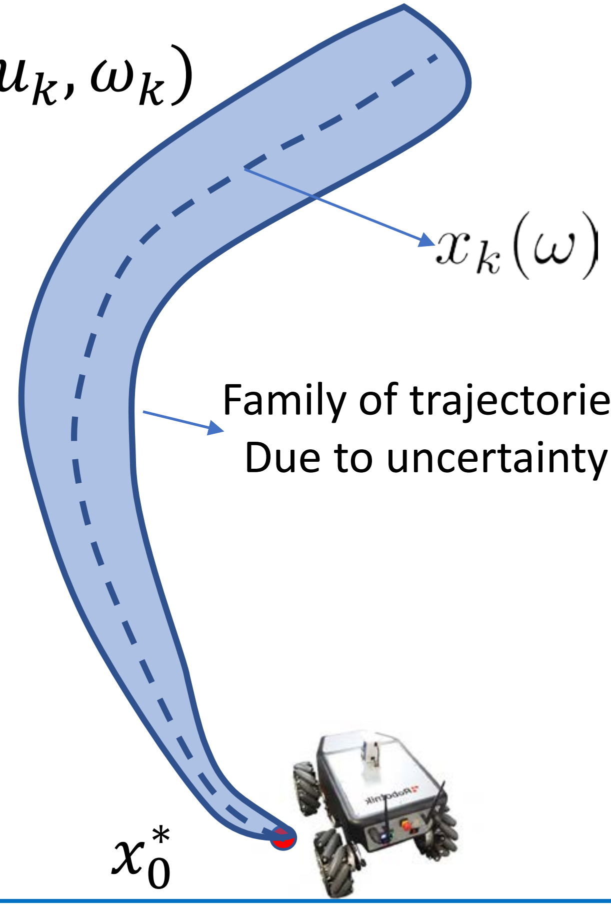
inputs

Uncertainty, $\omega_k \in \Omega$

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$$\omega_k \in \Omega$$

Uncertainty Set

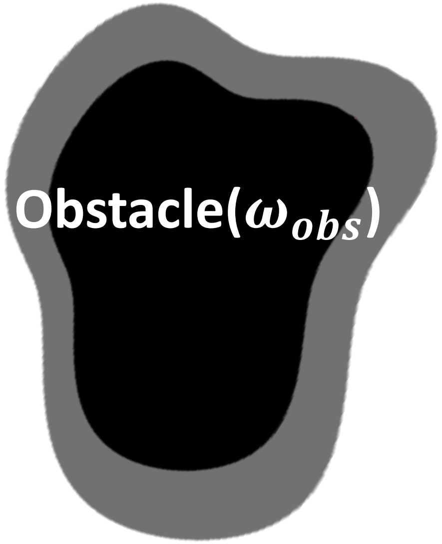


Uncertain Obstacle

$$g(x, \omega_{obs}) \geq 0$$

Uncertainty parameters of the obstacle

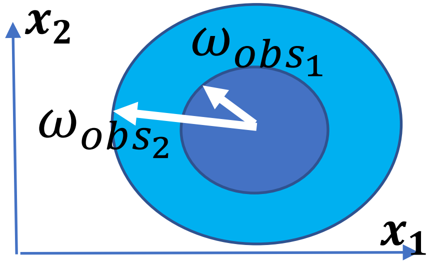
$\omega_{obs} \in \Omega_{\text{obs}}$



Example: Uncertain circular obstacle

$$\omega_{obs}^2 - x_1^2 - x_2^2 \geq 0$$

$$\omega_{obs} \in [\omega_{obs_1}, \omega_{obs_2}]$$

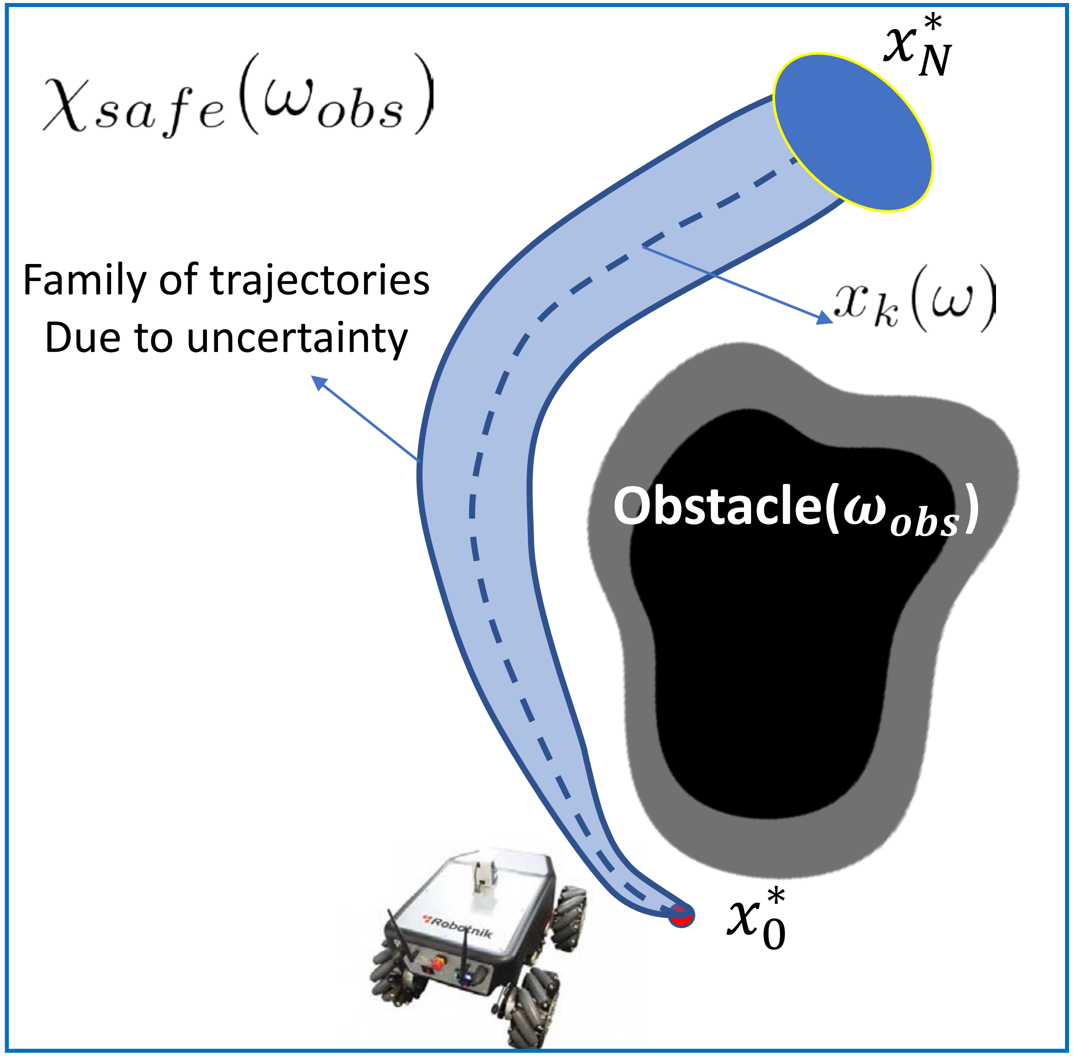


Example: Robust Trajectory Optimization

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$ to derive the robot to the goal region in the presence of uncertainties.

$$\underset{u_k|_{k=0}^{N-1}}{\text{minimize}} \quad \sum_{k=0}^{N-1} u^2(k)$$

$$\begin{aligned}
\text{subject to} \quad & x_0 = x_0^*, x_N \in x_N^* \\
& x_{k+1} = f(x_k, u_k, \omega_k) \\
& \boxed{\begin{aligned} & x_k \in \chi_{safe}(\omega_{obs}) \\ & \forall \omega_{x_k} \in \Omega_x, \; \forall \omega_{obs} \in \Omega_{obs} \end{aligned}} \\
& u_k \in \mathcal{U}
\end{aligned}$$



Robust Optimization Based Planning

$$\begin{aligned}&\underset{\text{design parameters}}{\text{minimize}} && \text{Objective Function}(\text{design parameters})\\ &\text{subject to} && \text{satisfy constraints for all possible valuse of uncetainties}\end{aligned}$$

Mathematical Formulation

$$\begin{aligned}&\underset{x\in\mathbb{R}^n}{\text{minimize}} && p(x)\\ &\text{subject to} && g_i(x,\underset{\substack{\downarrow\\ \text{Uncertainty}}}{\omega}) \geq 0,\ i=1,...,n_g, \quad \forall\omega\in\underset{\substack{\downarrow\\ \text{Uncertainty Set}}}{\Omega}\end{aligned}$$

3. Risk Aware Optimization Based Planning

Chance Optimization

$$\begin{array}{ll} \underset{\text{design parameters}}{\text{maximize}} & \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \\ \text{subject to} & \text{Constraints}(\text{design parameters}) \end{array}$$

Chance Constrained Optimization

$$\begin{array}{ll} \underset{\text{design parameters}}{\text{minimize}} & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} & \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta \end{array}$$

Acceptable risk level

Example: Chance Constrained Trajectory Optimization

Probabilistic Dynamical Systems and Probabilistic Safety Constraints

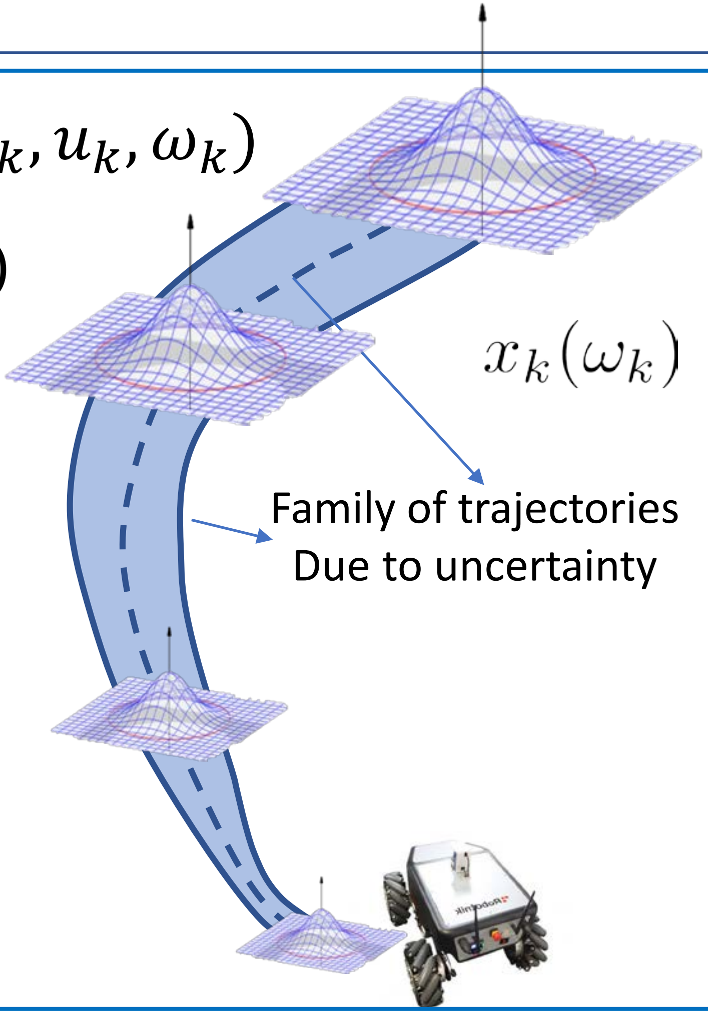
Continuous State space model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states inputs Uncertainty ~ $\text{pr}(\omega_k)$:probability distribution

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$$\omega_k \sim \text{pr}(\omega_k)$$

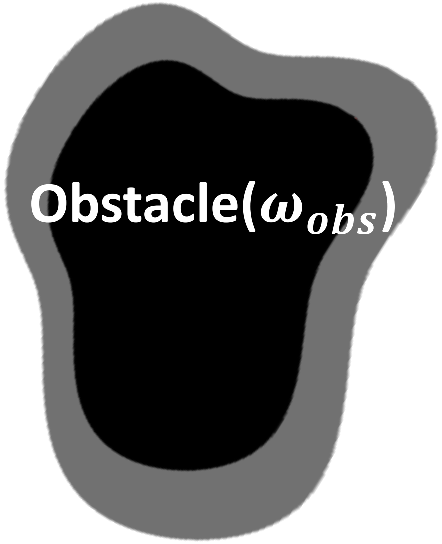


Family of trajectories
Due to uncertainty

Probabilistic Obstacle

$$p(x, \omega_{obs}) \geq 0$$

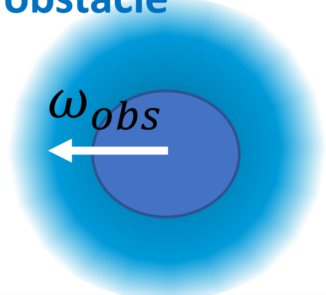
$$\omega_{obs} \sim \text{pr}(\omega_{obs})$$



Example: probabilistic circular obstacle

$$\omega_{obs}^2 - x_1^2 - x_2^2 \geq 0$$

$$\omega_{obs} \sim \text{pr}(\omega_{obs})$$

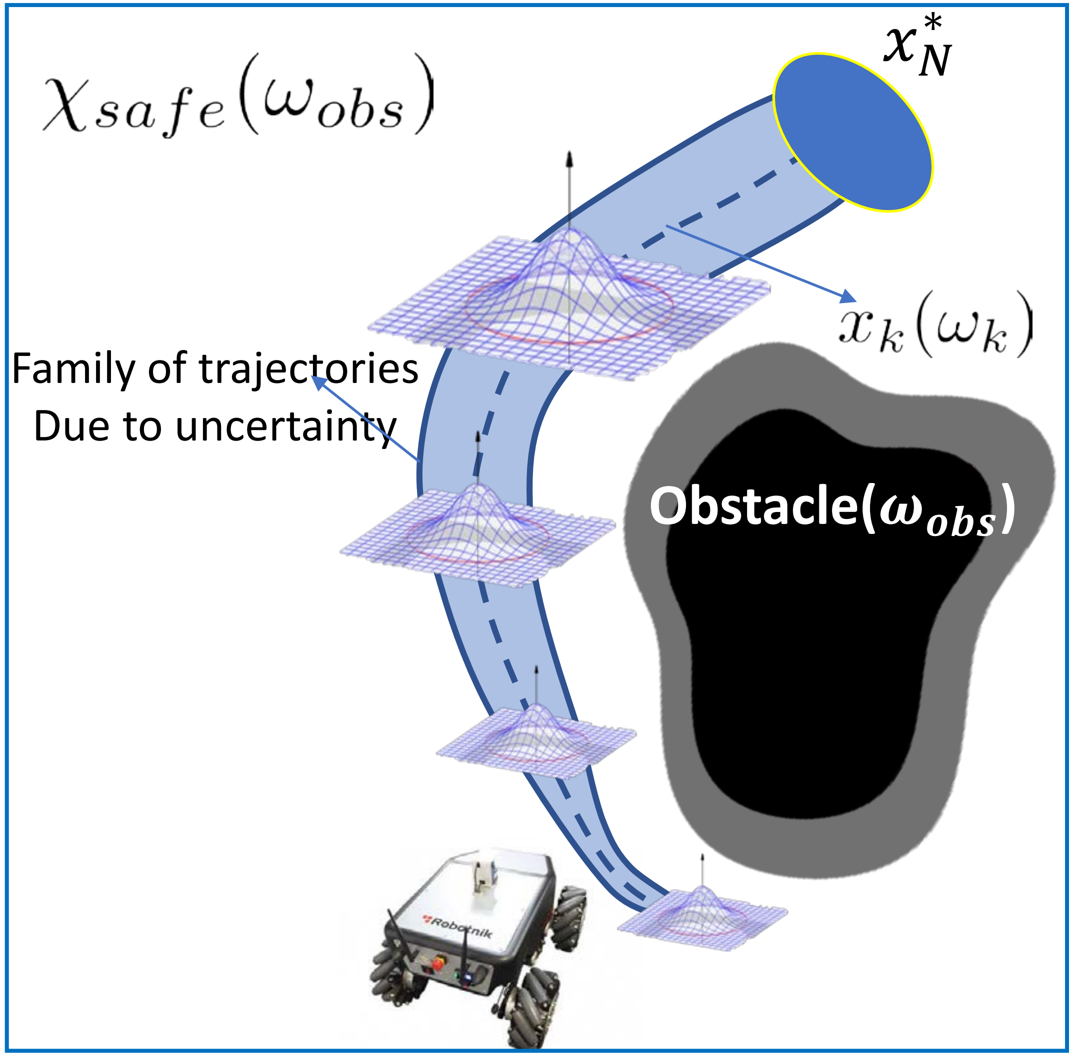


Example: Chance Constrained Trajectory Optimization

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$ to derive the robot to the goal region in the presence of probabilistic uncertainties.

$$
\begin{aligned}
\underset{u_k|_{k=0}^{N-1}}{\text{minimize}} \quad & \sum_{k=0}^{N-1} u^2(k) \\
\text{subject to} \quad & \mathrm{E}[x_N] \in x_N^* \\
& x_{k+1} = f(x_k, u_k, \omega_k) \\
& \mathrm{Prob}(x_k \in \chi_{safe}(\omega_{obs})) \geq 1 - \Delta \\
& x_0 \sim \mathrm{pr}(x), \ \ \omega_k \sim \mathrm{pr}(\omega_k) \\
& u_k \in \mathcal{U}
\end{aligned}
$$

- ***Success*** =Remaining Safe

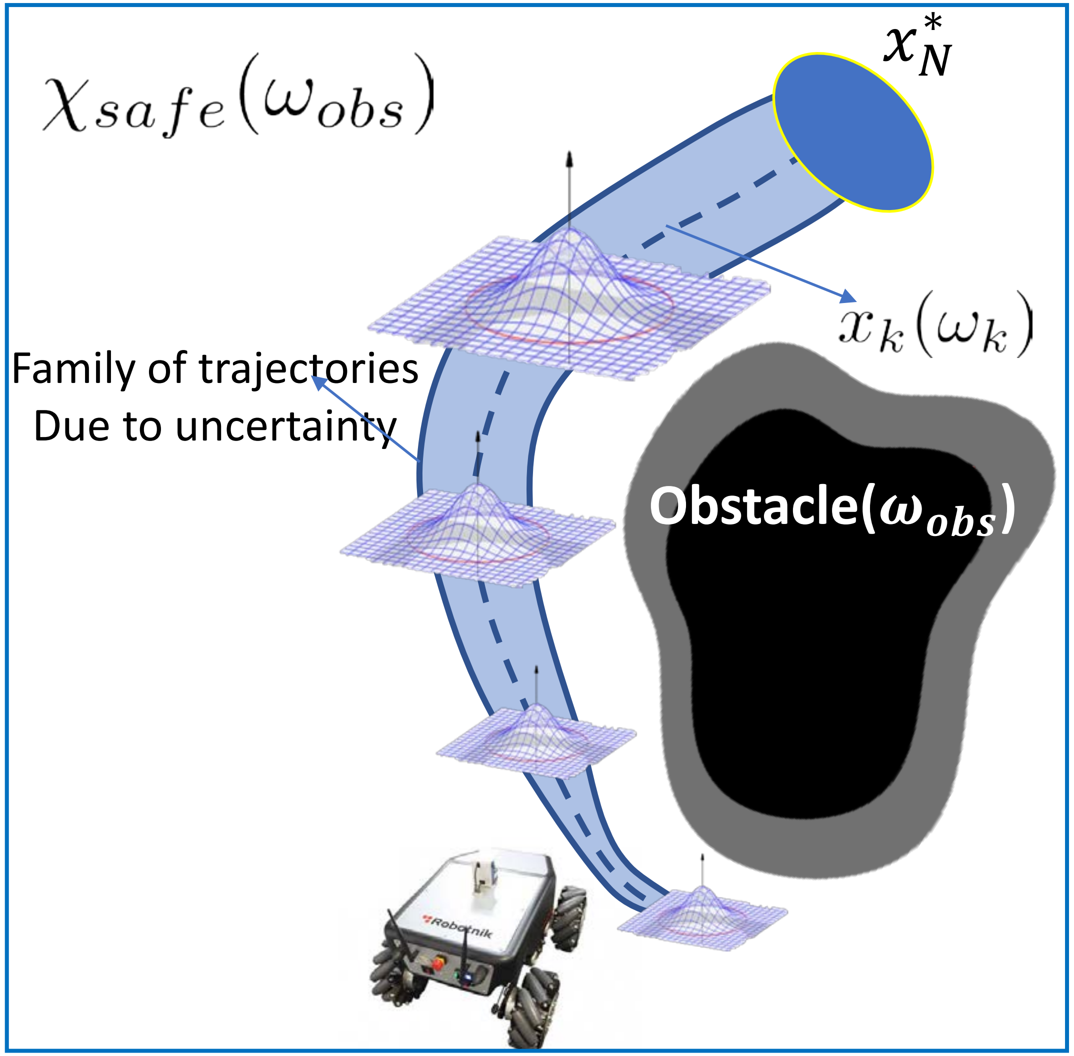


Example: Chance Trajectory Optimization

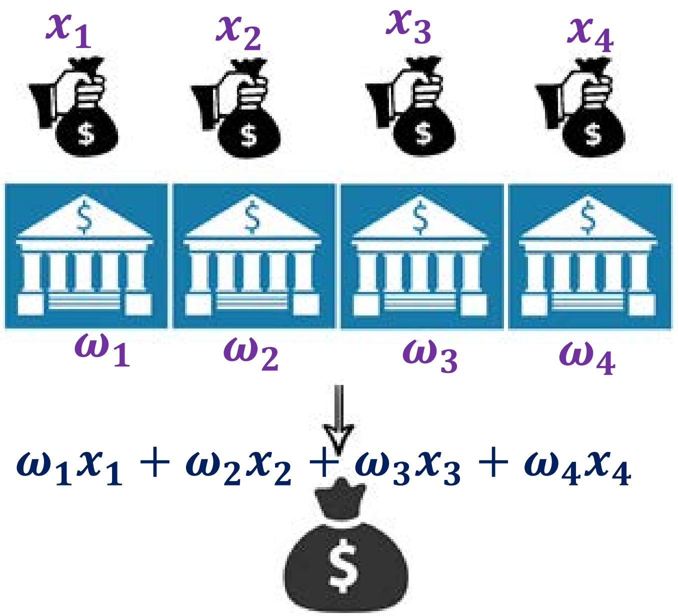
Find a sequence of control inputs $[u_0, \ldots, u_{N-1}]$ to derive the robot to the goal region in the presence of probabilistic uncertainties.

- ***Success*** =Remaining safe and reaching the goal

$$
\begin{array}{ll}
\underset{u_k|_{k=0}^{N-1}}{\text{maximize}} & \boxed{\text{Prob}(x_k \in \chi_{safe}(\omega_{obs}), x_N \in x_N^*)} \\
\text{subject to} & x_{k+1} = f(x_k, u_k, \omega_k) \\
& u_k \in \mathcal{U}
\end{array}
$$



Example: Portfolio Selection Problem



- Assets with uncertain rate of return $\omega_i \sim pr_i(\omega), i = 1, \dots, 4$
- x_i invested money in asset i
- $\boldsymbol{Success}$ = Achive a return higher than "r^*"
  $= \{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*\}$

Chance Optimization

$$\begin{aligned} \underset{x_1, x_2, x_3, x_4}{\text{maximize}} \quad & \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 \leq \chi \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{x_1, x_2, x_3, x_4}{\text{minimize}} \quad & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} \quad & \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \geq 1 - \Delta \end{aligned}$$

3. Risk Aware Optimization Based Planning

Mathematical Formulation

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}\big(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ \big) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i = 1, \ldots, n_g \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{\text{pr}(\omega)}\big(\ g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g\ \big) \geq 1 - \Delta \end{aligned}$$

4. Distributionally Robust Chance Constraint Optimization

- Probability distribution with uncertain parameters $a \in \mathcal{A}$

e.g., Gaussian probability distribution with uncertain mean and deviation

$$\underbrace{\frac{1}{\sqrt{2\pi}\, a_2} e^{\frac{-(x-a_1)^2}{2a_2^2}}, \qquad a_1 \in [l_1, u_2], a_2 \in [l_2, u_2]}_{\text{Family of probability distributions}}$$

$$\begin{aligned} \underset{\text{design parameters}}{\text{minimize}} \quad & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} \quad & \text{uncertainty} \sim \text{Probability distribution}(a),\ a \in \mathcal{A} \\ & \underbrace{\text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}}_{\text{Chance constraints should be satisfied for the family of the probability distributions of the uncertainties}} \end{aligned}$$

4. Distributionally Robust Chance Constrained Optimization

$$\begin{aligned} \underset{\text{design parameters}}{\text{minimize}} \quad & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} \quad & \text{uncertainty} \sim \text{Probability distribution}(a), \;\; a \in \mathcal{A} \\ & \underbrace{\text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}}_{\text{Chance constraints should be satisfied for the family of the probability distributions of the uncertainties}} \end{aligned}$$

Mathematical Formulation

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \omega \sim \text{pr}(\omega, a), \;\; a \in \mathcal{A} \\ & \text{Probability}_{\text{pr}(\omega,a)}\left(\; g_i(x,\omega) \geq 0, \;\; i = 1, \ldots, n_g \;\right) \geq 1 - \Delta, \quad \forall a \in \mathcal{A} \end{aligned}$$

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, ..., n_g \end{aligned}$$

Robust Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x, \omega) \geq 0, \;\; i = 1, ..., n_g, \quad \forall \omega \in \Omega \end{aligned}$$

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}(\; p_i(x, \omega) \geq 0, \;\; i = 1, ..., n_p \;) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, ..., n_g \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{\text{pr}(\omega)}(\; g_i(x, \omega) \geq 0, \;\; i = 1, ..., n_g \;) \geq 1 - \Delta \end{aligned}$$

Distributionally Robust Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \omega \sim \text{pr}(\omega, a), \;\; a \in \mathcal{A} \\ & \text{Probability}_{\text{pr}(\omega, a)}(\; g_i(x, \omega) \geq 0, \;\; i = 1, ..., n_g \;) \geq 1 - \Delta, \quad \forall a \in \mathcal{A} \end{aligned}$$

Purpose of this course:

- State-of-the-art techniques to efficiently solve nonlinear, robust, and risk aware optimization problems.
- Application in analyze and control of uncertain nonlinear dynamical systems.

Assumption

Objective function and constraints of optimization problems, p, g_i , are polynomial functions.

- Polynomial in "x" is a finite linear combination of powers of "x"

$$p(x_1) = 1 + 0.5x_1^2 + 0.75x_1^3$$

Polynomial of degree 3

$$p(x_1, x_2) = 0.56 + 0.5x_1 + 2x_2^2 + 0.75x_1^3x_2^2$$

Polynomial of degree 5

Polynomial: $p(x) : \mathbb{R}^n \to \mathbb{R}$ $\quad p(x) = \sum_\alpha p_\alpha x^\alpha$ $\quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

coefficients

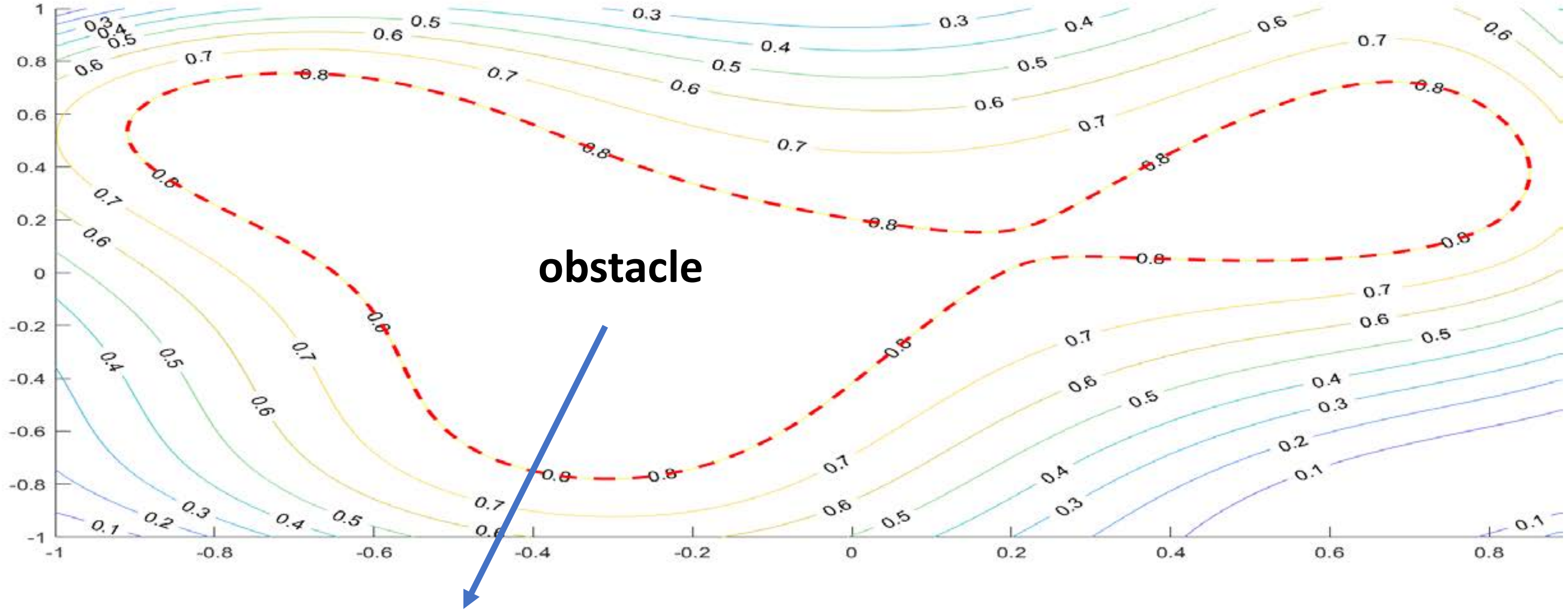
- **Stone-Weierstrass Theorem:** Every continuous function defined on a closed set can be uniformly approximated as closely as desired by a polynomial function.

Example

- Polynomial dynamical system

$$\begin{cases} x_1(k+1) = x_1(k) + 0.2\left(x_1(k) - \frac{x_1^3(k)}{3} - x_2(k) + 0.875\right) \\ x_2(k+1) = x_2(k) + 0.016(x_1(k) - 0.8x_2 + 0.7\ ) \end{cases}$$

- Polynomial constraint



obstacle

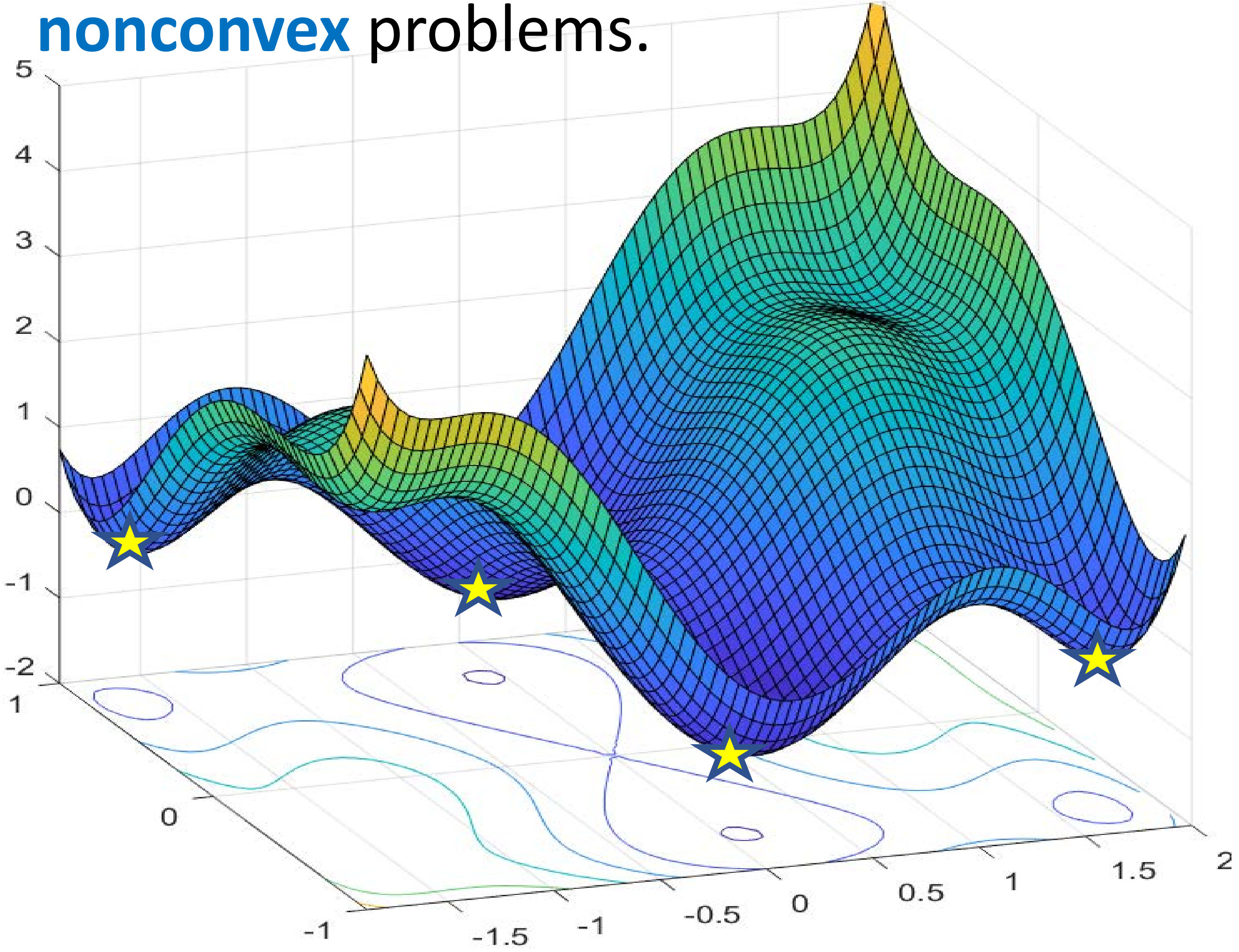
$$0.42x_1^5 - 1.2x_1^4x_2 - 0.48x_1^4 + 0.3x_1^3x_2^2 - 0.57x_1^3x_2 + 0.61x_1^3 - 0.66x_1^2x_2^3 + 0.17x_1^2x_2^2 + 1.9x_1^2x_2 + 0.066x_1^2 + 0.69x_1x_2^4 - 0.14x_1x_2^3 - 0.85x_1x_2^2 + 0.6x_1x_2 - 0.22x_1 + 0.011x_2^5 - 0.068x_2^4 - 0.07x_2^3 - 0.42x_2^2 - 0.084x_2 + 0.84 \geq 0.8$$

Optimization Based Planning Under Uncertainty

- **Challenges**

1. Challenge: **Nonconvexities**

- Nonlinear, robust, and risk aware optimization problems are in general **nonconvex** problems.

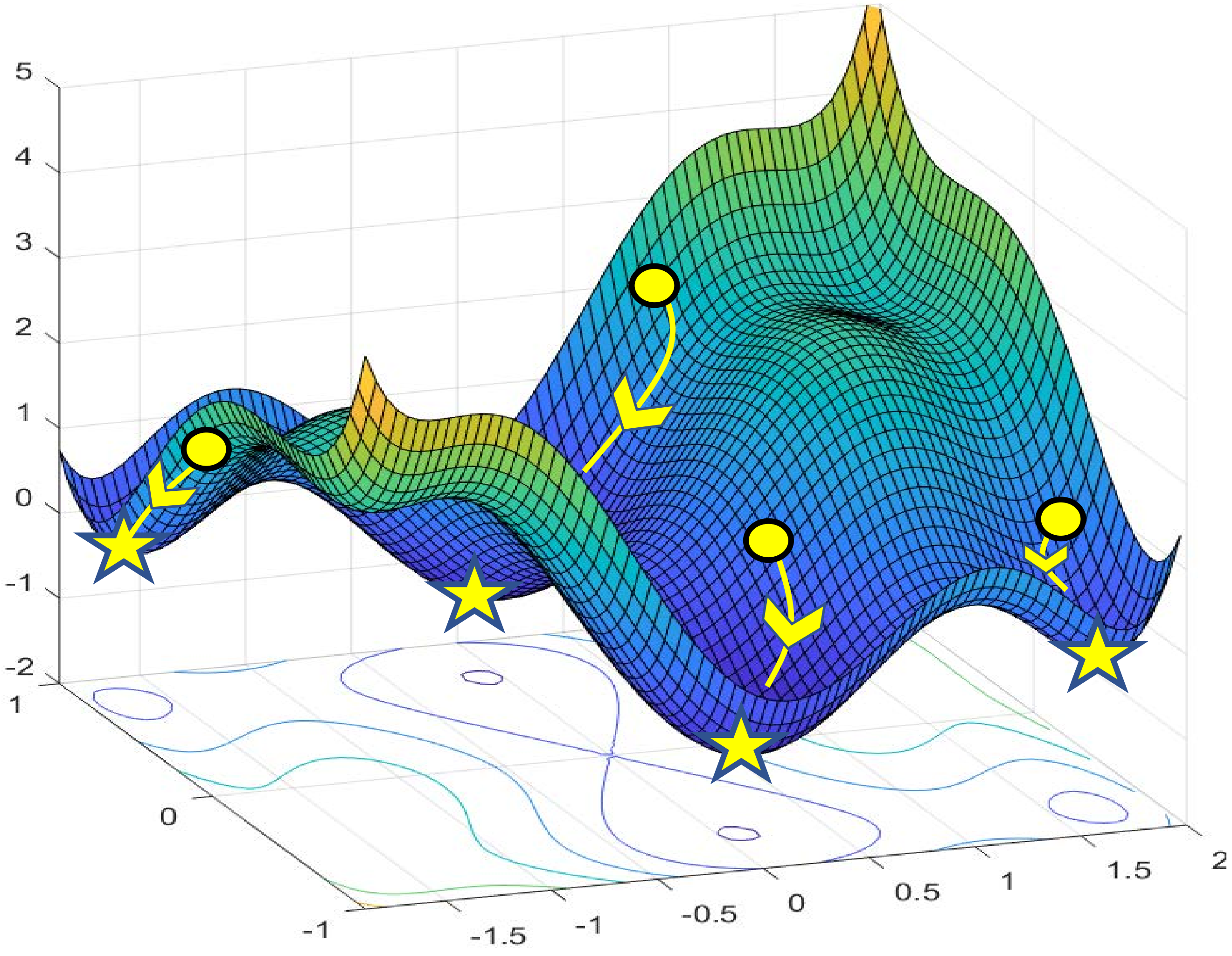


Nonconvex Optimization

★ Multiple local minima

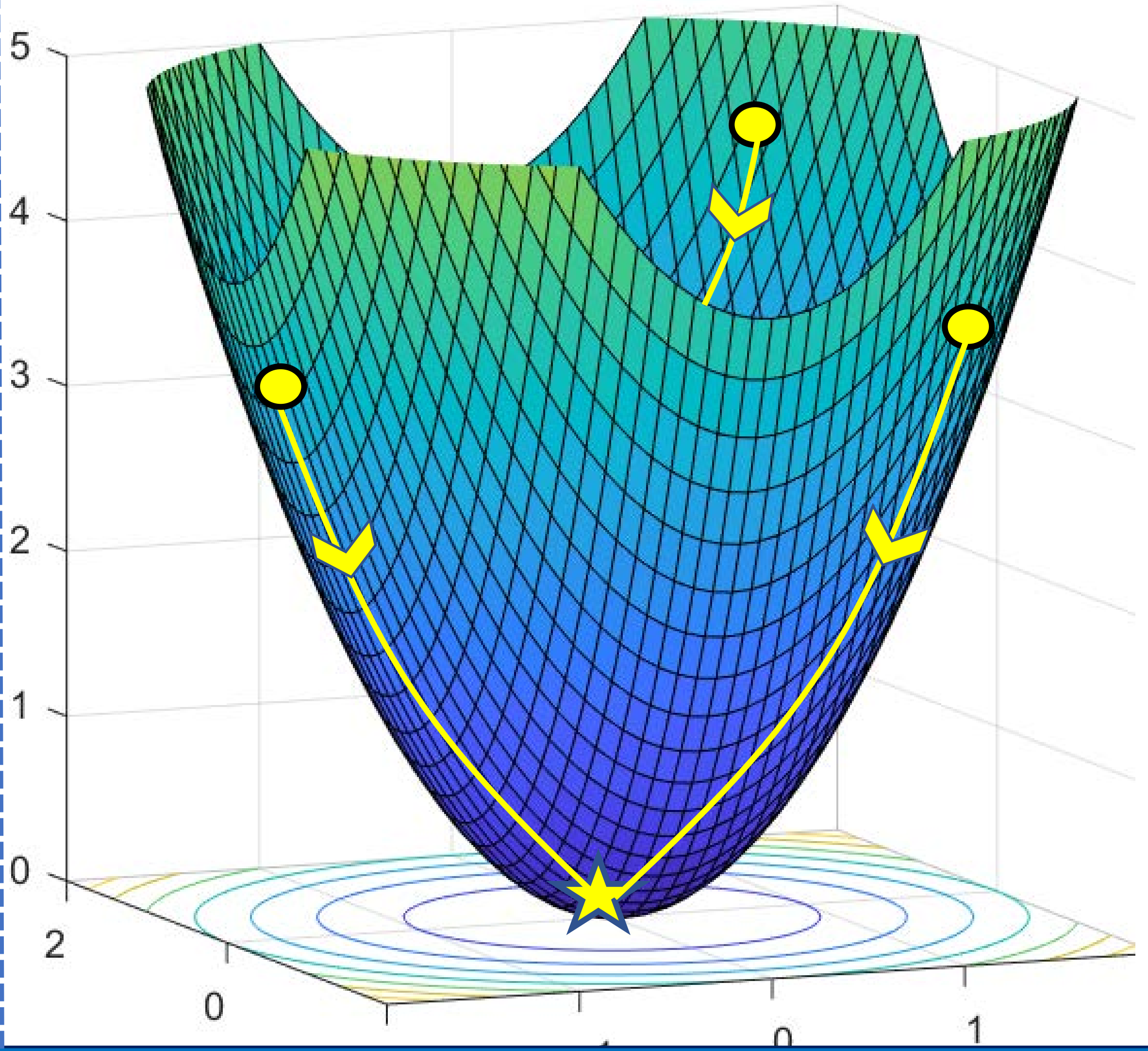
Nonconvex Optimization

- Multiple local minima ★
- Sensitive to initial point ●



Convex Optimization

- Unique minimum: global/local



2. Challenge: Chance and Robust Constraints Evaluation

Chance Constraint $\text{Probability}_{\text{pr}(\omega)}(\ g(x,\omega) \geq 0\ ) \geq 1 - \Delta$

Robust Constraint $g(x,\omega) \geq 0 \quad \forall \omega \in \Omega$

Chance Constraint Evaluation

Chance Constraint: $$\text{Probability}_{\text{pr}(\omega)}(\ g(x,\omega) \geq 0\ ) = \int_{g(x,\omega)\geq 0} \text{pr}(\omega) d\omega$$

- ➢ Multivariate integral
- ➢ In general, It does not have any analytical solution

- ➢ Sampling based methods (e.g., Monte-Carlo methods) DO NOT provide any guarantee.

Probability $\geq 1 - \Delta$ $\xrightarrow{\textbf{Replace}}$ Estimation of Probability $\geq 1 - \Delta$ ✗

Robust Constraint Evaluation

Robust Constraint $g(x,\omega) \geq 0 \quad \forall \omega \in \Omega$

- This results in Infinite number of constraints $g(x, \omega_i) \geq 0, \omega_i \in \Omega$

3. Challenge: Uncertainty Propagation

Continuous State space model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

uncertainties

$$x_0 \sim pr(x),$$
$$\omega_k \sim pr(\omega)$$

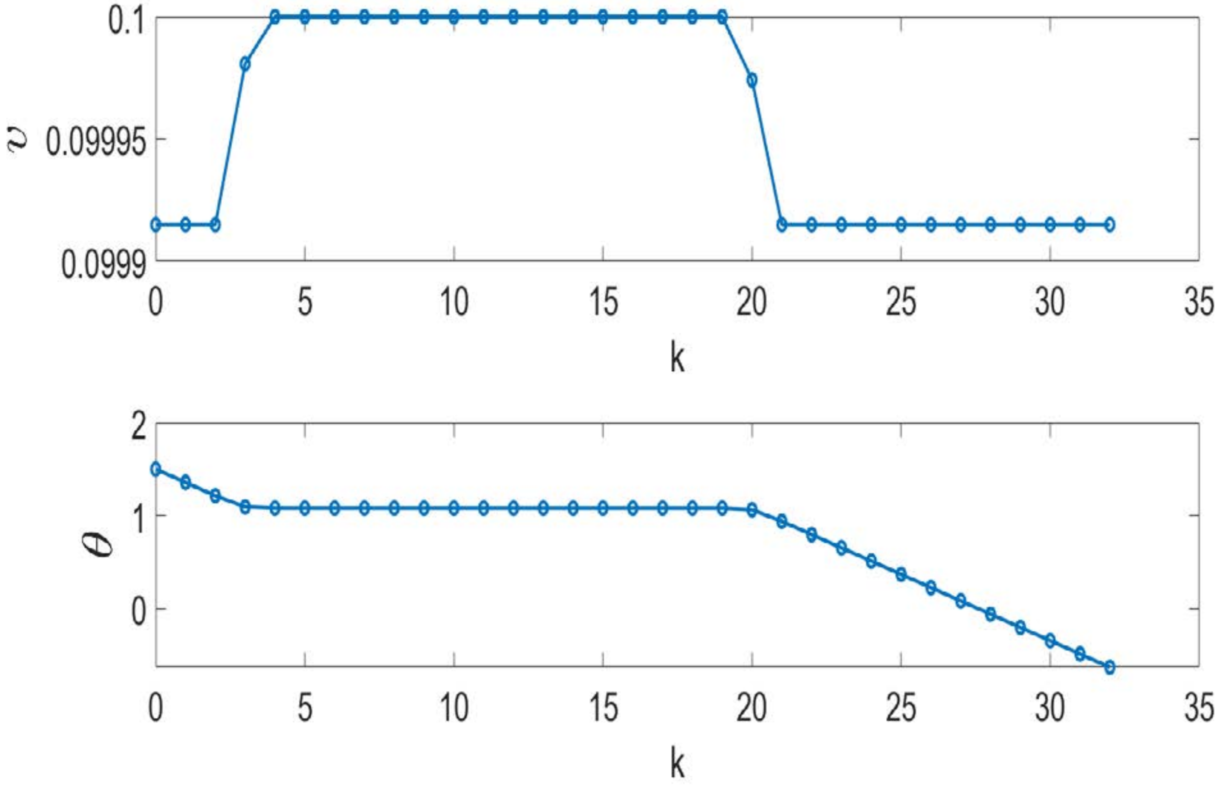
Uncertainty propagation

Example:

$$x_{k+1} = x_k + v_k \cos(\theta_k)$$
$$y_{k+1} = y_k + v_k \sin(\theta_k)$$

states: (x, y) position
control inputs: (θ, v) yaw angle and velocity

Planned Control Inputs:



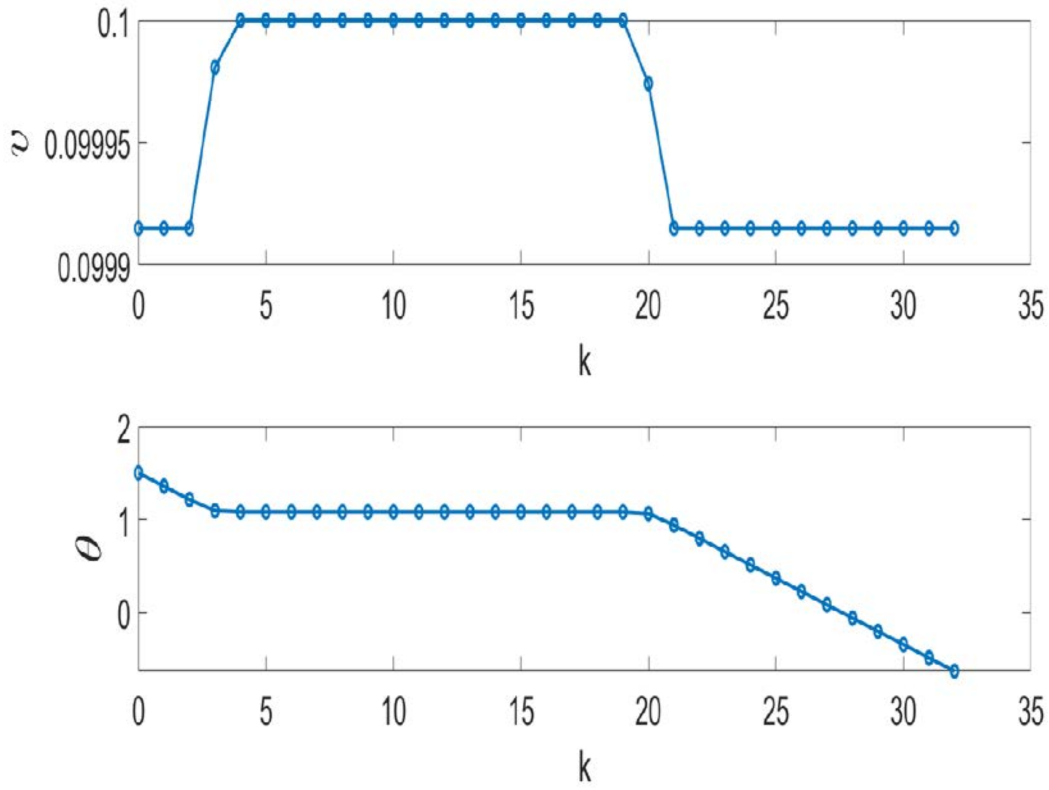
$$x_{k+1} = x_k + (v_k + \omega_{1_k}) \cos(\theta_k + \omega_{2_k}) + \omega_{3_k}$$
$$y_{k+1} = y_k + (v_k + \omega_{1_k}) \sin(\theta_k + \omega_{2_k}) + \omega_{4_k}$$

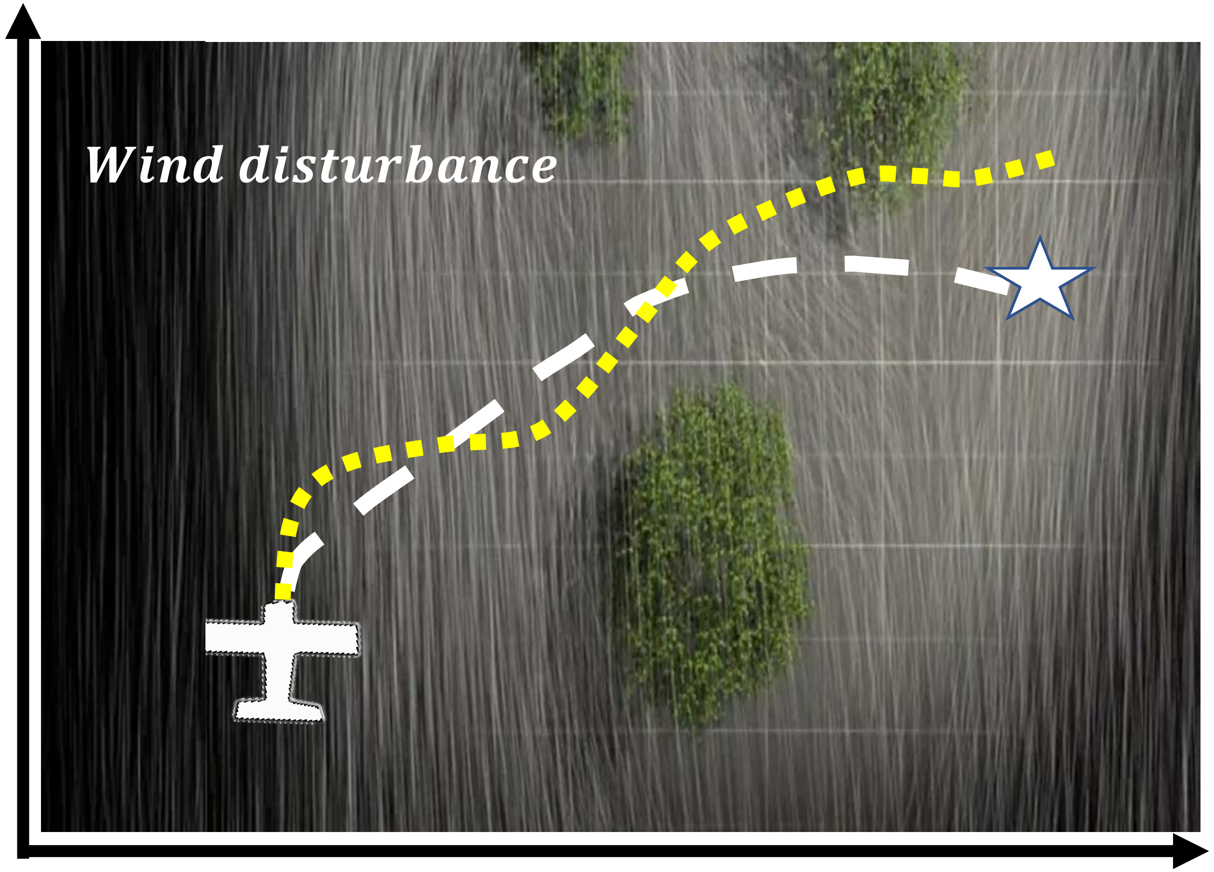
$\boldsymbol{states}$: (x, y) position

$\boldsymbol{control\ inputs}$: (θ, v) yaw angle and velocity

$\boldsymbol{uncertainty}$: $(\underbrace{\omega_1, \omega_2}_{\text{Control Noise}}, \underbrace{\omega_3, \omega_4}_{\text{Wind Disturbance}})$

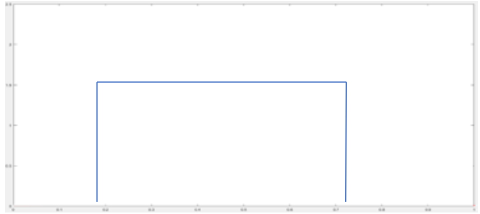
Planned Control Inputs:



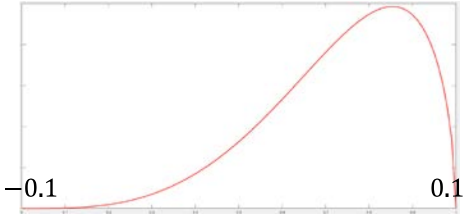


$$x_{k+1} = x_k + (v_k + \omega_{1_k})\cos(\theta_k + \omega_{2_k}) + \omega_{3_k}$$
$$y_{k+1} = y_k + (v_k + \omega_{1_k})\sin(\theta_k + \omega_{2_k}) + \omega_{4_k}$$

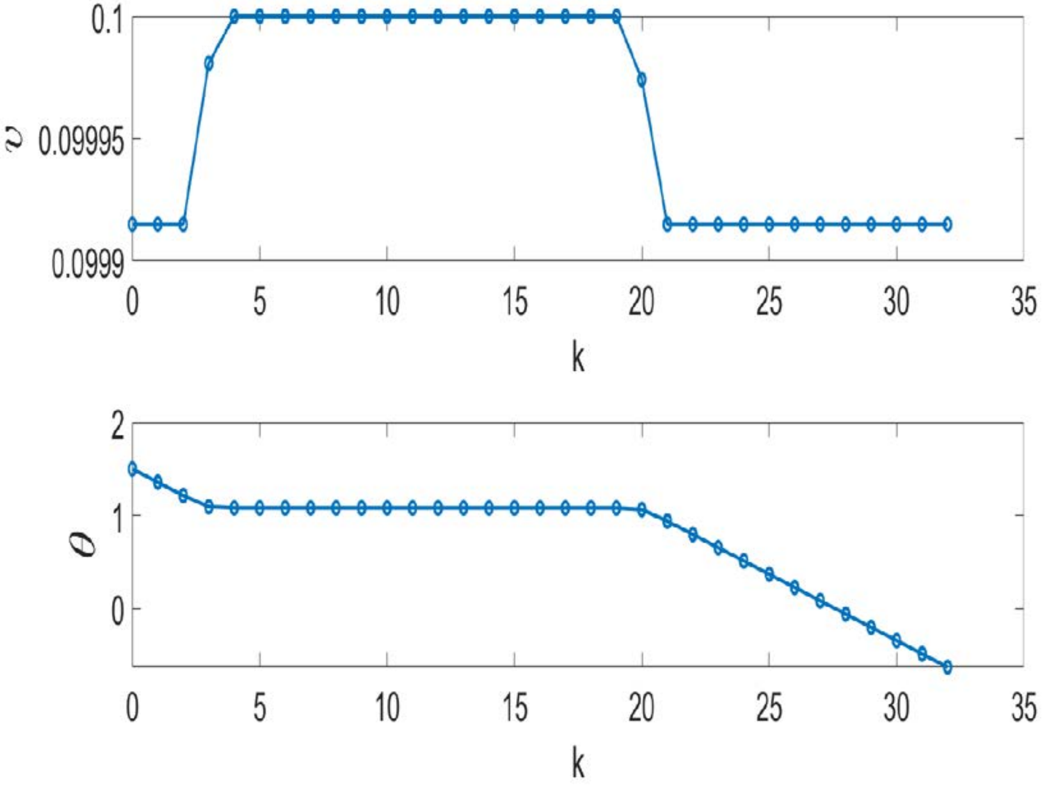
Probability distributions: $\begin{cases} \omega_1 \sim Uniform[-0.1,0.1] \\ \omega_2 \sim Uniform[-1,1] \end{cases}$

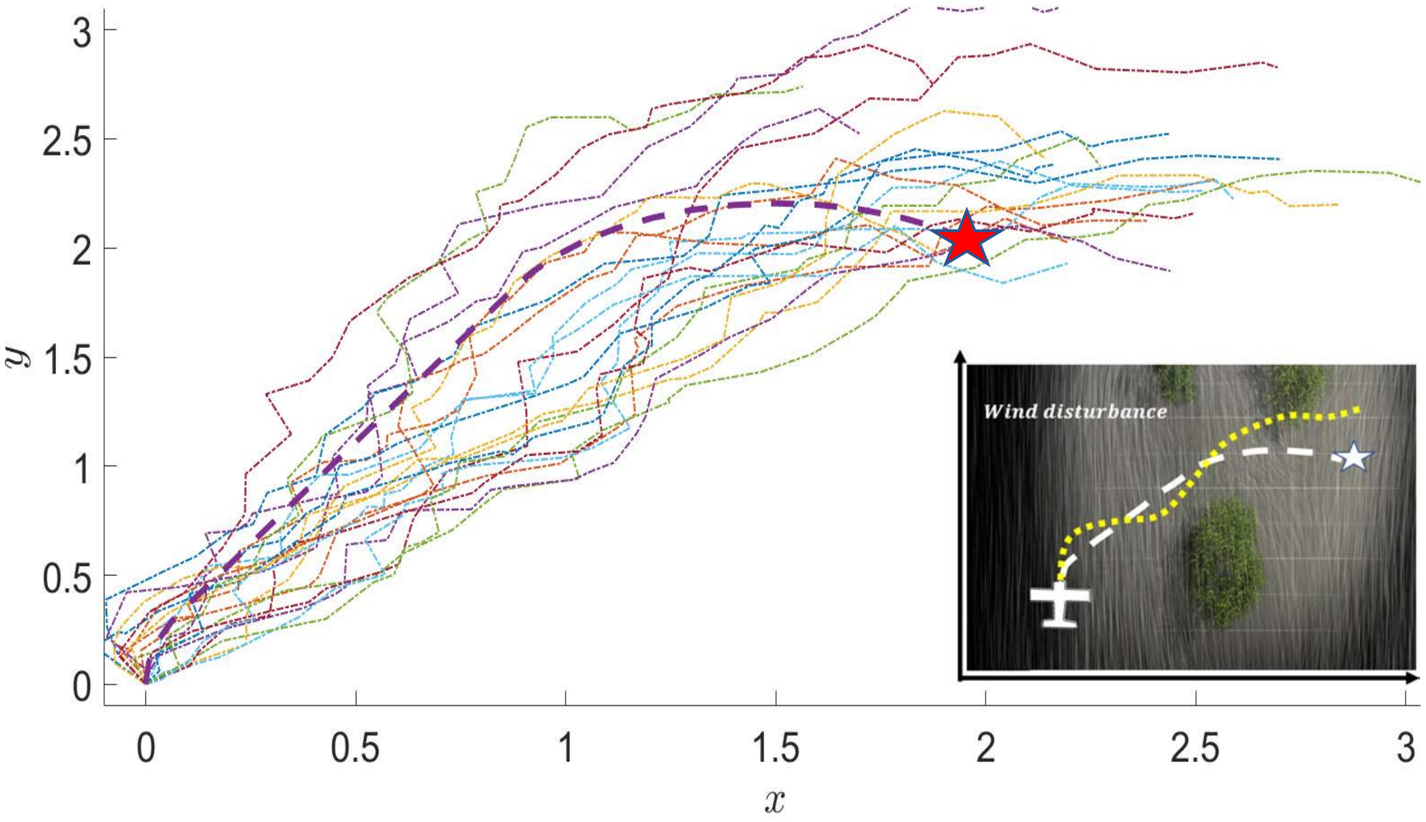


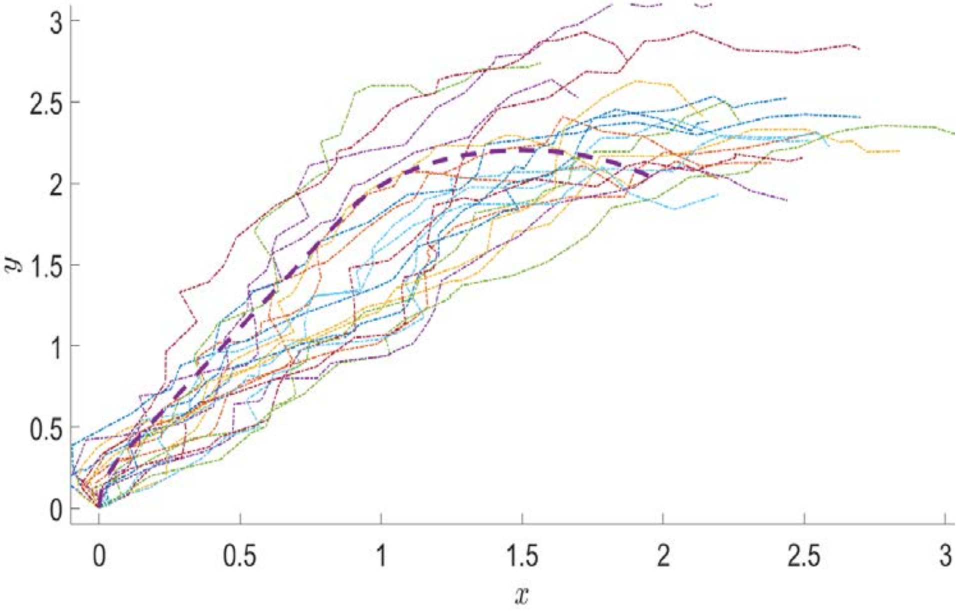
$\omega_3, \omega_4 \sim Beta[-0.1,0.1]$

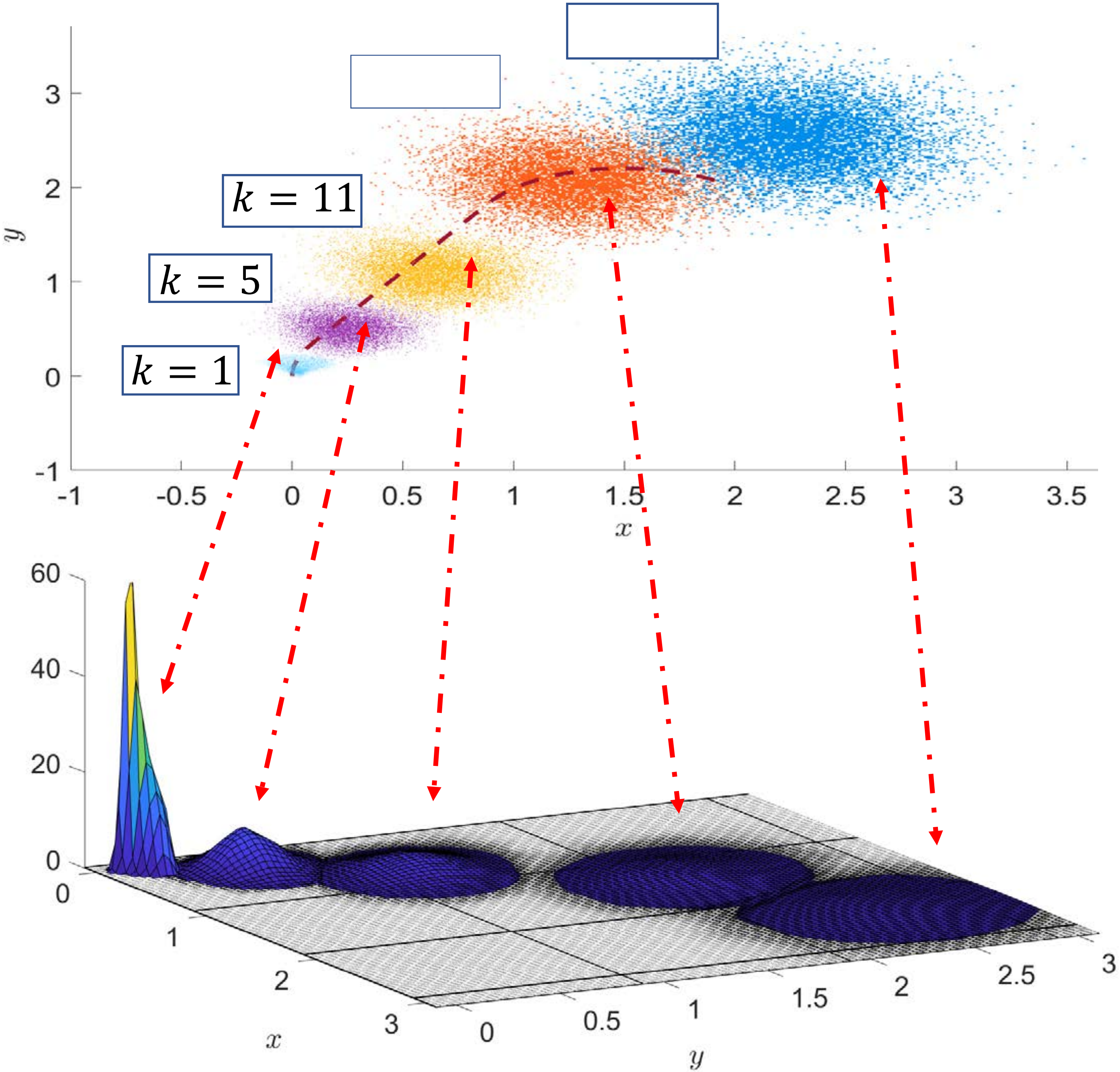


Planned Control Inputs:





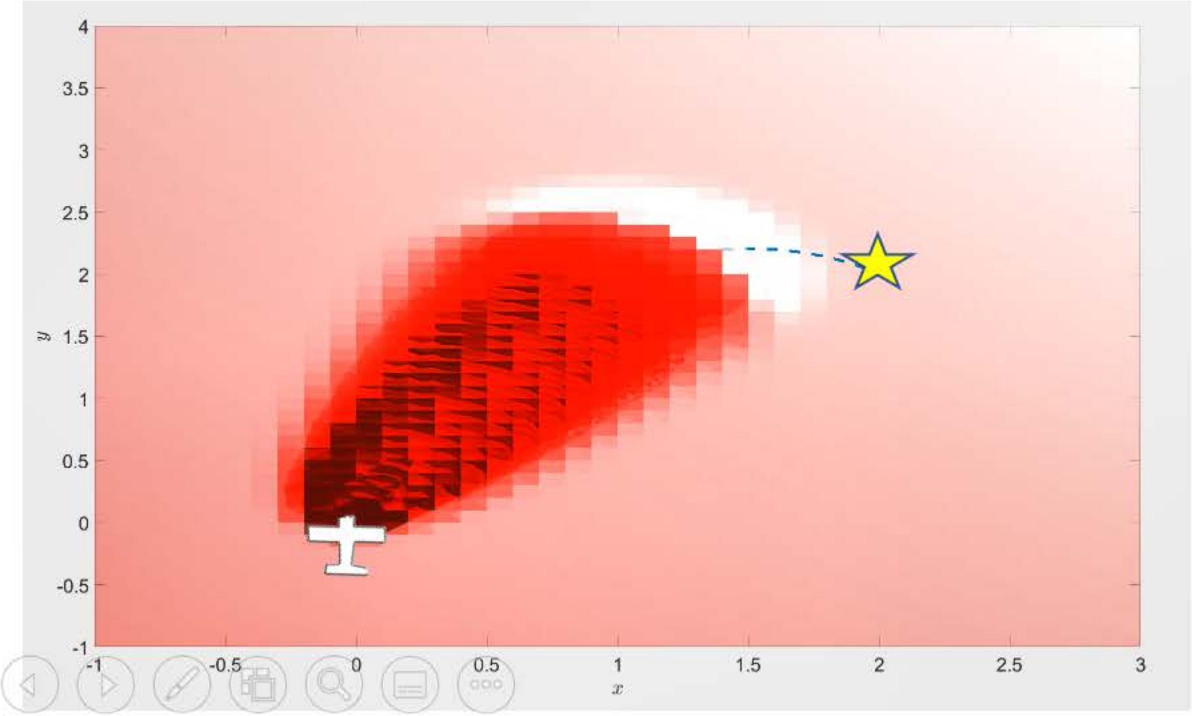


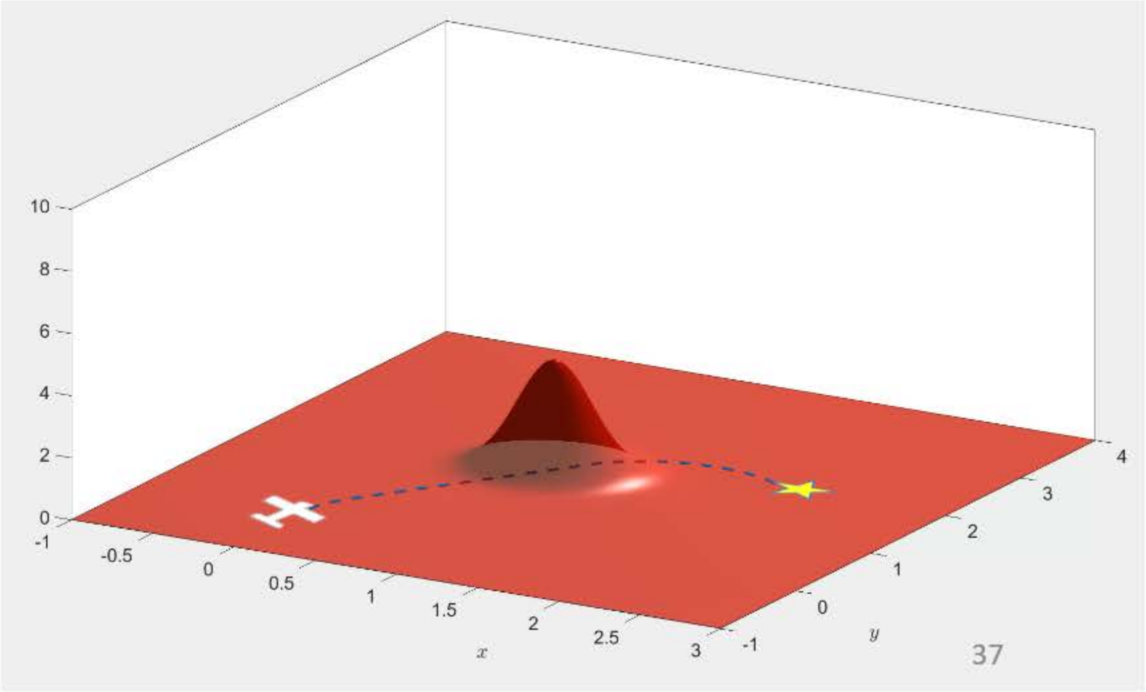


Probability distributions of states of the system

$$x_k \sim p(x_k)$$

Probability distributions of states of the system





- ➢ For safety verification, we need to obtain probability distributions of the states of the system.

- ➢ For this, we need to propagate initial probability distribution of the states through nonlinear dynamics of the system.

Optimization Based Planning

➢ Challenges:

1) Nonconvexities
2) Evaluation of Chance constraint and Robust Constraints
3) Uncertainty Propagation Through Nonlinear Systems

Topics:

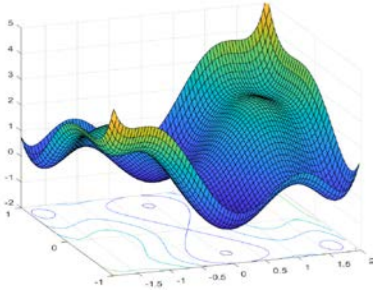
- Introduction to Planning Under Uncertainty
- **Approaches and Challenges**
- Technical Idea and Mathematical Tools
- Applications

Optimization Based Planning Under Uncertainty

- **Main Idea**

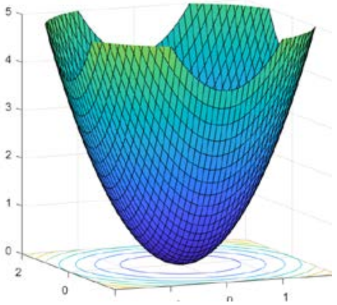
Main Idea: Convexification

- To efficiently solve the nonlinear, robust, and risk aware optimization problems, we look for convex relaxation of the optimization problems.



Nonconvex Optimization

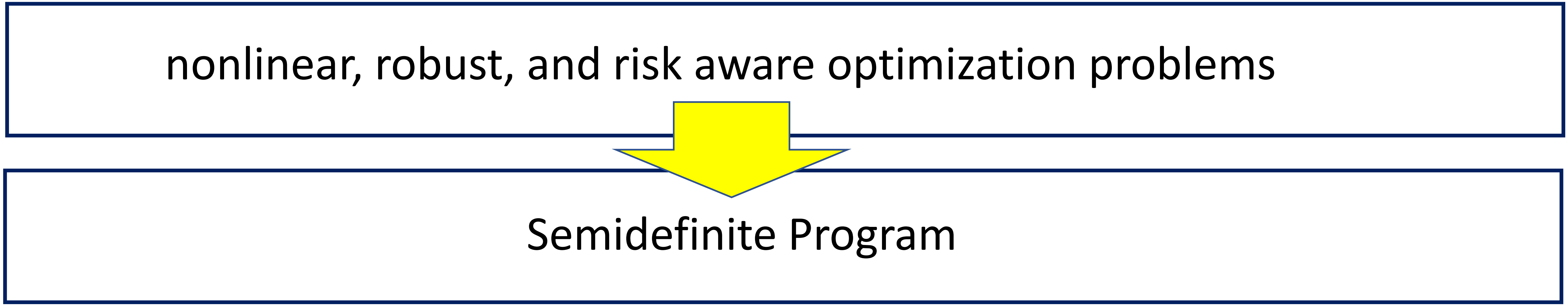
Multiple local optima



Convex Optimization

Unique optimum: global/local

- Convex optimization in form of Semidefinite Program(SDP).

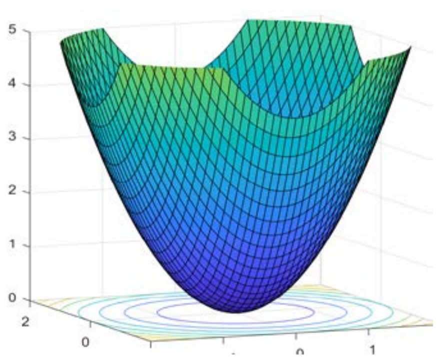


nonlinear, robust, and risk aware optimization problems

Semidefinite Program

Convex Optimization

Example



Unique optimum: global/local

Convex Optimization

Linear Program:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c^T x \longrightarrow \text{linear function}$$

$$\text{subject to} \quad \left.\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}\right\} \text{linear constraints}$$

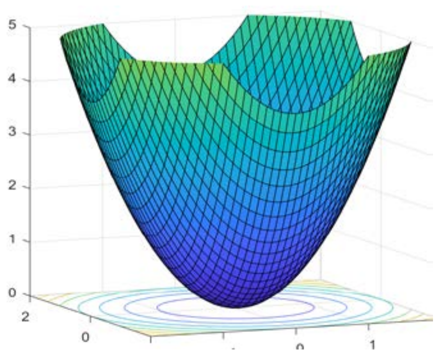
Example

Find $[x_1, x_2, x_3]$ to

$$\begin{aligned} &\min_{\mathrm{x}} 3x_1 + 5x_2 + x_3 \\ &\text{s.t.} \\ &\quad x_1 + 3x_2 + 5x_3 = 2 \\ &\quad x_1 + 9x_2 + 4x_3 = 1 \\ &\quad x_1 \geq 0, \;\; x_2 \geq 0 \end{aligned}$$

Convex Optimization

Example



Unique optimum: global/local

Convex Optimization

Linear Program:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \longrightarrow \text{linear function} \\ \text{subject to} \quad & \left.\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}\right\} \text{linear constraints} \end{aligned}$$

Semidefinite Program:

$$\begin{aligned} \underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad & C \bullet X \longrightarrow \text{linear function} \\ \text{subject to} \quad & A \bullet X = b \longrightarrow \text{linear constraints} \\ & X \succcurlyeq 0 \longrightarrow \text{linear matrix inequalities} \end{aligned}$$

Example

Find $[x_1, x_2, x_3]$ to

$$\begin{aligned} &\min_{\mathrm{x}} 3x_1 + 5x_2 + x_3 \\ &\text{s.t.} \quad x_1 + 3x_2 + 5x_3 = 2 \\ &\qquad x_1 + 9x_2 + 4x_3 = 1 \\ &\qquad x_1 \geq 0, \;\; x_2 \geq 0 \end{aligned}$$

Example

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

$$\begin{aligned} &\min_{\mathrm{x}} 3x_{11} + 5x_{12} + x_{22} \\ &\text{s.t.} \quad x_{11} + 3x_{12} + 5x_{22} = 2 \\ &\qquad x_{11} + 9x_{12} + 4x_{22} = 1 \\ &\qquad X \succcurlyeq 0 \end{aligned}$$

Main Idea: Convexification

- To efficiently solve the nonlinear, robust, and risk aware optimization problems, we look for convex relaxation of the optimization problems.

- Convex optimization in form of Semidefinite Program(SDP).

nonlinear, robust, and risk aware optimization problems

Semidefinite Program

Optimization Based Planning Under Uncertainty

- **Mathematical Tools**

nonlinear, robust, and risk aware optimization problems

Convexification:

Tools:

i) Theory of Nonnegative Polynomials

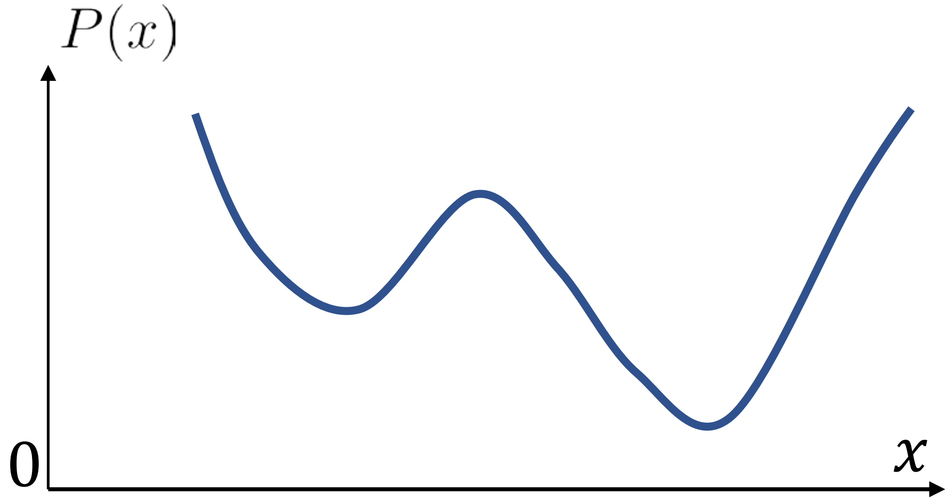
ii) Theory of Moments

Semidefinite Program

Nonnegative Polynomial Based SDP relaxation

Nonnegative Polynomials

$$P(x) \geq 0 \quad \forall x \in \mathbb{R}^n$$



Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \; i = 1, \ldots, n_g \end{aligned}$$

SOS based SDP Relaxation

SDP in terms of coefficients of $\mathrm{P}(x_1, x_2, \ldots., x_n) \geq 0$

Main Idea:

Instead of looking for decision parameters $(x_1, x_2, \ldots, x_n)$, we look for a nonnegative polynomial in terms of decision parameters, i.e. $\mathrm{P}(x_1, x_2, \ldots., x_n) \geq 0$

Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0,\ i = 1, \dots, n_g \end{aligned}$$

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- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial $\mathrm{P}(x_1, x_2, \dots, x_n)$.

Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0,\ i=1,\ldots,n_g \end{aligned}$$

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- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial $\mathrm{P}(x_1, x_2, \ldots., x_n)$.
- We use nonnegativity condition for polynomial $\mathrm{P}(x_1, x_2, \ldots., x_n)$. (i.e., sum of squares (SOS)condition)

Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \; i = 1, \ldots, n_g \end{aligned}$$

SOS based SDP Relaxation SDP in terms of coefficients of $\mathrm{P}(x_1, x_2, \ldots, x_n) \geq 0$

Main Idea:

Instead of looking for decision parameters $(x_1, x_2, \ldots, x_n)$, we look for a nonnegative polynomial in terms of decision parameters, i.e. $\mathrm{P}(x_1, x_2, \ldots, x_n) \geq 0$

- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial $\mathrm{P}(x_1, x_2, \ldots, x_n)$.
- We use nonnegativity condition for polynomial $\mathrm{P}(x_1, x_2, \ldots, x_n)$. (i.e., sum of squares (SOS)condition)
- This results in an SDP in terms of coefficients of $\mathrm{P}(x_1, x_2, \ldots, x_n)$. (SOS based SDP)

Moment Based SDP Relaxation

Nonlinear Optimization

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & g_i(x) \geq 0, \; i = 1, \ldots, n_g \end{array}$$

Main Idea:

- We treat decision variables $(x_1, x_2, \ldots, x_n)$ as random variable.
- Instead of looking for decision parameters $(x_1, x_2, \ldots, x_n)$, we look for its probability distribution, i.e. $\mathrm{pr}(x_1, x_2, \ldots, x_n)$
- Later, we extract the deterministic solution $(x_1, x_2, \ldots, x_n)$.

Moment Based SDP Relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \ i = 1, \ldots, n_g \end{aligned}$$

Main Idea:

- We treat decision variables $(x_1, x_2, \ldots, x_n)$ as random variable.
- Instead of looking for decision parameters $(x_1, x_2, \ldots, x_n)$, we look for its probability distribution, i.e. $\mathrm{pr}(x_1, x_2, \ldots., x_n)$
- Later, we extract the deterministic solution $(x_1, x_2, \ldots, x_n)$.

To obtain an SDP formulation, instead of looking for probability distribution $\mathrm{pr}(x_1, x_2, \ldots., x_n)$, we look for its statistics called moments.

Moments of probability distributions

moment of order α $$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha \mathrm{pr}(x)dx = \int x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \mathrm{pr}(x)dx$$

- 1-st moment (mean): $y_1 = \mathrm{E}[x^1] = \int x\mathrm{pr}(x)dx$
- 2-nd moment: $y_2 = \mathrm{E}[x^2] = \int x^2 \mathrm{pr}(x)dx$

Moments of probability distributions

moment of order α $\quad y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha \mathrm{pr}(x)dx = \int x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mathrm{pr}(x)dx$

- 1-st moment (mean): $y_1 = \mathrm{E}[x^1] = \int x\mathrm{pr}(x)dx$
- 2-nd moment: $y_2 = \mathrm{E}[x^2] = \int x^2\mathrm{pr}(x)dx$

⇒ $var = \mathrm{E}[(x - \mathrm{E}[x])^2] = y_2 - y_1^2$

Moment Based SDP Relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \; i = 1, \ldots, n_g \end{aligned}$$

Moment based SDP Relaxation SDP in terms of moments of $\text{pr}(x_1, x_2, \ldots, x_n)$

Moment Based SDP Relaxation

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \; i = 1, \ldots, n_g \end{aligned}$$

Moment based SDP Relaxation SDP in terms of moments of $\mathrm{pr}(x_1, x_2, \ldots., x_n)$

- We translate objective function and constraints of the original optimization problem in terms of the moments of probability distribution $\mathrm{pr}(x_1, x_2, \ldots., x_n)$.

- This results in an SDP in terms of the moment. (Moment based SDP)

- We will apply these techniques to the Uncertain optimization problems

Nonlinear Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i = 1, \ldots, n_g \end{aligned}$$

⇒ SOS / Moment Based SDP Relaxation

Robust Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g, \quad \forall \omega \in \Omega \end{aligned}$$

⇒ SOS / Moment Based SDP Relaxation

Chance Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{pr(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ ) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i = 1, \ldots, n_g \end{aligned}$$

⇒ SOS / Moment Based SDP Relaxation

Chance Constrained Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{pr(\omega)}(\ g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g\ ) \geq 1 - \Delta \end{aligned}$$

⇒ SOS / Moment Based SDP Relaxation

Distributionally Robust Chance Constrained Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \omega \sim \text{pr}(\omega, a),\ \ a \in \mathcal{A} \\ & \text{Probability}_{\text{pr}(\omega,a)}(\ g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g\ ) \geq 1 - \Delta, \quad \forall a \in \mathcal{A} \end{aligned}$$

⇒ SOS / Moment Based SDP Relaxation

1) Optimization Based Planning under Uncertainty

i) Nonlinear Optimization
ii) Robust Optimization
iii) Chance Optimization/Chance Constrained Optimization
iv) Distributionally Robust Chance Constrained Optimization

2) Challenges

i) Nonconvexities
ii) Evaluation of Chance constraint and Robust Constraints
iii) Uncertainty Propagation Through Nonlinear Systems

3) Main Idea:

Replace the nonconvex optimization with Convex optimization in the form of Semidefinite Program (SDP).

4) We solve Moment/SOS based SDP

Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications

Optimization Based Planning Under Uncertainty

- **Applications**

Optimization Based Planning Under Uncertainty

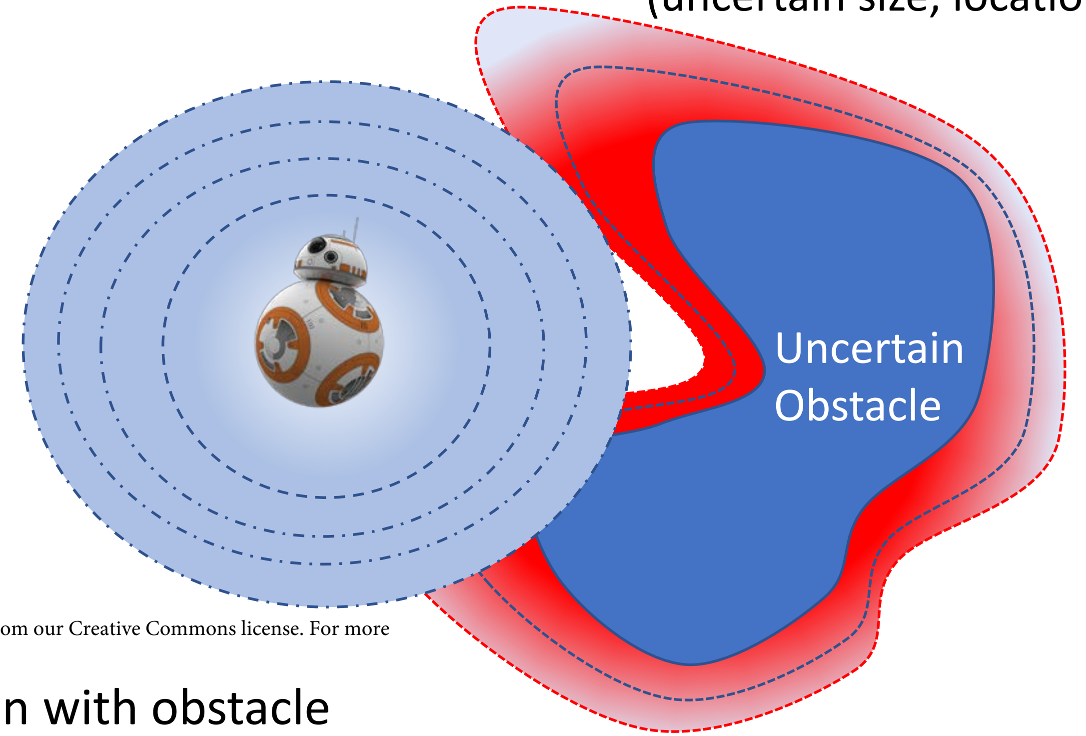
1. Safety Verification for Probabilistic Systems
2. Risk Aware Control and Planning
3. Dynamical system with Gaussian Uncertainties
4. Occupation Measure Based Control and Analyze
5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems

1. Safety Verification

1.1 Risk Estimation

- Probabilistic location of the robot

- Nonconvex obstacle with probabilistic uncertainties (uncertain size, location, geometry)



➢ **Risk:** probability of collision with obstacle

Find: Lower/Upper bounds of the risk

$$\mathbf{P}^{\mathbf{L}}_{\mathbf{risk}} \leq \mathbf{P}^{*}_{\mathbf{risk}} \leq \mathbf{P}^{\mathbf{U}}_{\mathbf{risk}}$$

⇨

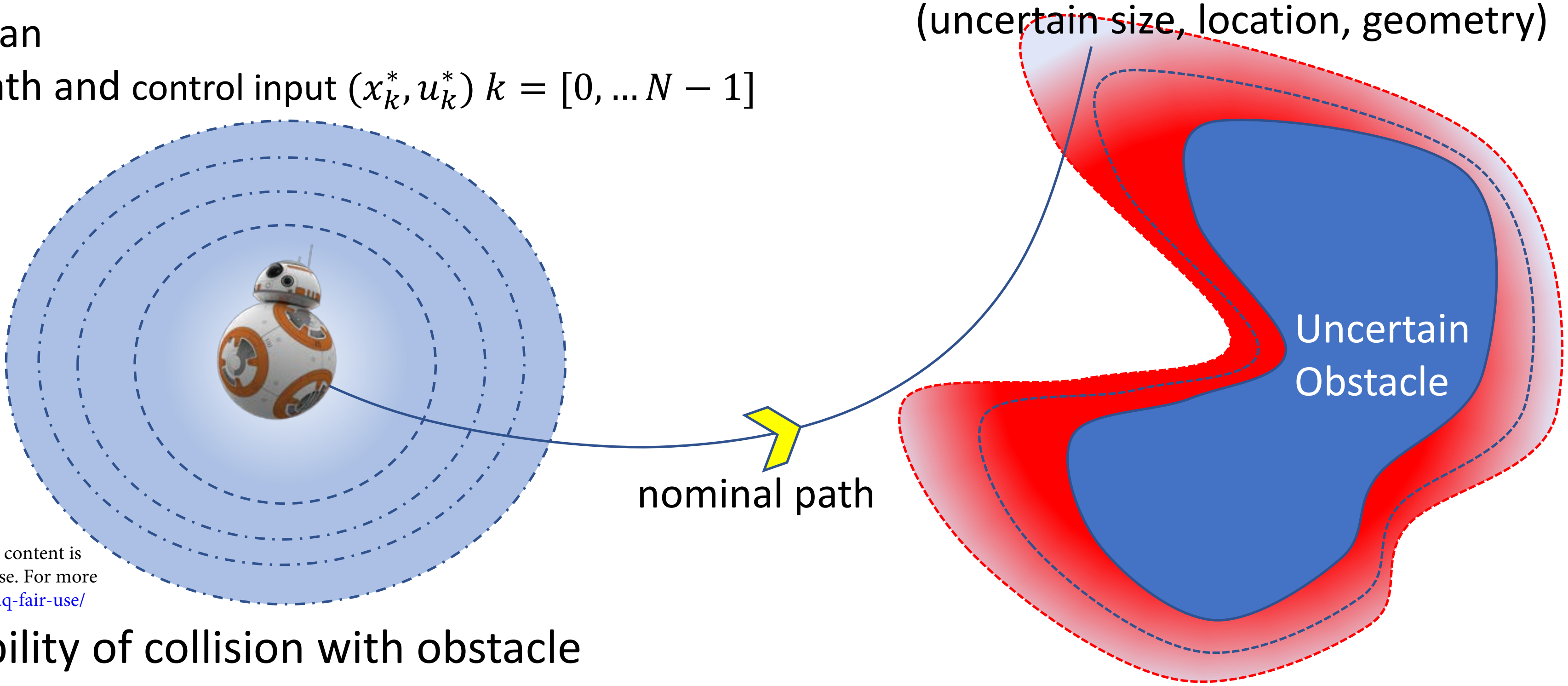
- Particular case of “chance optimization”
- SOS/Moment based SDP formulation

1.2 Risk Estimation and Uncertainty Propagation

- Initial Probabilistic location of the robot
- Candidate plan

e.g., nominal path and control input (x_k^*, u_k^*) $k = [0, \ldots N-1]$

- Nonconvex obstacle with probabilistic uncertainties (uncertain size, location, geometry)



➢ **Risk:** probability of collision with obstacle

Find: Lower/Upper bounds of the risk at time $k = [0, \ldots N-1]$ for given (x_k^*, u_k^*)

- We need to find $x_k \sim pr(x_k)$. We find moment sequence of $pr(x_k)$using the uncertain nonlinear dynamics
- Solve Risk estimation problem at each time k

1.3 Uncertainty Set Construction

- Initial Probabilistic location of the robot
- Candidate plan

e.g., nominal path and control input (x_k^*, u_k^*) $k = [0, \ldots N-1]$

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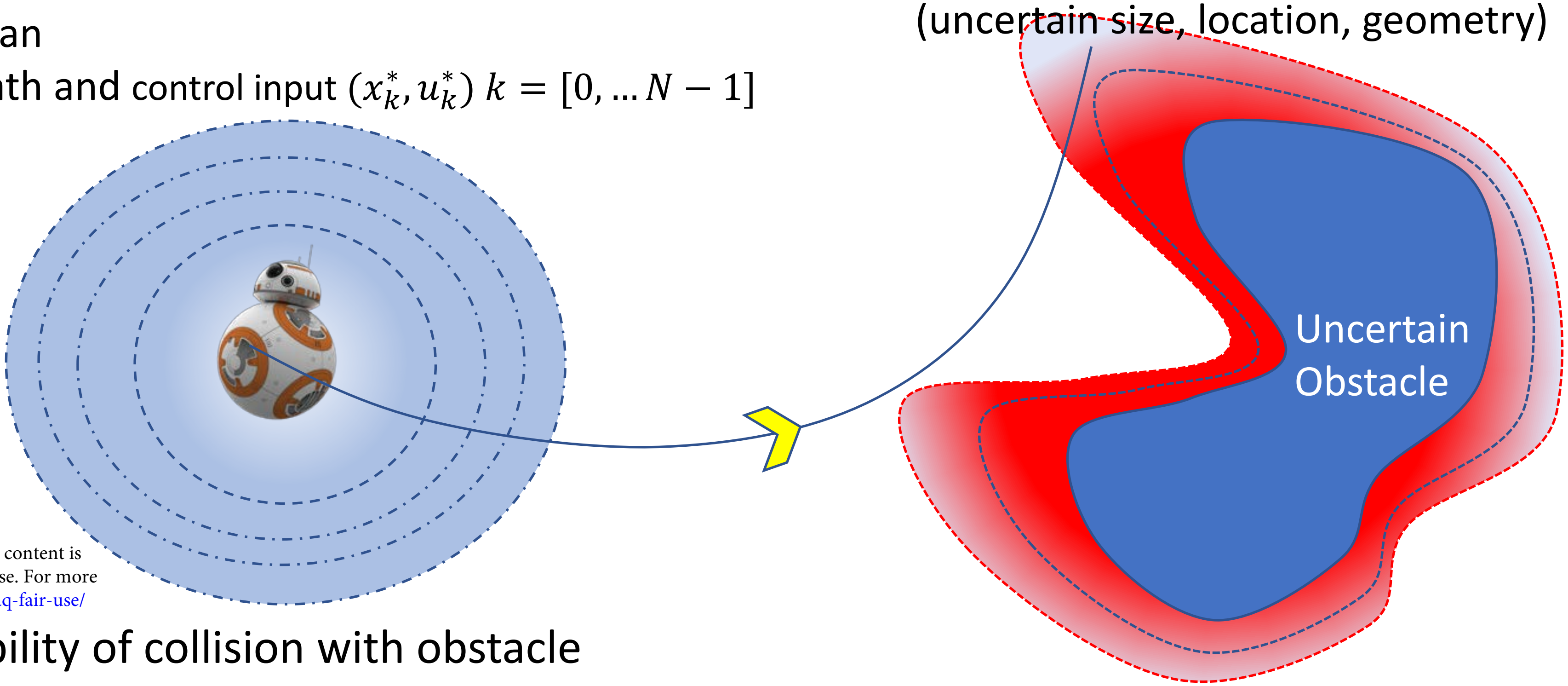


➢ **Risk:** probability of collision with obstacle

- Instead of looking for $x_k \sim \mathrm{pr}(x_k)$, we find the state uncertainty set $x_k \in \Omega_k$

Uncertainty set at time k

- Application:

"Robust safety validation"

"Reachable set Construction" for uncertain nonlinear systems

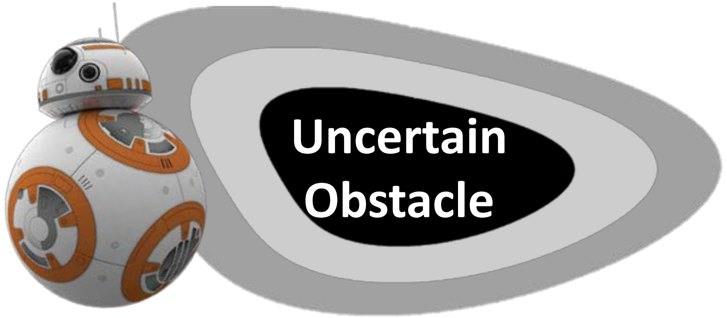
2. Risk Aware Control and Planning

2.1 Risk Bounded Trajectory Planning in Nonconvex Uncertain Environments

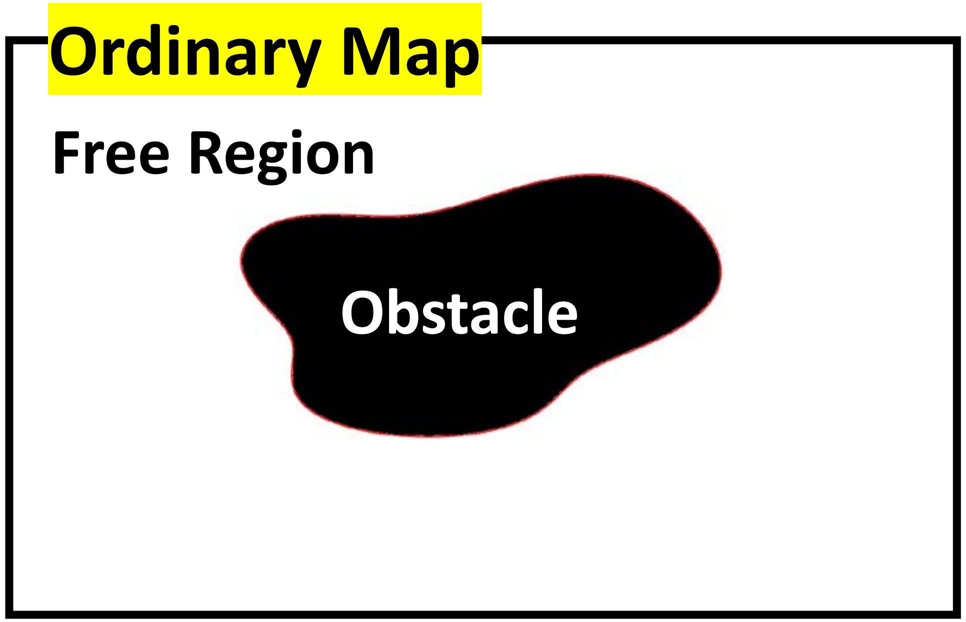
Goal: Risk Bounded Trajectory Planning in presence of perception uncertainties

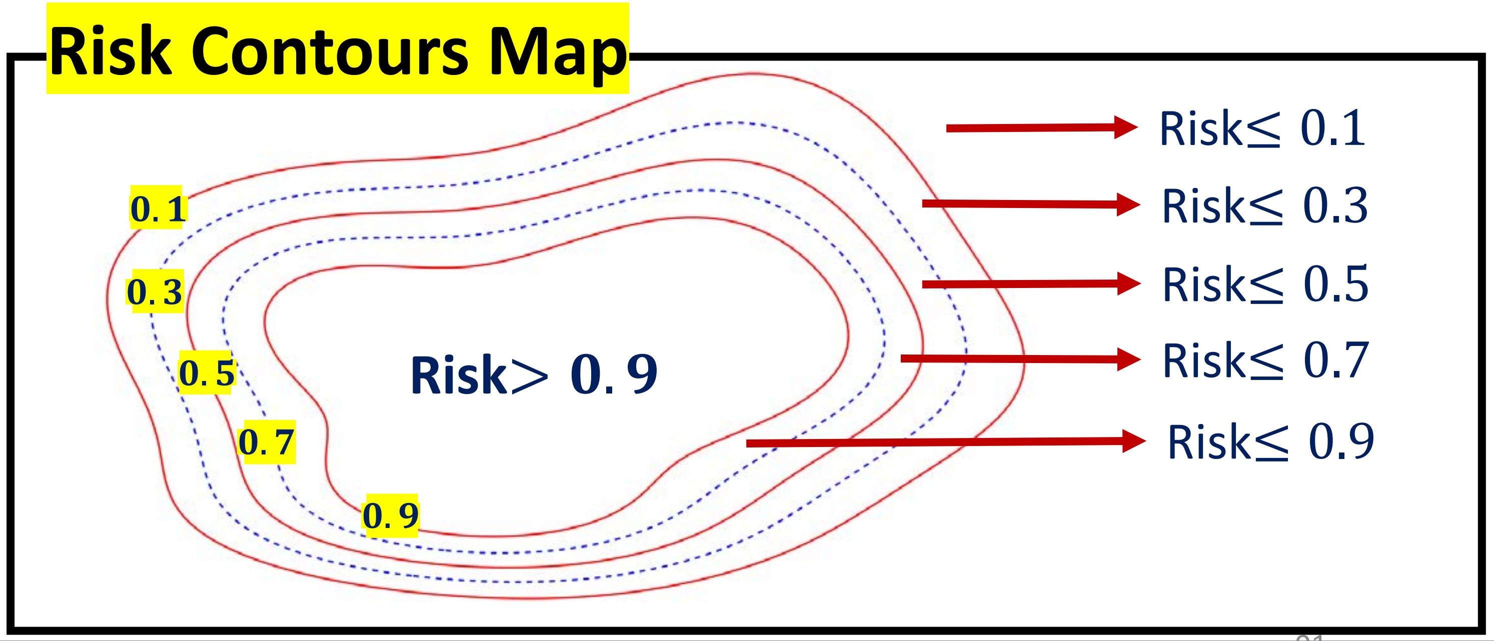
Perception Uncertainties:

Probabilistic uncertainties in location, size, and geometry of obstacles



Risk: Probability of collision of robot with obstacles in presence of probabilistic uncertainties.

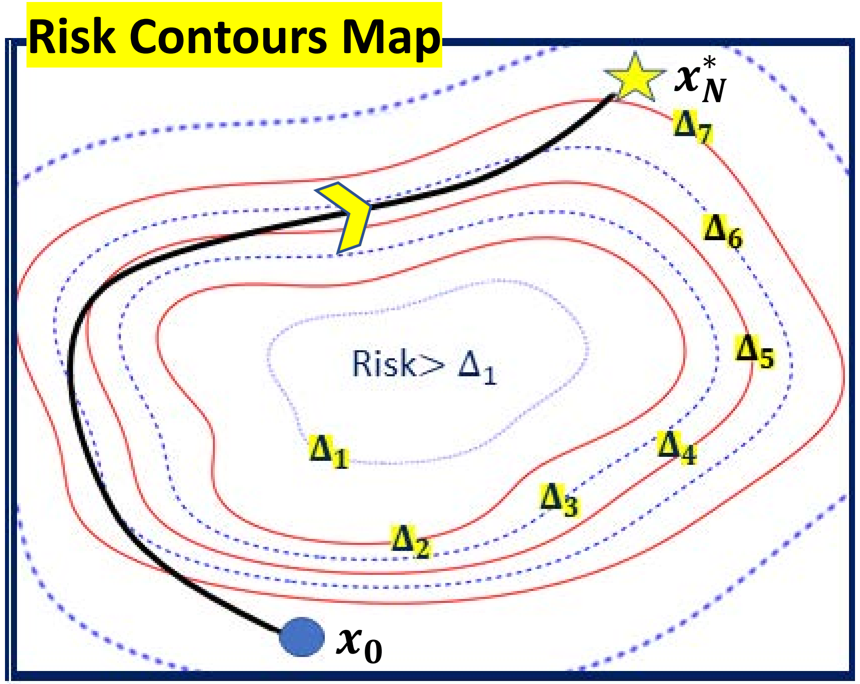




2.1 Risk Bounded Trajectory Planning in Nonconvex Uncertain Environments

- We construct a new map called "*risk contours map* (RCM)" that represents risk information of uncertain environment.

- We replace "*risk bounded trajectory planning*" with deterministic trajectory planning /path planning problem with respect to RCM.



2.2 Risk Aware Nonlinear Controller Design for Probabilistic Nonlinear Systems

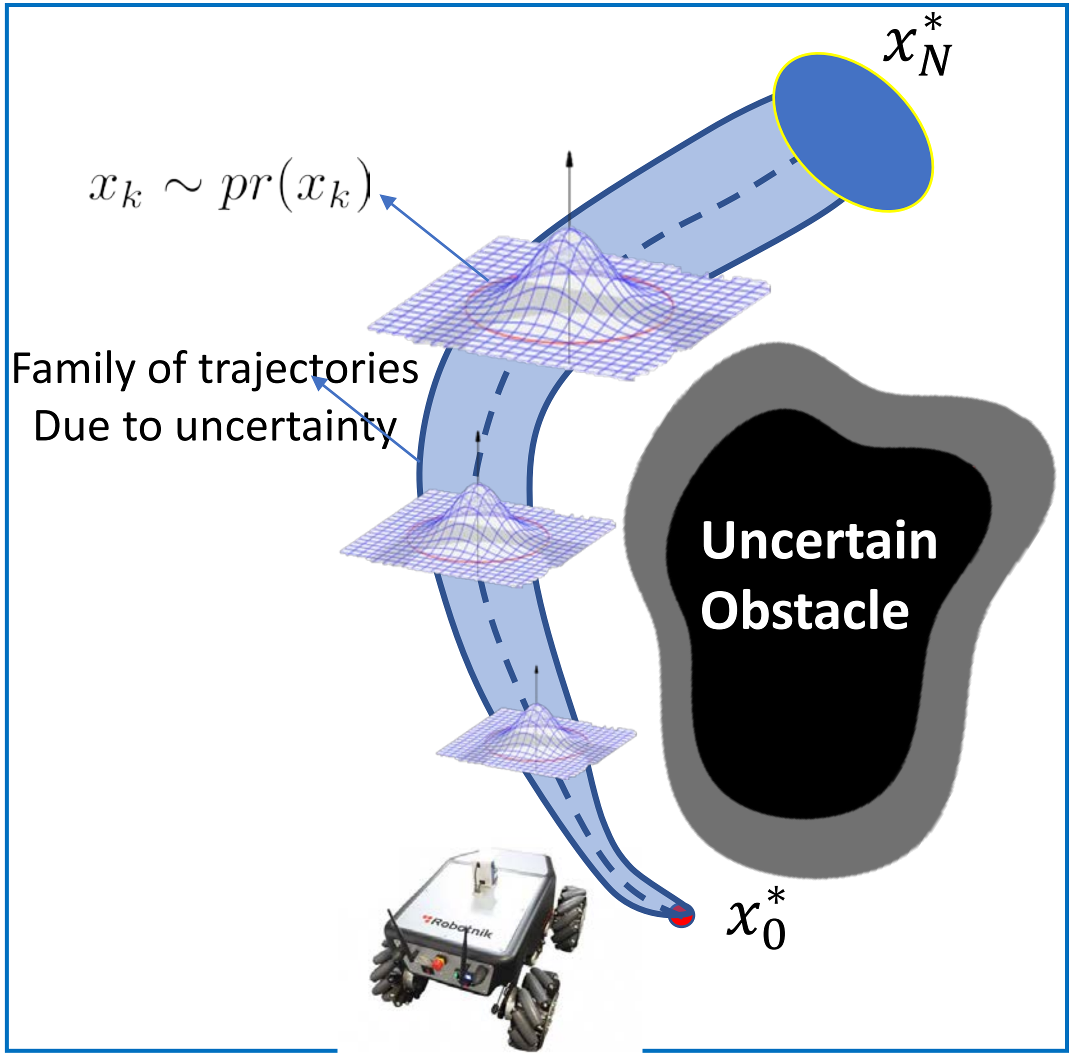
- We design closed-loop controller to:

i) drive the robot to the goal region
ii) avoid the obstacles
in the presence of system and environment uncertainties.

- closed-loop controller in the form of "*Polynomial State Feedback*", i.e., $u(x_k) = \sum_\alpha p_\alpha x_k^\alpha$

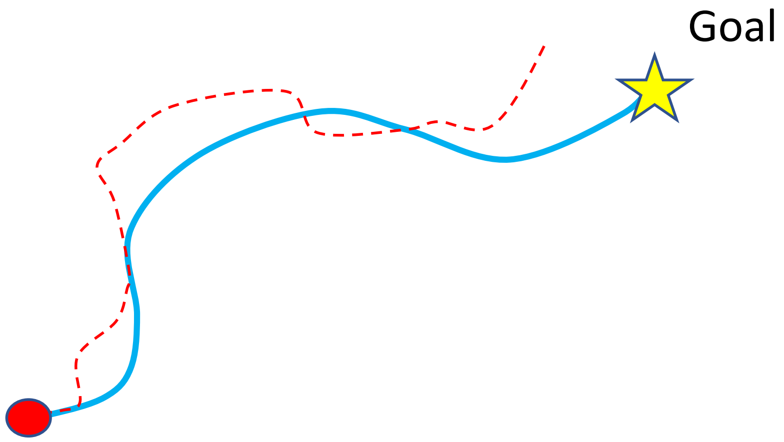
- Model Predictive Control (MPC) formulation: We look for "*open loop controller*" $u = [u_k, \ldots u_{k+N}]$

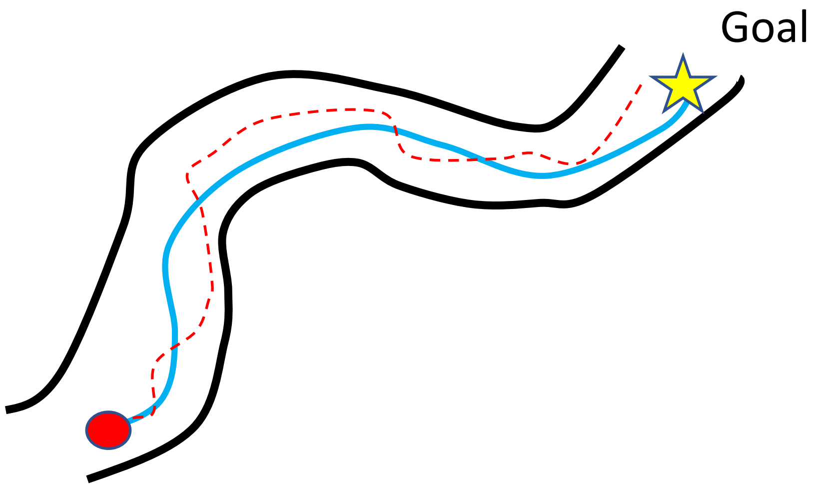
➢ Chance/Chance Constrained optimization formulation



2.3 Flow-Tube Based Control Of Probabilistic Nonlinear Systems

- We design a **closed-loop controller** (*Polynomial State Feedback*), to follow the given nominal trajectory (x_k^*, u_k^*) $k = [0, \dots N-1]$ in presence of uncertainties.





🟦 Planned Trajectory from initial pose to the goal pose.

🟥 Actual trajectory due to disturbances.

⬛ Flow-Tube around the planned trajectory

- To cope with uncertainties, we design a closed-loop controller (*Polynomial State Feedback*) to,

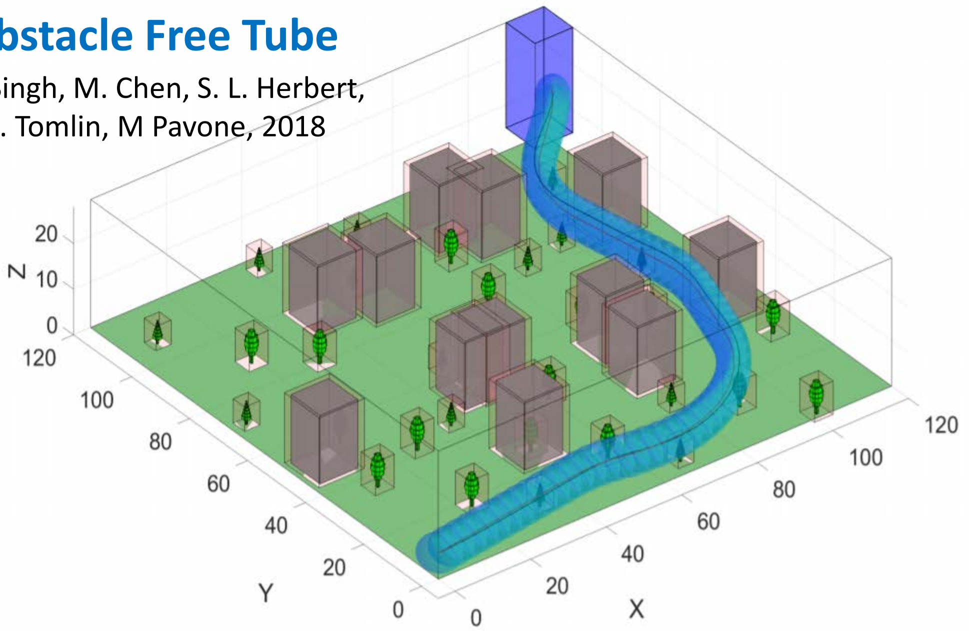
i) follow the given nominal trajectory

ii) for safety purposes remain in the tube around the nominal trajectory, despite all uncertainties

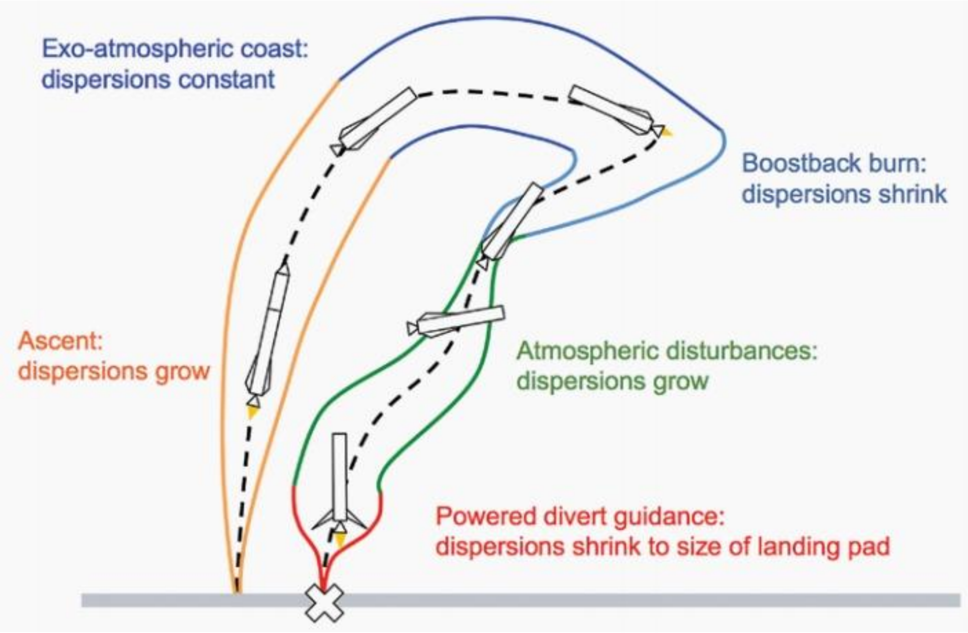
2.3 Flow-Tube Based Control Of Probabilistic Nonlinear Systems

Obstacle Free Tube

S. Singh, M. Chen, S. L. Herbert,
C. J. Tomlin, M Pavone, 2018

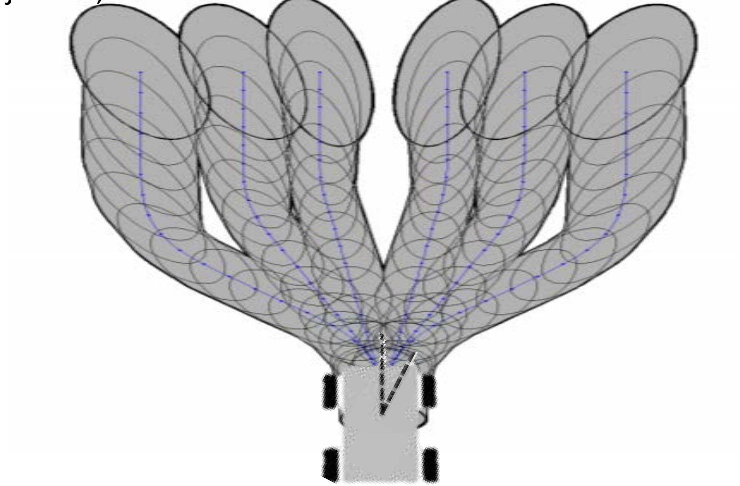


SpaceX

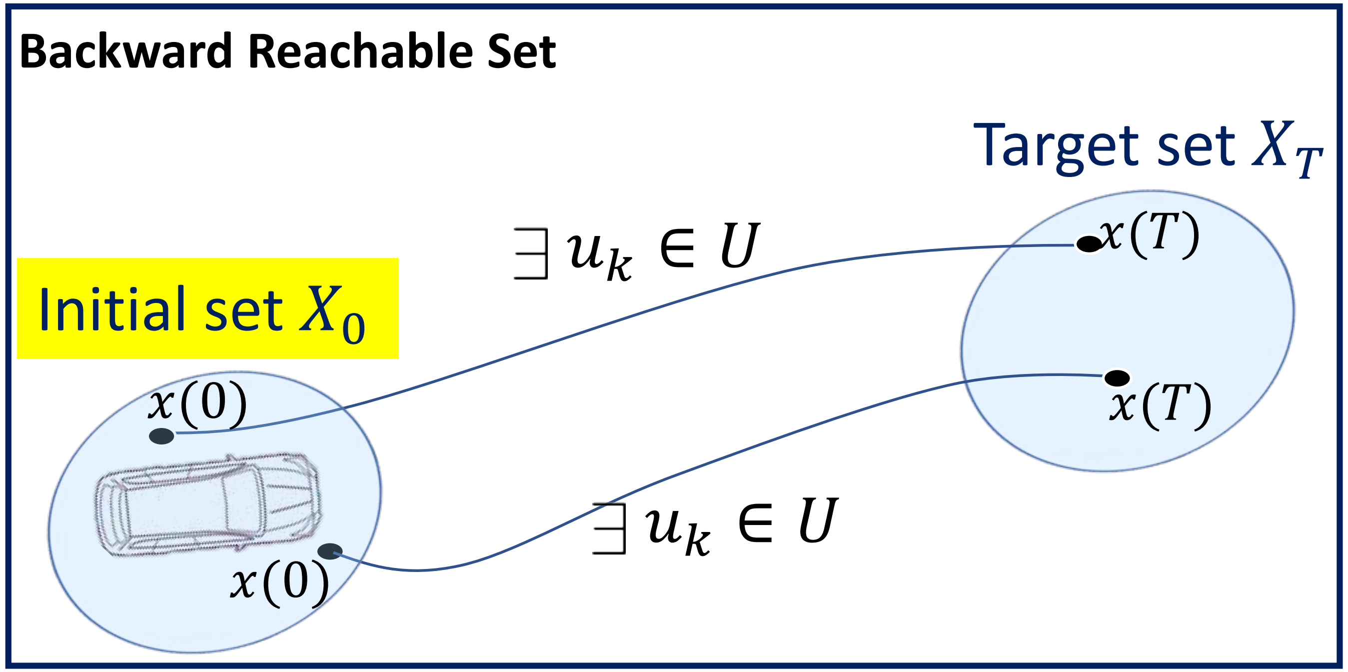


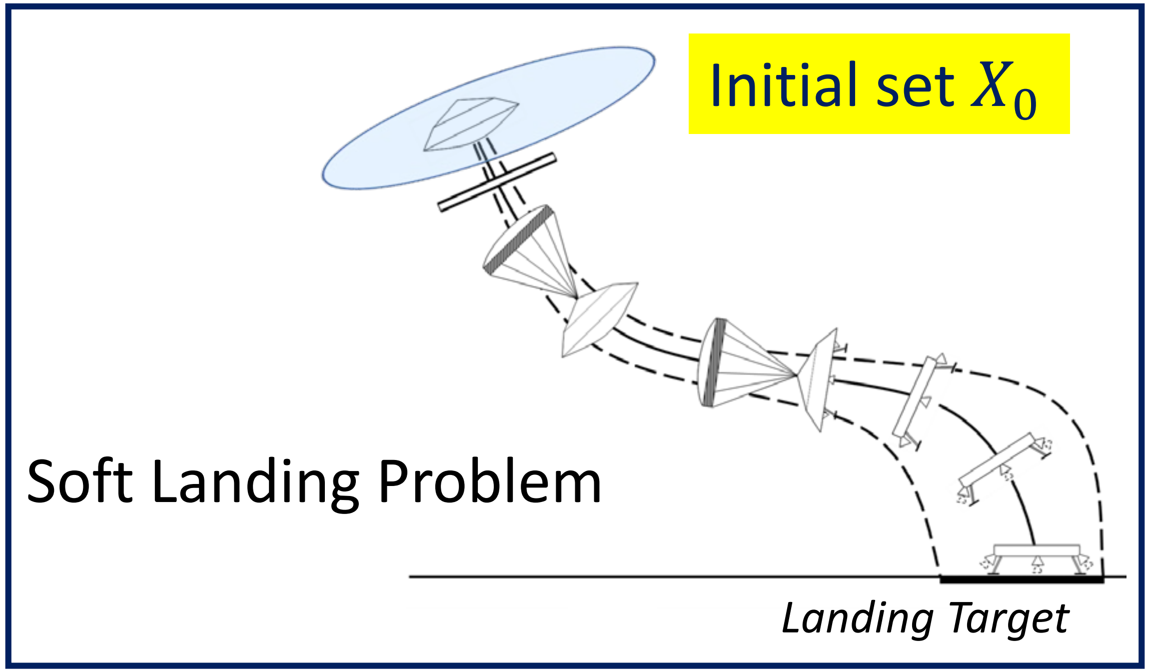
Library of tubes for real-time motion planning

A. Majumdar, R. Tedrake 2017



2.4 Chance Constrained Backward Reachable Sets For Probabilistic Nonlinear Systems





- **Backward Reachable Set:** a set of initial states X_0 for which target set X_T is reachable in T time steps under input constraints.

- **Chance Constrained Backward Reachable Set:** a set of initial states X_0 for which Probability of reaching the target set X_T in T time steps under input constraints is greater than $1 - \Delta$.

➢ Chance Constrained Optimization Formulation

3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties

3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties :

- Dynamical Systems with Gaussian Uncertainties:

$$x_{k+1} = Ax_k + Bu_k + \omega_k$$
$$\omega_k \sim N(0, \Sigma_k)$$

$$x_{k+1} = f(x_k, u_k, \omega_k)$$
$$\omega_k \sim N(0, \Sigma_k)$$

- Stochastic Differential Equations (SDE)

$$dx(t) = f(x)dt + g(x)d\omega(t)$$

ω: Brownian motion

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- **Distributionally Robust Chance Constrained Control**

Given mean m^* and covariance Σ^* of uncertainties, we plan for worst-case probability distribution.

- $Pr(m^*, \Sigma^*)$ =Family of probability distributions with mean m^* and covariance Σ^*.
- worst-case scenario: Probability distribution $\mathrm{Pr} \in Pr(m^*, \Sigma^*)$ that causes highest risk in the system.
- We make sure that worst-case risk is bounded by $1 - \Delta$.

4. Occupation Measure and Liouville Equation

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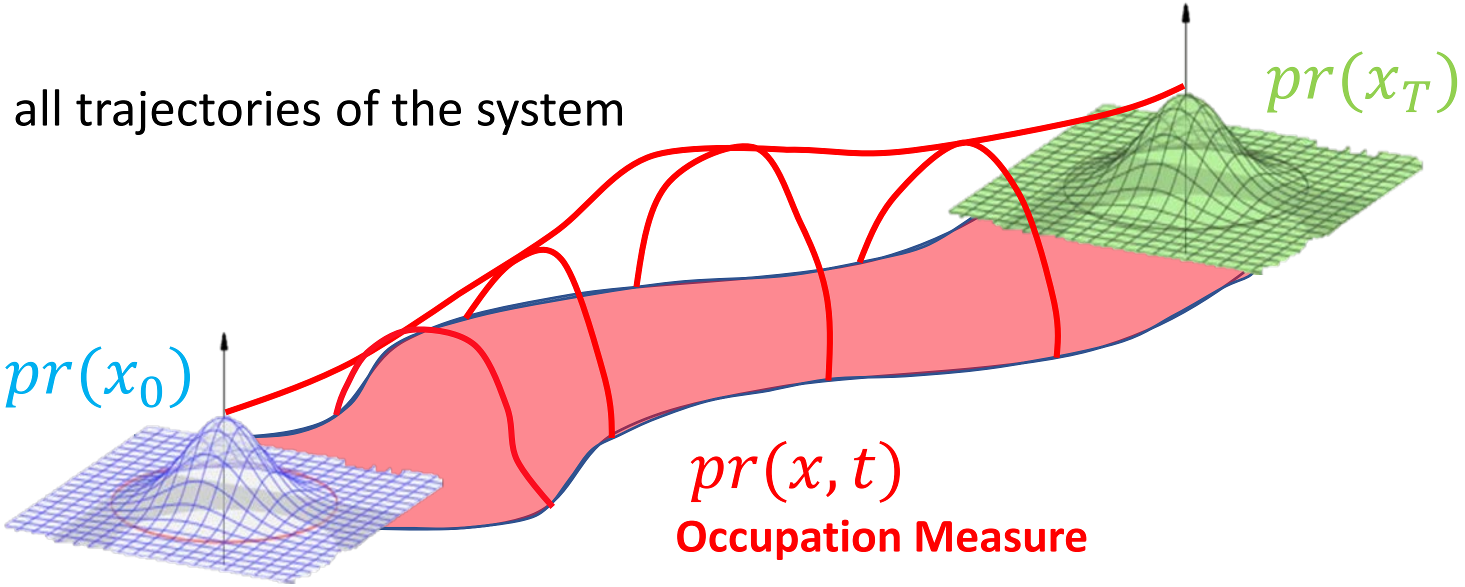
- We will consider nonlinear ordinary differential equation (ODE) with uncertain initial condition

$$\dot{x}(t) = f(x(t), t) \qquad x(0) \sim pr(x_0)$$

- **Liouville's Equation: Linear** Partial Differential Equation (PDE) that describes propagation of initial uncertainty through nonlinear ODE.

- **Occupation Measure:** distribution defined on all trajectories of the system



Distributions $pr(x_0)$, $pr(x_T)$, $pr(x,t)$ are connected through Liouville's equation

4. Occupation Measure and Liouville Equation

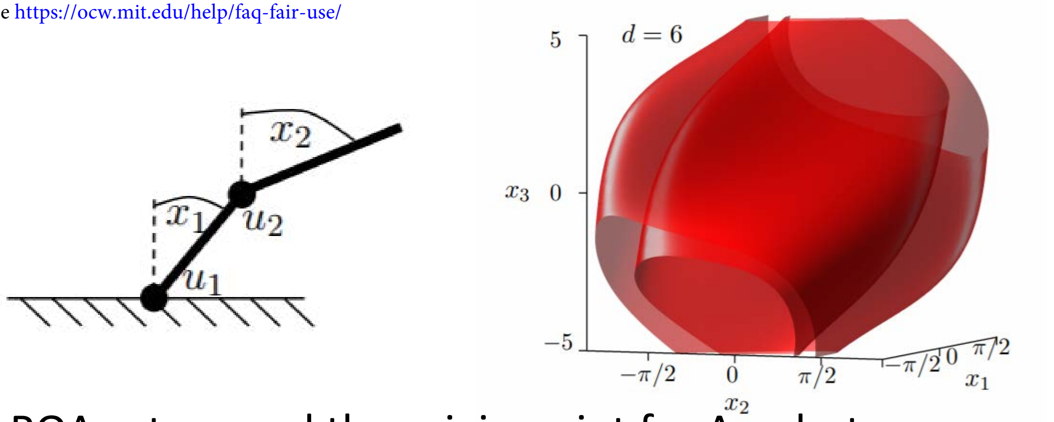
- We leverage Liouville’s Equation, Occupation Measures, and Moment Theory to analyze and control of nonlinear dynamical systems.

- Safety Verification

- Region of Attraction (ROA) Set Computation

i.e., the set of all initial conditions that can be steered to the target set in an admissible way

- Optimal Control



ROA set around the origin point for Acrobot

D. Henrion, M. Korda , 2013

5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems

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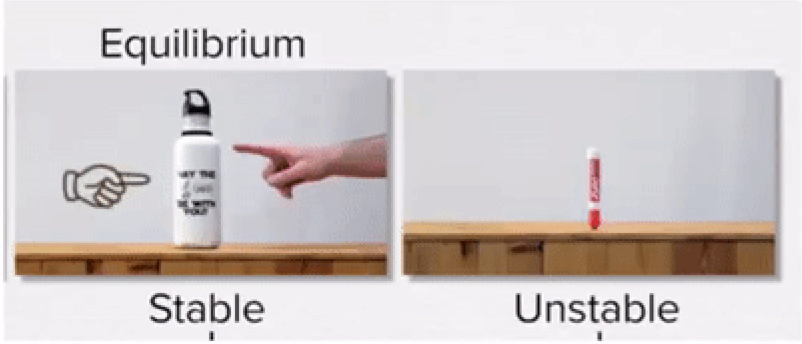
- Relies on classical definition of stability of nonlinear systems
- Lyapunov stability certificate
- SOS SDP formulation



Applications:

5.1 Lyapunov Based Stability and Region of Attraction Set,
5.2 Barrier Function Based Safety Verification,
5.3 Robust Control

Summary of Applications

1. **Probabilistic Safety Verification**
2. **Risk Aware Control and Planning**
3. **Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties**
4. **Occupation Measure Based Control and Analyze of Nonlinear Systems**
5. **Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems**

Challenges of SDP Based Planning

- We can formulate many problems in different domains as a special cases of provided optimization frameworks.

- Convex formulations enable us to solve the optimization problems efficiently.

- **What is the Cost of Convexification ?**

Nonlinear Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0,\ i=1,\dots,n_g \end{aligned}$$

- ➢ Number of decision variables: n , $(x_1, \dots, x_n)$

- In the SOS based SDP we look for a polynomial $P(x) \geq 0$ of order d.
  - ➢ Number of decision variables: Coefficients of polynomial $\binom{n+d}{n} = \frac{(n+d)!}{n!d!}$
- In the Moment based SDP, we look for a moments of probability distribution $\mathrm{pr}(x)$ up to order d.
  - ➢ Number of decision variables: Moments up to order d $\binom{n+d}{n} = \frac{(n+d)!}{n!d!}$

➢ Convexification increases the space of decision variables

- In the absence of problem structure, sum of squares problems are currently limited, roughly speaking, to a several thousands variables (variables in SDP).

- How to address large scale problems?

DSOS and SDSOS Optimization: More Tractable Alternatives to Sum of Squares and Semidefinite Optimization, A.A. Ahmadi, A. Majumdar, SIAM J. Appl. Algebra Geom. 2017

1) Modified SOS optimization that results in

i) smaller SDP’s or ii) other types of convex constraints like LP.

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2) Taking advantage of structure of the problem like sparsity

This results in following techniques:

1) Spars Sum-of-Squares Optimization (SSOS)
2) Bounded Degree Sum-of-Squares Optimization (BSOS, SBSOS)
3) (Scaled) Diagonally Dominant Sum-of-Squares Optimization (DSOS, SDSOS)

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Example:

DSOS and SDSOS Optimization: More Tractable Alternatives to Sum of Squares and Semidefinite Optimization, A.A. Ahmadi, A. Majumdar, SIAM J. Appl. Algebra Geom. 2017

SOS based SDP problem that takes 1526.5 (s) ⇒ DSOS runtime: 9.67 (s)
SDSOS runtime:25.9 (s)

Y. Zheng, G. Fantuzzi, A. Papachristodoulou, “Sparse sum-of-squares (SOS) optimization: A bridge between DSOS/SDSOS and SOS optimization for sparse polynomials”, 2018

SOS based SDP problem that takes 262.08 (s) ⇒ SSOS runtime:0.76 (s)
DSOS runtime: 2.89 (s)
SDSOS runtime:5 (s)

Application
Atlas Robot with 30 states and 14 inputs.

Main Benefit: They can scale to problems where SOS programming ceases to run due to memory/computation constraints.

3) Reformulating original optimization problem to reduce the size of the problem

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Example:

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Example: flow-tube based control

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Example: replacing risk estimation problem involving n uncertain parameters (multivariate SOS) with univariate risk estimation problem (univariate SOS)

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- Planning in subspace:

Example: instead of constructing reachable set in n-dimensional state space, i.e., $(x_1, \ldots, x_n)$ construct reachable set in the subspaces of $(x_i, x_{i+1}), i = 1, \ldots, n-1$

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4) Efficient Algorithms for Large Scale SDP's (Guest Lecture)

Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications

Week		Lecture Topic
1		**Introduction and Overview of the Course**
2		**Overview of Nonlinear and Convex Optimization:** i) Optimality Conditions, ii) Newton’s Method, iii) Interior Point Method, iv) Dual Optimization, v) Convex Optimization, vi) Linear Program, vii) Semidefinite Program
3		**Nonlinear Optimization Using the Theory of Nonnegative Polynomials,** Sum-of-Squares Formulation (SOS)
4		**Nonlinear Optimization Using the Theory of Measure and Moments**
5		**Duality:** i) Duality of Moments and Polynomials , ii) Duality of Measures and Continuous Functions
6		**Modified Sum-of-Squares Optimization:** i) Spars Sum-of-Squares Optimization (SSOS), ii) Bounded Degree Sum-of-Squares Optimization (BSOS), iii) (Scaled)Diagonally Dominant Sum-of-Squares Optimization (SDSOS, DSOS)
7		**Chance Optimization and Chance Constrained Optimization**: i) Measure and Moments Formulation , ii) Sum-of-Squares Formulation
8		i) **Robust Optimization Using Sum-of-Squares Optimization** ii) **Distributionally Robust Chance-Constrained Optimization**
9		**Algorithms for Large Scale Semidefinite Programs (Guest Lecture)**
10		**Safety Verification of Probabilistic Systems:** i) Risk Estimation, ii) Probabilistic Uncertainty Propagation, iii) Uncertainty Set Construction, iv) Forward Reachable Sets
11		**Risk Aware Planning and Control:** i) Risk Bounded Trajectory Planning, ii) Risk Aware Nonlinear Control, iii) Flow-Tube Based Control, iv) Backward Reachable Sets
12		**Dynamical Systems with Gaussian Uncertainties**: i) Chance Constrained Control, ii) Safety Verification, iii) Distributionally Robust Chance Constraints
13		**Occupation Measure Based Analyze and Control**: i) Safety Verification, ii) Region of Attraction Set, iii) Optimal Control
14		**Sum-of-Squares Optimization for Uncertain Nonlinear Systems:** i) Lyapunov Based Stability and Region of Attraction Set, ii) Barrier Function Based Safety Verification, iii) Robust Control
15		**Final Project Presentation**

Prerequisites: Linear Algebra (e.g., 18.06), Convex Optimization (e.g., 6.215, 6.251, 6.255), Probability Theory (e.g., 6.431), Dynamical Systems (e.g., 6.241) or permission of the instructor.

Assignments and Grading: 50% Problem Sets, 50% Research Project

Problem sets will be posted in the course website and will be due one week later.

Bibliography: Variety of book and recent papers will be introduced for each lecture.

Research Project

- Apply the provided techniques to your research problems.
- Implementation of other techniques that address uncertain nonlinear problems.
- Research Projects, i.e. improving and extending the state-of-the-art

16.S498 Risk Aware and Robust Nonlinear Planning
Fall 2019