14.12 Recitation 2

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Concepts

1. Rationality: formally, a player is said to be rational if and only if he maximizes the expected value of his payoffs (given his beliefs about the other players' strategies.)

2. Dominance: A strategy s_i^* *strictly dominates* s_i if and only if

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

3. Best response: For any player i, a strategy s_i^{BR} is a *best response* to s_{-i} if and only if

$$u_i(s_i^{BR}, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_i \in S_i$$

This definition is identical to that of a dominant strategy except that it is not for all $s_{-i} \in S_{-i}$ but for a specific strategy s_{-i}. If it were true for all s_{-i}, then S_i^{BR} would also be a dominant strategy, which is a stronger requirement than being a best response against some strategy s_{-i}.

4. Nash Equilibrium: strategy profile $(s_1^{NE}, ...s_N^{NE})$ is a *Nash Equilibrium* if and only if s_i^{NE} is a best response to $s_{-i}^{NE} = (s_1^{NE}, ...s_{i-1}^{NE}, s_{i+1}^{NE}, ...s_N^{NE})$ for each i. That is, for all i, we have that

$$u_i(s_i^{NE}, s_{-i}^{NE}) \geq u_i(s_i, s_{-i}^{NE}) \quad \forall s_i \in S_i.$$

Problem 1 (Similar to HW1-4)

Suppose there is a (polluting) firm and a (pollution-averse) consumer. The firm either pollutes or is shut down. One way for the (rich) government to resolve the externality is as follows:

1. Ask the firm to state the monetary benefit $\hat{b}$ of generating pollution

2. Ask the consumer to state the monetary equivalent of the cost of suffering pollution, $\hat{c}$.

3. Shut the firm down iff $\hat{c} \geq \hat{b}$. If the firm is open, give the consumer $\hat{b}$ and charge the firm $\hat{c}$

The players are 1) the firm, and 2) the consumer. True benefit and cost are b and c, respectively.

(a) Write this in the normal form.

The strategies are $\hat{b} \in [0, \infty]$ and $\hat{c} \in [0, \infty]$. Utility (payoffs) from strategy profile $\left(\hat{b}, \hat{c}\right)$ of players are $u_f\left(\hat{b}, \hat{c}\right) = (b - \hat{c})\left[\hat{b} > \hat{c}\right]\left(= (b - \hat{c})\, 1_{[\hat{b} > \hat{c}]}\right)$ and $u_c\left(\hat{b}, \hat{c}\right) = \left(\hat{b} - c\right)\left[\hat{b} > \hat{c}\right]$.

(b) Check if there is a dominant strategy equilibrium, and compute it if there is one.

First, check the firm. Suppose $\hat{b} > b$. Then, three cases:

i) $\hat{b} > b > \hat{c}$: $u_f\left(\hat{b}, \hat{c}\right) = (b - \hat{c}) = u_f(b, \hat{c})$

ii) $\hat{b} > \hat{c} \geq b$: $u_f\left(\hat{b}, \hat{c}\right) = -(\hat{c} - b) < 0 = u_f(b, \hat{c})$

iii) $\hat{c} \geq \hat{b} > b$: $u_f\left(\hat{b}, \hat{c}\right) = 0 = u_f(b, \hat{c})$

Now, suppose $b > \hat{b}$

i) $b > \hat{b} > \hat{c}$: $u_f\left(\hat{b}, \hat{c}\right) = (b - \hat{c}) = u_f(b, \hat{c})$

ii) $b > \hat{c} \geq \hat{b}$: $u_f\left(\hat{b}, \hat{c}\right) = 0 < b - \hat{c} = u_f(b, \hat{c})$

iii) $\hat{c} \geq b > \hat{b}$: $u_f\left(\hat{b}, \hat{c}\right) = 0 = u_f(b, \hat{c})$

Hence, for any $\hat{b}$, $u_f(b, \hat{c}) \geq u_f\left(\hat{b}, \hat{c}\right)$ and there exist $\hat{c}^*\left(\hat{b}\right)$ such that $u_f(b, \hat{c}^*) > u_f\left(\hat{b}, \hat{c}^*\right)$.

Now, check the consumer. Suppose $\hat{c} > c$. Then,

i) $\hat{c} > c \geq \hat{b}$: $u_c\left(\hat{b}, \hat{c}\right) = 0 = u_c\left(\hat{b}, c\right)$

ii) $\hat{c} \geq \hat{b} > c$: $u_c\left(\hat{b}, \hat{c}\right) = 0 < \hat{b} - c = u_c\left(\hat{b}, c\right)$

iii) $\hat{b} > \hat{c} > c$: $u_c\left(\hat{b}, \hat{c}\right) = \hat{b} - c = u_c\left(\hat{b}, c\right)$

Finally, suppose $c > \hat{c}$:

i) $c > \hat{c} \geq \hat{b}$: $u_c\left(\hat{b}, \hat{c}\right) = 0 = u_c\left(\hat{b}, c\right)$

ii) $c \geq \hat{b} > \hat{c}$: $u_c\left(\hat{b}, \hat{c}\right) = -\left(c - \hat{b}\right) < 0 = u_c\left(\hat{b}, c\right)$

iii) $\hat{b} > c > \hat{c}$: $u_c\left(\hat{b}, \hat{c}\right) = \hat{b} - c = u_c\left(\hat{b}, c\right)$

Therefore, both the consumer and the firm have truth-telling as a weakly dominant strategy.

One problem: The government suffers a deficit of $\hat{b} - \hat{c}$, (or it may suffer a surplus if the firm is closed if the government taxes everyone beforehand).

Problem 2 (2011 Midterm 1-1)

(a) Compute the set of all rationalizable strategies in the following game.

	w	x	y	z
a	0,3	0,1	3,0	0,1
b	3,0	0,2	2,4	1,1
c	2,4	3,2	1,2	10,1
d	0,5	5,3	1,2	0,10

Answer: Iterated Elimination of Strictly Dominated Strategies: eliminate all the strictly dominated strategies and iterate this k-times. In this procedure, one eliminates all the strictly dominated strategies and iterates this k times. Two main points are:

1. One must eliminate only the strictly dominated strategies. One cannot eliminate a strategy if it is weakly dominated but not strictly dominated.

2. One must eliminate the strategies that are stricly dominated by mixed strategies (but not necessarily by pure strategies).

Strategy x is strictly dominated by the mixed strategy σ_2 with $\sigma_2(w) \in \left(\frac{1}{3},\ \frac{1}{2}\right)$ and $\sigma_2(y) = 1 - \sigma_2(w)$. In the first round, x is therefore eliminated. (No other strategy is eliminated in that round.) In the second round, d is strictly dominated by b and eliminated. In the third round, z is strictly dominated by σ_2 above and eliminated. In the fourth round, c is strictly dominated by b and eliminated. There are no other elimination, and the set of rationalizable strategies is {a,b}×{w,y}. (Note: explain how to find which strategy to eliminate by checking best responses)

(b) Compute the set of all Nash equilibria.

Answer: The only Nash equilibrium is σ^* where $\sigma_1^*(a) = \frac{4}{7}$, $\sigma_1^*(b) = \frac{3}{7}$ and $\sigma_2^*(w) = \frac{1}{4}$, $\sigma_2^*(y) = \frac{3}{4}$. (Note: explain how to derive this - in order for one player to mix strategies and play multiple strategies with positive probabilities, he must be indifferent between those strategies, or have the same expected utility for all choices)

Problem 3 (2009 HW2-1)

Consider the following investment game. There are two firms, each of them has to decide how much to invest. If we let $k_i \geq 0$ be the investment choice of firm i, then the profits of the firm are given by:

$$\pi_i(k_i,\ k_j) = Ak_i - \frac{k_i^2}{2}$$

where the productivity of firm i is given by $A = \alpha + (k_i + k_j)(1 - \alpha)$, where $\alpha \in (\frac{2}{3},\ 1]$. Note how productivity depends on the investment level of both firms.

(a) What are the best response function for each firm as a function of α?

Each firm will maximize its profit given the other firm's strategy. The first order condition is

$$\frac{\partial \pi_i}{\partial k_i} = \alpha + k_j(1-\alpha) + k_i(1-2\alpha) = 0$$

In other words,

$$k_i = BR_i(k_j) = \frac{\alpha + k_j(1-\alpha)}{2\alpha - 1}$$

(b) Find the Nash equilibrium of the game, call it $(k(\alpha),\ k(\alpha))$. Note that in the Nash equilibrium both firms choose the same investment levels.

The Nash equilibrium is the point at which the two best response functions intersect. By symmetry, we set $k_i = k_j = k$ and solve

$$k = \frac{\alpha + k(1-\alpha)}{2\alpha - 1}$$

The NE is $\left(\frac{\alpha}{3\alpha-2},\ \frac{\alpha}{3\alpha-2}\right)$.

(c) What happens when $\alpha \to 1$? Does the equilibrium investment level increase or decrease? Do you have intuition for this result?

$k = \frac{\alpha}{3\alpha-2} = \frac{1}{3} + \frac{2/3}{3\alpha-2}$. Thus, as $\alpha \to 1$, the investment level decreases. Intuition is clear: as α increases, productivity depends less on the total investment level and firms want to invest less.

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